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NORMALITY CRITERIONS ON SHARING A HOLOMORPHIC FUNCTION

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Abstract. Let $\mathcal{F}$ be a family of holomorphic functions in a domain D, whose zeros have multiplicity at least 4 in D. Let $\psi(\not\equiv 0)$ be a holomorphic function in D, whose zeros have multiplicity at most 2. If for each pair of functions $\{f, g\} \subset \mathcal{F}$, f' and g' share ψ in D, then $\mathcal{F}$ is normal in D.

1. Introduction and main result

Let $\mathbb{C}$ be the complex plane and D be a domain on $\mathbb{C}$. For $z_0 \in \mathbb{C}$ and $r > 0$, we write $\Delta(z_0, r) := \{z \,|\, |z - z_0| < r\}$, $\Delta := \Delta(0, 1)$ and $\Delta'(z_0, r) := \{z \,|\, 0 < |z - z_0| < r\}$. We write $f_n \overset{\chi}{\Rightarrow} f$ in D to indicate that the sequence $\{f_n\}$ converges to f in the spherical metric uniformly on compact subsets of D and $f_n \Rightarrow f$ in D if the convergence is in the Euclidean metric. We also write $\mathbb{P}_f := f^{-1}\{\infty\}$ for the set of poles of a meromorphic function f.

Let f and g be two meromorphic functions and φ be a holomorphic function in D. We say that f and g share φ in D if $\{z \,|\, f(z) = \varphi(z), z \in D\} = \{z \,|\, g(z) = \varphi(z), z \in D\}$.

Our point of departure is the following criterions for normality due to Xu [1].

Theorem 1. [1, Lemma 7] *Let $\mathcal{F}$ be a family of holomorphic functions in D, and let $\psi(\not= 0)$ be a holomorphic function in D. Suppose that, for each $f \in \mathcal{F}$, f has no zeros in D. If for each pair of functions $\{f, g\} \subset \mathcal{F}$, f' and g' share ψ in D, then $\mathcal{F}$ is normal in D.*

Theorem 2. [1, Theorem 1.1] *Let $\mathcal{F}$ be a family of meromorphic functions in D, and let $\psi(\not\equiv 0)$ be a holomorphic function in D whose zeros are simple. Suppose that, for each $f \in \mathcal{F}$, f has no zeros and has only multiple poles in D. If for each pair of functions $\{f, g\} \subset \mathcal{F}$, f' and g' share ψ in D, then $\mathcal{F}$ is normal in D.*

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In this paper, we show the condition that f has no zeros for each $f \in \mathcal{F}$ in Theorem 1 can be weakened. As for a family of holomorphic functions $\mathcal{F}$, we show that the condition f has only multiple poles is not necessary and the condition that f has no zeros for each $f \in \mathcal{F}$ can be weakened in Theorem 2. In this paper, we prove the following theorems.

Theorem 1.1. *Let $\mathcal{F}$ be a family of meromorphic functions in D, and let $\psi(\neq 0)$ be a holomorphic function in D. Suppose that, for each $f \in \mathcal{F}$, all zeros of f have multiplicity at least 4 in D. If for each pair of functions $\{f, g\} \subset \mathcal{F}$, f' and g' share ψ in D, then $\mathcal{F}$ is normal in D.*

Remark 1.2. The following example shows the condition that all zeros of f have multiplicity at least 4 in Theorem 1.1 is sharp.

Set $D := \Delta$, $\psi(z) := 1$ and $f_n(z) := \frac{(z+\frac{1}{n})^3}{(z-\frac{2}{n})^2}$, where $n = 1, 2, 3, \cdots$. Obviously, $\{f_n(z)\}$ fails to be normal at 0.

In fact $f_n'(z) - 1 = -\frac{27z}{n^2(z-\frac{2}{n})^3}$ has only one zero $z = 0$ in $\mathbb{C}$.

Theorem 1.3. *Let $\mathcal{F}$ be a family of holomorphic functions in D, and let $\psi(\not\equiv 0)$ be a holomorphic function in D, whose zeros have multiplicity at most 2. Suppose that, for each $f \in \mathcal{F}$, all zeros of f have multiplicity at least 4 in D. If for each pair of functions $\{f, g\} \subset \mathcal{F}$, f' and g' share ψ in D, then $\mathcal{F}$ is normal in D.*

Remark 1.4. The following example shows the condition that all zeros of $\psi(z)$ have multiplicity at most 2 in Theorem 1.3 is sharp.

Set $D := \Delta$, $\psi(z) := z^3$ and $f_n(z) := nz^4$, where $n = 1, 2, 3, \cdots$. Obviously, $\{f_n(z)\}$ fails to be normal at 0.

Remark 1.5. The condition that all zeros of f have multiplicity at least 4 in Theorem 1.3 is also sharp as shows by the following example.

Set $D := \Delta$, $\psi(z) := z^2$ and $f_n(z) := nz^3$, where $n = 1, 2, 3, \cdots$. Obviously, $\{f_n(z)\}$ fails to be normal at 0.

2. Lemmas

Lemma 2.1. [2, Lemma 2] *Let $\mathcal{F}$ be a family of functions meromorphic in D, whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if $\mathcal{F}$ is not normal at z_0, there exist, for each $0 \leq \alpha \leq k$,*

(a) *points z_n, $z_n \to z_0$;*
(b) *functions $f_n \in \mathcal{F}$; and*
(c) *positive numbers $\rho_n \to 0$*

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) = g_n(\zeta) \stackrel{\chi}{\Rightarrow} g(\zeta)$ in $\mathbb{C}$, where g is a nonconstant meromorphic function in $\mathbb{C}$, whose zeros have multiplicity at least k, such that $g^{\#}(\zeta) \leq g^{\#}(0) = kA+1$.

The following result follows from Lemma 2.4 in [3] and Lemma 5 in [4].

Lemma 2.2. *Let $R(z)$ be a nonconstant rational function, whose zeros have multiplicity at least 4. Then $R'(z) - 1$ has at least two distinct zeros.*

Lemma 2.3. [5, Theorem 1] *Let f be a transcendental meromorphic function in $\mathbb{C}$, all but finitely many of whose zeros are multiple, and let $R(\not\equiv 0)$ be a rational function. Then $f' - R$ has infinitely many zeros.*

Lemma 2.4. [6, Theorem 1] *Let $h(z)(\not\equiv 0)$ be a holomorphic function in D, and let $\mathcal{F}$ be a family of meromorphic functions in D, whose zeros have multiplicity at least 4. If for each $f \in \mathcal{F}$ and each $z \in D$, $f'(z) \neq h(z)$, then $\mathcal{F}$ is normal in D.*

Lemma 2.5. *Let $l(\leq 2)$ be a positive integer, and $P(z)$ be a nonconstant polynomial, whose zeros have multiplicity at least 4. Then $P'(z) - z^l$ has at least two distinct zeros in $\mathbb{C}$.*

Proof. Obviously, $P'(z) - z^l$ has at least one zero in $\mathbb{C}$. We argue by contradiction. Suppose that $P'(z) - z^l$ only has a single zero z^* (not counting multiplicity) and derive a contradiction.

Then we can write

$$P(z) = C_1 \prod_{i=1}^{s} (z - a_i)^{\alpha_i},$$

where C_1 is a nonzero constant, s, $\alpha_1(\geq 4)$, $\alpha_2(\geq 4)$, $\cdots$, $\alpha_s(\geq 4)$ are positive integers, and $a_1, a_2, \cdots a_s$ are distinct complex numbers. By the condition of this lemma, we have

$$P'(z) - z^l = C_2(z - z^*)^t, \tag{2.1}$$

where C_2 is a nonzero constant, $t(= -1 + \sum_{i=1}^{s} \alpha_i)$ is a positive integer.

We claim that $P(z)$ has only a single zero (not counting multiplicity) in $\mathbb{C}$. Taking the $(l-1)$-th and l-th derivative of both sides of the equation (2.1) separately, we obtain

$$P^{(l)}(z) = C_3(z - z^*)^{t-l+1} + C_4 z,$$
$$P^{(l+1)}(z) = (t - l + 1)C_3(z - z^*)^{t-l} + C_4,$$

where C_3, C_4 are nonzero constants. Let α^* is a zero of $P(z)$. Clearly, α^* is also a zero of $P^{(l)}(z)$ and $P^{(l+1)}(z)$. Then we have

$$\begin{cases} C_3(\alpha^* - z^*)^{t-l+1} + C_4\alpha^* = 0, \\ (t - l + 1)C_3(\alpha^* - z^*)^{t-l} + C_4 = 0. \end{cases} \tag{2.2}$$

Solving the system of equations (2.2), we obtain $\alpha^* = \frac{z^*}{l-t}$. This implies $P(z)$ has only has a single zero (not counting multiplicity) in $\mathbb{C}$.

Now we can write

$$P(z) = C_1(z - a_1)^{\alpha_1},$$

and then

$$P'(z) - z^l = C_1\alpha_1(z - a_1)^{\alpha_1 - 1} - z^l = C_2(z - z^*)^t.$$

Comparing the coefficients of the term z^t, we see that $C_1\alpha_1 = C_2$ and hence $z^l \equiv 0$. We obtain a contradiction. ☐

Lemma 2.6. *Let $\psi(\not\equiv 0)$ be a holomorphic function in Δ such that $\psi(z) = z^l\widehat{\psi}(z)$, where l is a positive integer, $\widehat{\psi}(0) = 1$ and $\widehat{\psi}(z) \neq 0, \infty$ in Δ. Let $\{f_n\}$ be a family of meromorphic functions in Δ, whose zeros have multiplicity at least 4. Suppose that*

(a) *for any two distinct positive integers m and n, f'_m and f'_n share ψ in Δ;*
(b) *no subsequence of $\{f_n\}$ is normal at 0.*

Then $\{\frac{f_n(\eta_n z)}{\eta_n^{l+1}}\}$ *is normal in* $\mathbb{C}\setminus\{0\}$*, where* η_n *is a positive number such that* $\eta_n\to 0$ *as* $n\to\infty$.

We argue by contradiction. Suppose that $\{\frac{f_n(\eta_n z)}{\eta_n^{l+1}}\}$ is not normal at $z^*\in\mathbb{C}\setminus\{0\}$.

By Lemma 2.1, there exist points $z_n\to z^*$, and positive numbers $\rho_n\to 0$ and a subsequence of $\{f_n\}$ (still denoted by $\{f_n\}$) such that

$$F_n(\zeta)=\frac{f_n[\eta_n(z_n+\rho_n\zeta)]}{\eta_n^{l+1}\rho_n}\overset{\chi}{\Rightarrow}F(\zeta)\mathrm{in}\mathbb{C},$$

where $F(\zeta)$ is a nonconstant meromorphic function in $\mathbb{C}$, whose zeros have multiplicity at least 4. Then we have

$$\begin{aligned}\frac{f_n'[\eta_n(z_n+\rho_n\zeta)]-\psi[\eta_n(z_n+\rho_n\zeta)]}{\eta_n^l}&=F_n'(\zeta)-(z_n+\rho_n\zeta)^l\widehat{\psi}[\eta_n(z_n+\rho_n\zeta)]\\&\overset{\chi}{\Rightarrow}F'(\zeta)-(z^*)^l\mathrm{in}\mathbb{C}\setminus\mathbb{P}_F.\end{aligned}$$

Clearly, $F'(\zeta)-(z^*)^l\not\equiv 0$ in $\mathbb{C}$ (Otherwise, $F(\zeta)$ is a polynomial of order 1 in $\mathbb{C}$ which contradicts the fact that all zeros of F have multiplicity at least 4). By lemma 2.2 and lemma 2.3, $F'(\zeta)-(z^*)^l$ has at least two distinct zeros. Let ζ_1 and ζ_2 be two distinct zeros of $F'(\zeta)-(z^*)^l$. It now follows from Hurwitz's Theorem that there exist sequences $\zeta_{n,1}\to\zeta_1$ and $\zeta_{n,2}\to\zeta_2$ such that, for sufficiently large n, $f_n'(z_{n,1})-\psi(z_{n,1})=0$ and $f_n'(z_{n,2})-\psi(z_{n,2})=0$, where $z_{n,1}=\eta_n(z_n+\rho_n\zeta_{n,1})$ and $z_{n,2}=\eta_n(z_n+\rho_n\zeta_{n,2})$. For any fixed positive integer m, we consider the function f_m. According to the hypothesis of Theorem 1.1, we have $f_m'(z_{n,1})-\psi(z_{n,1})=0$ and $f_m'(z_{n,2})-\psi(z_{n,2})=0$. Noting that $z_{n,1}\to 0$, $z_{n,2}\to 0$ and $z_{n,1}\neq z_{n,2}$ for sufficiently large n, we see that, for any arbitrarily small (but fixed) positive number δ, $f_m'(z)-\psi(z)$ has at least two distinct zeros $z_{n,1},z_{n,2}$ in $\Delta(0,\delta)$ when n is sufficiently large. By the uniqueness theorem of analytic functions, $f_m'(z)\equiv\psi(z)$ in Δ. By the hypothesis of Theorem 1.1, we see that $f_n'(z)\equiv\psi(z)$ and thus $f_n'(z)\neq\psi(z)+1$ for all n in Δ. By Lemma 2.4, $\{f_n\}$ is normal in Δ. This is a contradiction.

3. Proof of Theorem 1.1

We argue by contradiction. Suppose that $\mathcal{F}$ is not normal at $z^*\in D$.

By Lemma 2.1, there exist points $z_n\to z^*$, and positive numbers $\rho_n\to 0$ and a subsequence $\{f_n\}$ of $\mathcal{F}$ such that

$$F_n(\zeta)=\frac{f_n(z_n+\rho_n\zeta)}{\rho_n}\overset{\chi}{\Rightarrow}F(\zeta)\mathrm{in}\mathbb{C},\tag{3.1}$$

where $F(\zeta)$ is a nonconstant meromorphic function in $\mathbb{C}$. Clearly, all zeros of F have multiplicity at least 4 in $\mathbb{C}$. Then we have

$$F_n'(\zeta)-\psi(z_n+\rho_n\zeta)=f_n'(z_n+\rho_n\zeta)-\psi(z_n+\rho_n\zeta)\overset{\chi}{\Rightarrow}F'(\zeta)-\psi(z^*)\mathrm{in}\mathbb{C}\setminus\mathbb{P}_F.$$

Clearly, $F'(\zeta)-\psi(z^*)\not\equiv 0$ in $\mathbb{C}$ (Otherwise, $F(\zeta)$ is a polynomial of order 1 in $\mathbb{C}$ which contradicts the fact that all zeros of F have multiplicity at least 4). By lemma 2.2 and lemma 2.3, $F'(\zeta)-\psi(z^*)$ has at least two distinct zeros. Let ζ_1 and ζ_2 be two distinct zeros of $F'(\zeta)-\psi(z^*)$. It now follows from Hurwitz's Theorem that there exist sequences $\zeta_{n,1}\to\zeta_1$ and $\zeta_{n,2}\to\zeta_2$ such that, for sufficiently large n, $f_n'(z_{n,1})-\psi(z_{n,1})=0$ and $f_n'(z_{n,1})-\psi(z_{n,1})=0$, where $z_{n,1}=z_n+\rho_n\zeta_{n,1}$ and $z_{n,2}=z_n+\rho_n\zeta_{n,2}$. For any fixed $g \in \mathcal{F}$, we consider the function g. According to the hypothesis of Theorem 1.1, $g'(z_{n,1}) - \psi(z_{n,1}) = 0$ and $g'(z_{n,2}) - \psi(z_{n,2}) = 0$ for all n. Noting that $z_{n,1} \to z^*$, $z_{n,2} \to z^*$ and $z_{n,1} \neq z_{n,2}$ for sufficiently large n, we see that, for any arbitrarily small (but fixed) positive number δ, $g'(z) - \psi(z)$ has at least two distinct zeros $z_{n,1}, z_{n,2}$ in $\Delta(z^*, \delta)$ when n is sufficiently large. By the uniqueness theorem of analytic functions, $g'(z) \equiv \psi(z)$. By the hypothesis of Theorem 1.1, for all $f \in \mathcal{F}$, we see that $f'(z) \equiv \psi(z)$, and then $f'(z) \neq \psi(z)+1$ in D. By Lemma 2.4, $\mathcal{F}$ is normal in D. This is a contradiction.

4. Proof of Theorem 1.3

We argue by contradiction. Suppose that $\mathcal{F}$ is not normal at 0 and derive a contradiction.

Set

$$\mathbb{E} := \{z \,|\, \psi(z) = 0, z \in D\}.$$

By Theorem 1.1, $\mathcal{F}$ is normal in $D \setminus \mathbb{E}$. Since normality is a local property, it remains to prove that $\mathcal{F}$ is normal at each point of $\mathbb{E}$.

Without loss of generality, for arbitrary $p^* \in \mathbb{E}$, we may assume that $p^* = 0$, $\Delta \subset D$, $\Delta' \cap \mathbb{E} = \emptyset$, and

$$\psi(z) = z^l + a_{l+1}z^{l+1} + \cdots = z^l \widehat{\psi}(z),$$

where $l \leq 2$ is a positive integer, $\widehat{\psi}(0) = 1$ and $\widehat{\psi}(z) \neq 0, \infty$ in Δ.

By the definition of normality, we can take a subsequence $\{f_n\}$ of $\mathcal{F}$ such that no subsequence of $\{f_n\}$ is normal at 0.

We show that there exists $\delta_0 \in (0,1)$ such that $f_n'(0) = 0$ and $f_n'(z) \neq \psi(z)$ in $\Delta'(0, \delta_0)$ for all n. For any fixed positive integer m, we consider the function f_m. Firstly, assume that $f_m'(z) \equiv \psi(z)$ in Δ. By the hypothesis of Theorem 1.3, we see that $f_n'(z) \equiv \psi(z)$ for all n in Δ, and hence $f_n'(z) \neq \psi(z) + 1$ for all n in Δ. By Theorem 2.4, $\{f_n\}$ is normal Δ. We obtain a contradiction. Secondly, assume that $f_m'(0) \neq \psi(0) = 0$ in Δ. Thus there exists $\delta_1 \in (0,1)$ such that $f_m'(z) \neq \psi(z)$ in $\Delta(0, \delta_1)$. By the hypothesis of Theorem 1.3, we obtain that $f_n'(z) \neq \psi(z)$ for all n in $\Delta(0, \delta_1)$. By Theorem 2.4, $\{f_n\}$ is normal $\Delta(0, \delta_1)$. We also obtain a contradiction. Thus there exists $\delta_0 > 0$ such that $f_m'(0) = 0$ and $f_m'(z) \neq \psi(z)$ in $\Delta'(0, \delta_0)$. The hypothesis of Theorem 1.3 implies that $f_n'(0) = 0$ and $f_n'(z) \neq \psi(z)$ in $\Delta'(0, \delta_0)$ for all n.

By Theorem 1.1, $f_n(z)$ is normal in $\Delta'(0, \delta_0)$. Taking a suitable subsequence and renumbering, we may assume $f_n(z) \Rightarrow f^*(z)$ in $\Delta'(0, \delta_0)$, where $f^*(z)$ is a holomorphic function or identically equal to infinity in $\Delta'(0, \delta_0)$. In fact, we must have $f^*(z) \equiv \infty$ in $\Delta'(0, \delta_0)$. Otherwise, applying the maximum principle to $f_n(z)$ in $\Delta(0, \delta_0)$, we obtain that $\{f_n\}$ is normal at 0. This is a contradiction.

We claim that there exist a subsequence of $\{f_n\}$ (we continue to call $\{f_n\}$) and a sequence of points $z_n \to 0$ such that $f_n(z_n) = 0$. Otherwise, there exists $\delta \in (0, \delta_0)$ such that $\{f_n\}$ has no zero in $\Delta(0, \delta)$. Applying the maximum principle to $\frac{1}{f_n(z)}$ in $\Delta(0, \delta)$, we obtain that $\frac{1}{f_n(z)} \Rightarrow 0$ in $\Delta(0, \delta)$. Thus $\{f_n\}$ is normal at 0. This is a contradiction.

Taking a subsequence and renumbering if necessary, we may assume that a_n is the zero of $\{f_n\}$ of smallest modulus. Obviously, $a_n \to 0$ as $n \to \infty$. Since $\frac{f_n(a_n)}{a_n^{l+1}} = 0$ and $\frac{f_n(z)}{z^{l+1}} \Rightarrow \infty$ in $\Delta'(0, \delta_0)$, there exists point $b_n (\neq 0)$ (for sufficiently large n) such that

(a) $\left|\frac{f_n(b_n)}{b_n^{l+1}}\right| = 1$ and $\frac{b_n}{a_n}$ is a positive real number greater than 1;

(b) for any positive real number $\lambda_n \in (1, \frac{b_n}{a_n})$, the inequality $\left|\frac{f_n(\lambda_n a_n)}{(\lambda_n a_n)^{l+1}}\right| < 1$ holds;

(c) $b_n \to 0$ as $n \to \infty$.

Set $H_n(\zeta) := \frac{f_n(b_n\zeta)}{b_n^{l+1}}$ for $z \in \Delta(0, \delta_0)$. By Lemma 2.6, $H_n(\zeta)$ is normal in $\mathbb{C} \setminus \{0\}$. Taking a suitable subsequence and renumbering, we may assume $H_n(\zeta) \Rightarrow H^*(\zeta)$ in $\mathbb{C} \setminus \{0\}$, where $H^*(\zeta)$ is a holomorphic function or identically equal to infinity in $\mathbb{C} \setminus \{0\}$. Noting that $|H^*(1)| = \lim_{n\to\infty} |H_n(1)| = 1$, we obtain that $H^*(z)$ is a holomorphic function in $\mathbb{C} \setminus \{0\}$. Applying the maximum principle to $H_n(\zeta)$ in Δ, we obtain that $H_n(\zeta) \Rightarrow H^*(z)$ in Δ, where $H^*(z)$ can be extended to a holomorphic function in $\mathbb{C}$. Then we have $H_n(\zeta) \Rightarrow H^*(z)$ in $\mathbb{C}$, whose zeros have multiplicity at least 4. Taking a suitable subsequence and renumbering, we may assume $\frac{a_n}{b_n} \to \alpha^*$ as $n \to \infty$. Clearly, we have $0 \le \alpha^* \le 1$ and $H^*(\alpha^*) = \lim_{n\to\infty} H_n(\frac{a_n}{b_n}) = 0$. A simple calculation shows

$$\frac{f_n'(b_n\zeta) - \psi(b_n\zeta)}{b_n^l} = H_n'(\zeta) - \zeta^l \widehat{\psi}(b_n\zeta) \Rightarrow H'(\zeta) - \zeta^l \text{in}\mathbb{C}. \tag{4.0}$$

We claim that $H'(\zeta) - \zeta^l \neq 0$ in $\mathbb{C} \setminus \{0\}$. Clearly, $H'(\zeta) - \zeta^l \not\equiv 0$ in $\mathbb{C} \setminus \{0\}$ (Otherwise, $H(\zeta)$ is a polynomial of order $l+1$ in $\mathbb{C}$ which contradicts the fact that all zeros of H have multiplicity at least 4). Let $\zeta_0 \in \mathbb{C} \setminus \{0\}$ be a zero of $H'(\zeta) - \zeta^l$. By Hurwitz's Theorem and (4), there exist sequences $\zeta_n \to \zeta_0$ such that $f_n'(b_n\zeta_n) - \psi(b_n\zeta_n) = 0$. Obviously, $b_n\zeta_n \in \Delta'(0, \delta_0)$ for sufficiently large n which contradicts the fact $f_n'(z) \neq \psi(z)$ in $\Delta'(0, \delta_0)$ for all n.

The fact $H'(\zeta) - \zeta^l \neq 0$ in $\mathbb{C} \setminus \{0\}$ implies that $H'(\zeta) - \zeta^l$ has at most one zero(ignoring multiplicities) in $\mathbb{C}$. By Lemma 2.3 and Lemma 2.5. $H^*(\zeta)$ is a constant function. This contradicts the facts $|H^*(1)| = 1$ and $H^*(\alpha^*) = 0$.

Open questions

Proposition. Let $\mathcal{F}$ be a family of meromorphic functions in D, and let $\psi(\not\equiv 0)$ be a holomorphic function in D, whose zeros have multiplicity at most 2. Suppose that, for each $f \in \mathcal{F}$, all zeros of f have multiplicity at least 4 in D. If for each pair of functions $\{f, g\} \subset \mathcal{F}$, f' and g' share ψ in D, then $\mathcal{F}$ is normal in D?

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