Poincare Journal of Analysis & Applications
Vol. 11, No. 2 (2024), 197-210

DOI: 10.46753/pjaa.2024.v011i02.009
Online Published on 28. 11. 2024

NUMERICAL SOLUTION OF NON-LINEAR LIÉNARD EQUATION USING HAAR WAVELET METHOD

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Date of Receiving	:	01. 11. 2024
Date of Revision	:	15. 11. 2024
Date of Acceptance	:	18. 11. 2024

Abstract. In this article, we propose the Haar wavelet method (HWM) for the numerical solution of linear and non-linear Liénard equations. This method transforms the Liénard equation into a system of algebraic equations using truncated Haar wavelet expansions, which is then solved using Newton's iteration method. Numerical experiments demonstrate the validity and accuracy of HWM, particularly when compared to other existing methods. The desired accuracy for solutions to non-linear Liénard equations is achieved with only a small number of basis points, highlighting HWM as an effective tool for solving both linear and non-linear Liénard equations.

1. Introduction

The real world problems, especially, nonlinear problems, which arise in many physical and scientific fields such as solid-state physics, chemical kinetics, plasma physics, fluid mechanics, and mathematical biology [37, 38], are modelled with nonlinear differential equations [19, 20]. These differential equations, in general, are non-solvable in nature or very hard to achieve the solution. The researchers use various techniques to get the approximate solution of these nonlinear equations. The Liénard equation is one of the differential equations of second order, which is named after the French physicist Alfred-Marie Liénard, in 1928. The general form of the Liénard equation is:

$$y''(x) + f(y)y'(x) + g(y) = b(x) \tag{1.1}$$

where $f(y)$, $g(y)$ are functions of y and $b(x)$ is a function of x.
The equation (1.1) is actually generalization of the damped pendulum equation, where the term $f(y)y'(x)$ is known as the damping force and $f(y)$ is known as the restoring force. It is a special kind of differential equation in mathematics as well as in natural

2010 *Mathematics Subject Classification.* 34A34, 65T60.
Key words and phrases. Liénard Equation, Haar wavelet method, Newton iteration method.
Communicated by: Carlo Cattani
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science and engineering, where the variety of examples, directly or indirectly related to it. Many researchers have studied the solution of Liénard equation. In 2002, Feng [7] obtained explicit and exact solutions to the Liénard equation, and illustrated the applications of these solutions in the context of the nonlinear Schrödinger equation and the Pochhammer-Chree equation. In 2003, Sun et al. [35] studied more explicit exact solitary wave solutions for the generalized Pochhammer–Chree equation by qualitatively investigating the homoclinic and heteroclinic orbits associated with this specific category of Lińard equation. Later in 2006, Aghajani and Moradifam [1] explored the behavior of trajectories in the Liénard system, specifically focusing on whether they intersect the vertical isocline, which is essential for the existence of periodic solutions. In 2012, Gaiko [8] addressed the challenging problem of determining limit cycles in a general Liénard polynomial system, offering a novel approach by introducing a canonical system with field rotation parameters. In 2014, Harko et al. [11] furnished a valuable contribution to the study of Liénard-type nonlinear differential equations by developing a class of exact solutions. Later, in 2015, Li et al. [24] investigated the existence of positive periodic solutions for the second-order forced Liénard equation.

An explicit exact solution for the Liénard equation was derived in [17]. Utilizing this solution, the author obtained an explicit exact solitary wave solution for the Rangwala-Rao equation, the Ablowitz equation, the Gerdjikov-Ivanov equation, among others. In recent years, numerous researchers have endeavored to create various numerical methods for addressing Liénard differential equations.The Liéenard equation was further studied by several authors in [3, 9, 13].

Various numerical methods have been employed by researchers to solve specific forms of Liénard's equation. These include the differential transform method [27], hybrid heuristic computation method [25], block pulse functions method [10], residual power series method [36], and the Chebyshev operational matrix method [21, 32]. Shah and Irfan [30] developed a generalized wavelet-based collocation method to effectively solve the fractional Pennes bioheat transfer model in the context of hyperthermia treatment. Unlike existing operational matrix methods that rely on orthogonal functions, they derived Haar wavelet operational matrices for general-order integration, eliminating the necessity for block pulse functions. Kumar and Rai [22] developed a fractional hyperbolic bioheat transfer model by employing the fractional Taylor series formula alongside the single-phase-lag constitutive equation. They proposed a novel hybrid numerical approach that combines the multi-resolution and multi-scale computational features of Legendre wavelets, utilizing a fractional operational matrix to derive the numerical solution for this problem.

Marasi and Derakhshan [26] proposed a numerical framework that leverages the Haar wavelet collocation method to resolve variable-order Caputo-Prabhakar fractional integrodifferential equations. Ahsan et al. [2] presented a hybrid collocation strategy that integrates finite-difference and Haar wavelet methods to resolve the ill-posed nonlinear inverse Cauchy problem. In this paper, we have presented the application of the Haar wavelet collocation method to obtain an approximate solution for the nonlinear Liénard equation. The method is distinguished by its ease of use, efficiency, and its ability to produce accurate solutions with a small number of grid points.

2. Preliminaries

Wavelets are generated from a single function known as mother wavelet, by dilating and translating it. When both the translational and dilational variables are continuous, the family of continuous wavelets are defined as follows:

$$\psi_{a,b}(x) = |a|^{-\frac{1}{2}}\psi\left(\frac{x-b}{a}\right), \quad a, b \in \mathbb{R}, \quad a \neq 0.$$

When reducing the variables a and b to discrete quantities, the discrete wavelets family is defined as $a = a_0^{-m}, b = nb_0a_0^{-m}, a_0 > 1$, and $b_0 > 1$, where n and m are natural numbers. Hence,

$$\psi_{a,b}(x) = |a_0|^{\frac{m}{2}}\,\psi\left(a_0^m x - nb_0\right),$$

where $\psi_{a,b}$ is a wavelet basis of $L^2(\mathbb{R})$. In particular case, when $a_0 = 2$ and $b_0 = 1$, the $\psi_{a,b}(x)$ forms an orthonormal basis.

Haar function was first time introduced by Alfred Haar in 1910. The first Haar wavelet is define as

$$h_0(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

The Haar wavelet family [31] is defined as

$$h_i(x) = \begin{cases} 1, & a \le x < b \\ -1, & b \le x < c \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

where $a = \frac{k}{m}$ $\quad b = \frac{k+0.5}{m}$ and $\quad c = \frac{k+1}{m}$. The first and second integral of Haar family function, defined in [4], are as follows

$$p_{1,i}(x) = \int_0^x h_i(t)dt = \begin{cases} x-a, & a \le x < b \\ c-x, & b \le x < c \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

and

$$p_{2,i}(x) = \int_0^x p_{1,i}(t)dt = \begin{cases} 0, & 0 \le x < a \\ \frac{(x-a)^2}{2}, & a \le x < b \\ \frac{1}{4m^2} - \frac{(c-x)^2}{2}, & b \le x < c \\ \frac{1}{4m^2}, & c \le x < 1 \end{cases} \tag{2.4}$$

For more details on wavelets, one may refer to [6,12,14,15,23,28,29].

One may note that any continuous function $f(x) \in L^2(\mathbb{R})$ on $[0, 1)$ can be written as:

$$f(x) = \sum_{i=0}^{\infty} c_i h_i(x) \tag{2.5}$$

where $c_i = (2^{j-1})\int_0^1 f(x)h_i(x)dx$. If we truncate the infinite series (2.5) upto the 2^{j-1} term, then f can be written as

$$f(x) = \sum_{i=0}^{2^{J-1}} c_i h_i(x) = C^T h(x) \tag{2.6}$$

where C and $h(x)$ are given by

$$C = [c_1, c_2 \ldots, c_{2^{j-1}}]^T \text{ and } h(x) = [h_1(x), h_2(x), \ldots, h_{2^{j-1}}(x)]^T.$$

3. Haar wavelet method for Liénard equation

Consider the Liénard equation

$$y''(x) + f(y)y'(x) + g(y) = b(x) \tag{3.1}$$

with initial conditions $y(0) = k_1$ and $y'(0) = k_2$. Let

$$y''(x) = \sum_{i=0}^{2^{J-1}} c_i h_i(x). \tag{3.2}$$

After integrating this equation twice with respect to 'x' and with limit from 0 to x, we get

$$y'(x) = \sum_{i=0}^{2^{J-1}} c_i p_{1,i}(x) + y'(0) \tag{3.3}$$

and

$$y(x) = \sum_{i=0}^{2^{J-1}} c_i p_{2,i}(x) + y'(0)x + y(0). \tag{3.4}$$

Therefore, equation (3.1) reduces to

$$\begin{aligned}\sum_{i=0}^{2^{J-1}} c_i h_i(x) + h(\sum_{i=0}^{2^{J-1}} c_i p_{2,i}(x) + y'(0)x + y(0))(\sum_{i=0}^{2^{J-1}} c_i p_{2,i}(x) + y'(0) + y(0)) \\ + g(\sum_{i=0}^{2^{J-1}} c_i p_{2,i}(x) + y'(0) + y(0)) = b(x).\end{aligned} \tag{3.5}$$

Collocate (3.5) at 2^{J-1} points, we get a system of 2^{J-1} equations. Solving these equations using Newton's iterative method in MATLAB, given in [23], we get the value of the unknown vector C. After substituting these values of c_i into (3.4), we can obtain the Haar wavelet approximate solution for the given problem.

Next, we give the convergence analysis of the proposed method.

Lemma 3.1 ([31]). *Assume that $y(x) \in L^2(\mathbb{R})$ with the bounded first derivative on $(0, 1)$, then the error norm at J^{th} level satisfies the following inequality*

$$||e_J(x)|| \leq \sqrt{\frac{K}{7}} C 2^{-(3/2)M}. \tag{3.6}$$

From the above lemma it is clear that the error bound is inversely proportional to square of J^{th} level resolution of Haar wavelet. Hence, more level of resolution gives more accurate approximation of function.

4. Numerical experiments

In this section, we examine three test problems to evaluate the efficiency and accuracy of the proposed scheme. We use the HWM to determine the solution at specified points. To assess the accuracy of the numerical method, we compute the absolute error with respect to HWM and other techniques. All computational tasks are performed using MATLAB software.

4.1. Test Problem 1. Let us consider (1.1) with $f(y) = y(x)$, $g(y) = y(x) + y^2(x)$, and $b(x) = \cos^2(x) - \cos(x)\ \sin(x)$. The initial values are $y'(0) = 0$ and $y(0) = 1$, and the exact solution is $y(x) = \cos(x)$. Then, (1.1) can be rewritten as

$$y''(x) + y(x)y'(x) + y(x) + y^2(x) = \cos^2(x) - \cos(x)\ \sin(x). \tag{4.1}$$

Let

$$y''(x) = \sum_{i=0}^{2^{J-1}} c_i h_i(x). \tag{4.2}$$

After integrating (4.2) twice with respect to 'x' with limit from 0 to x, we get

$$y'(x) = \sum_{i=0}^{2^{J-1}} c_i p_{1,i}(x) + y'(0) \text{ and } y(x) = \sum_{i=0}^{2^{J-1}} c_i p_{2,i}(x) + y'(0)x + y(0). \tag{4.3}$$

Therefore, (4.1) becomes

$$\begin{aligned} &\sum_{i=0}^{2^{J-1}} c_i h_i(x) + (\sum_{i=0}^{2^{J-1}} c_i p_{2,i}(x) + y'(0)x + y(0))(\sum_{i=0}^{2^{J-1}} c_i p_{1,i}(x) + y'(0)) \\ &\quad + (\sum_{i=0}^{2^{J-1}} c_i p_{2,i}(x) + y'(0)x + y(0))^2 = \cos^2(x) - \cos(x)\ \sin(x). \end{aligned} \tag{4.4}$$

Now, for $J = 4$, the truncated solution of the proposed HWM is of the form,

$$y(x) = \sum_{i=0}^{8} c_i p_{2,i}(x) + y'(0)x + y(0).$$

Therefore, after applying the procedure as proposed in Section 3, we get system of 8 equations. After solving them using the Newton method, we get the desired wavelet coefficients.

The comparison of numerical values obtained by this method and exact values are given in Table 1.

x	HWM at $N = 3$	MSM	Exact Sol.
0	1	1	1
0.1	0.995002	0.994860075659013	0.995004
0.2	0.980043	0.9789118152487274	0.980067
0.3	0.955239	0.9514346422581428	0.955336
0.4	0.920817	0.9118477951509781	0.921061
0.5	0.877109	0.8597626237202116	0.877583

Table 1. Comparison of exact solution w.r.t. HWM and other known methods for Test Problem 1.

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x	Error(MSM) h=0.001	Error(HWM) N=3	Error(HWM) N=4
0.1	1.4409 e-04	1.77615 e-06	9.53737 e-07
0.2	1.15476 e-03	2.34771 e-05	2.44408 e-06
0.3	3.90185 e-03	9.74899 e-05	6.3936 e-06
0.4	9.2132 e-03	2.4442 e-05	9.10331 e-06
0.5	1.78199 e-02	4.73574 e-04	1.33305 e-06

TABLE 2. Comparison of absolute errors w.r.t. HWM and other known methods for Test Problem 1.

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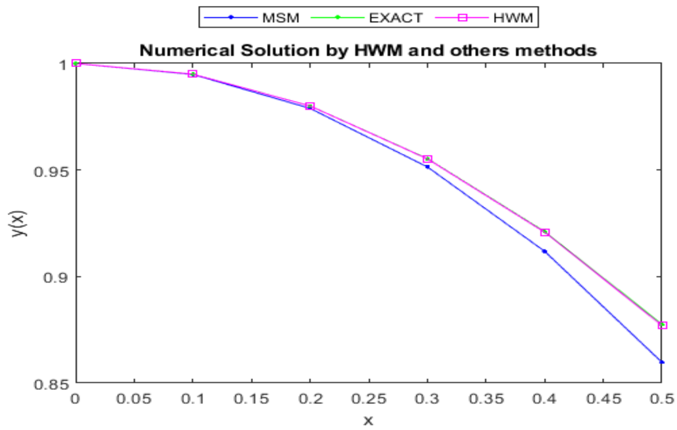

FIGURE 1. Haar wavelet solution for the Test Problem 1.

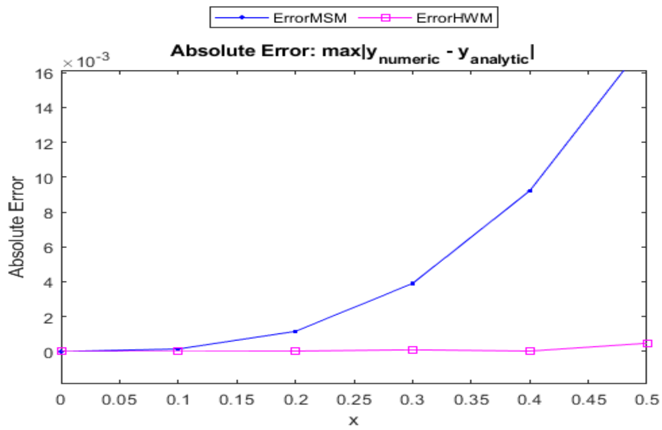

FIGURE 2. Comparison of Absolute errors for the Test Problem 1.

From Figure 1, HWM gives more accurate result than Magnus Series Method (MSM) [16]. Also, it is evident from Figure 2 and Table 1 that the absolute error between MSM and HWM is very low and of the order of 10^{-3}.

4.2. Test Problem 2. Let us discuss the problem (1.1) with $f(y) = 0$, $g(y) = ay(x) + by^3(x) + cy^5(x)$, $b(x) = 0$, where, a, b and c are real numbers. The equation (1.1) becomes

$$y''(x) + ay(x) + by^3(x) + cy^5(x) = 0 \tag{4.5}$$

Let $a = -1, b = 4$ and $c = -3$ along with the initial value $y(0) = \frac{1}{\sqrt{2}}$ and $y'(0) = \frac{\sqrt{2}}{4}$. Eq.(4.5) has the exact solution $y(x) = \sqrt{0.5(1 + \tanh(x))}$. Let

$$y''(x) = \sum_{i=0}^{2^{J-1}} c_i h_i(x). \tag{4.6}$$

After integrating this equation twice with respect to 'x' with limit from 0 to x, we get

$$y'(x) = \sum_{i=0}^{2^{J-1}} c_i p_{1,i}(x) + y'(0) \tag{4.7}$$

and

$$y(x) = \sum_{i=0}^{2^{J-1}} c_i p_{2,i}(x) + y'(0)x + y(0). \tag{4.8}$$

Therefore, (4.5) becomes

$$\begin{aligned} &\sum_{i=0}^{2^{J-1}} c_i h_i(x) - \sum_{i=0}^{2^{J-1}} c_i p_{2,i}(x) + y'(0)x + y(0) + 4(\sum_{i=0}^{2^{J-1}} c_i p_{2,i}(x) + y'(0) + y(0))^3 \\ &\quad - 3(\sum_{i=0}^{2^{J-1}} c_i p_{2,i}(x) + y'(0) + y(0))^5 = 0. \end{aligned} \tag{4.9}$$

Now, for $J = 4$, the truncated solution of the proposed HWM is of the form

$$y(x) = \sum_{i=0}^{8} c_i p_{2,i}(x) + y'(0)x + y(0).$$

Therefore, applying the procedure as proposed in section 3, we obtain system of 8 equations. After solving them using a Newton method, we get the desired wavelet coefficients.

The precise solution by the proposed method and other existing methods (Khalouta differential transform method (KHDTM) [5], Vieta Lucas operational matrix (VLOM) [33], Legendre artificial neural network (LNN) [39], Jacobi spectral colocation method (JSCM) [34] and homotopy analysis transform method (HATM) [18]) is compared with the exact solution in Table 3.

x	Exact Sol.	KHDTM	HWM	VLOM	LNN	JSCM	HATM
0.01	0.710633	-	0.710633	0.7106081	0.710722	0.7106155	-
0.02	0.714142	0.71414	0.71414	0.7140406	0.714231	0.7140699	0.7141419094
0.03	0.717632	-	0.717628	0.7174032	0.717722	0.7174686	-
0.04	0.721103	0.72110	0.721097	0.7206950	0.721193	0.7208102	0.7211028634
0.05	0.724554	-	0.724547	0.7239150	0.724645	0.7240935	-
0.06	0.727986	0.72799	0.727978	0.7270623	0.728077	0.7273171	0.7279862988
0.07	0.731398	-	0.73139	0.7301360	0.731489	0.7304797	-
0.08	0.734789	0.73479	0.734782	0.7331350	0.734881	0.7335800	0.7347890065
0.09	0.738159	-	0.738156	0.7360586	0.738251	0.7366167	-
0.1	0.741508	0.74151	0.741511	0.7389057	0.741601	0.7395886	0.7415079207

TABLE 3. Comparison of absolute errors w.r.t. HWM and other known method for Test Problem 2.

x	LNN	HATM	VLOM	KHDTM	HWM	JSCM
0.01	8.85 e-05	-	2.53 e-05	-	2.34746 e-08	1.79 e-05
0.02	8.90 e-05	1.86691 e-06	1.013 e-04	5.0793 e-09	8.88986 e-08	7.19 e-05
0.03	9.02 e-05	-	2.286 e-04	-	1.81273 e-07	1.63 e-04
0.04	9.01 e-05	6.2706 e-06	4.078 e-04	8.2374 e-08	2.756 e-07	2.93 e-04
0.05	9.06 e-05	-	6.394 e-04	-	3.36887 e-07	4.61e-04
0.06	9.08 e-05	4.94502 e-05	9.239 e-04	4.2249 e-07	3.20151 e-07	6.69 e-04
0.07	9.12 e-05	-	1.2619 e-03		1.7042 e-07	9.18 e-04
0.08	9.20 e-05	1.161249 e-04	1.654 e-03	1.3522 e-06	1.7726 e-07	1.21 e-03
0.09	9.19 e-05	-	2.1005 e-03		7.97818 e-07	1.54 e-03
0.1	9.31 e-05	2.24737 e-04	2.6022 e-03	3.3415 e-06	1.77615 e-07	1.91 e-03

TABLE 4. Comparison of absolute errors w.r.t. HWM and others known methods for Test Problem 2.

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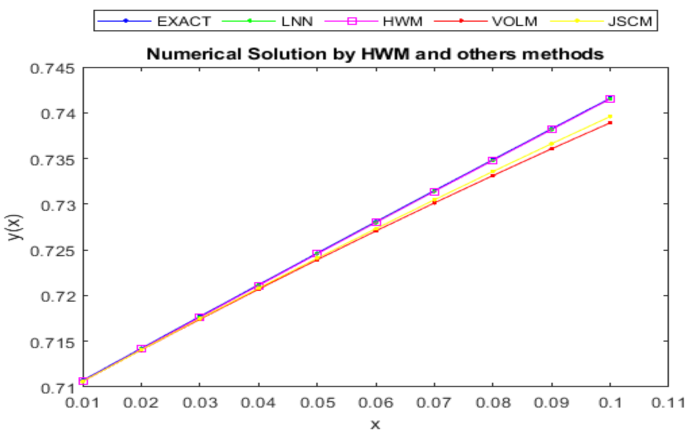

FIGURE 3. Haar wavelet solution for the Test Problem 2.

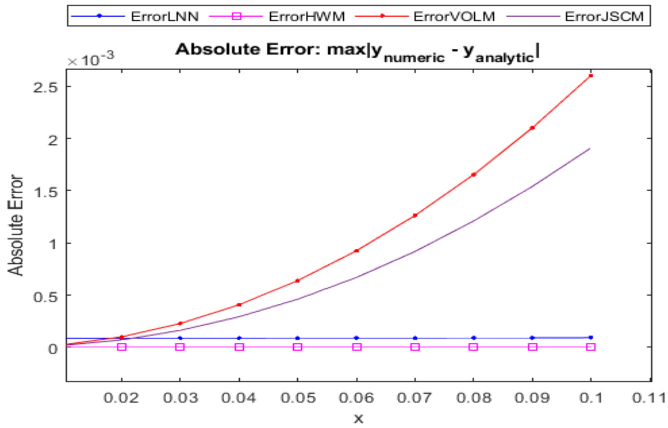

FIGURE 4. Comparision of Absolute errors for the Test Problem 2.

HWM and exact solutions of Test Problem 2 shows excellent agreement while larger error with other methods as evident from Figure 3 and 4.

4.3. Test Problem 3. Let us consider the problem (1.1) with $f(y) = 0$, $g(y) = ay(x) + by^3(x) + cy^5(x)$, $b(x) = 0$, and $a = -1$ $b = 4$ and $c = 3$. The the initial values are given by $y(0) = \frac{1}{\sqrt{\sqrt{2}+1}}$ and $y'(0) = 0$. It has the exact solution $y(x) = \sqrt{\frac{\sec^2 h(x)}{2\sqrt{2}+(1-\sqrt{2})\sec^2 h(x)}}$. We can write the equation as

$$y''(x) - y(x) + 4y^3(x) + 3y^5(x) = 0. \tag{4.10}$$

By fixing the values $J = 4$, we can solve (4.10) by the proposed HWM as discussed in problem 1. The solution is almost the exact solution for the larger values of N. The approximate solution of problem 3 is compared with the exact solution in Table 3.

x	Exact Sol.	KHDTM	HWM	LNN	JSCM	VLOM
0.01	0.643557	-	0.643557	0.643524	0.643524	0.6435619
0.02	0.643443	0.64344	0.643444	0.643313	0.643314	0.6434668
0.03	0.643255	-	0.643256	0.642959	0.642965	0.6433074
0.04	0.642992	0.64299	0.642993	0.642462	0.642477	0.6430830
0.05	0.642653	-	0.642655	0.641821	0.641894	0.6427927
0.06	0.64224	0.64224	0.642241	0.641033	0.641082	0.6424360
0.07	0.641752	-	0.641753	0.640100	0.640176	0.6420119
0.08	0.64119	0.64118	0.641189	0.639019	0.639130	0.6415199
0.09	0.640554	-	0.64055	0.637790	0.637946	0.6409593
0.1	0.639845	0.63982	0.639836	0.636413	0.636623	0.6403293

TABLE 5. Comparison of absolute errors w.r.t. HWM and other known methods for Test Problem 3.

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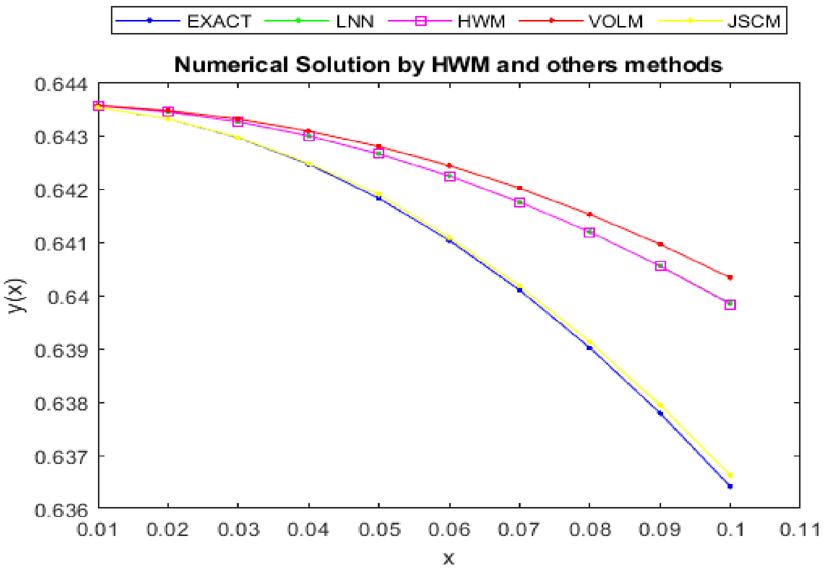

FIGURE 5. Haar wavelet solution for the Test Problem 3.

x	LNN	VLOM	KHDTM	HWM	JSCM
0.01	3.2 e-05	5.4 e-06	-	1.17924 e-07	3.2210 e-05
0.02	1.3 e-04	2.34 e-05	3.2888 e-08	4.47031 e-07	1.3055 e-05
0.03	2.96 e-04	5.23 e-05	-	9.13368 e-07	2.9626 e-04
0.04	5.29 e-04	9.15 e-05	5.2585 e-07	1.39383 e-06	5.3072 e-04
0.05	8.32 e-04	1.397 e-04	-	1.71635 e-06	8.3522 e-04
0.06	1.206 e-03	1.964 e-04	2.6590 e-06	1.66025 e-06	1.2110 e-03
0.07	1.651 e-03	2.601 e-04	-	9.5654 e-07	1.6592 e-03
0.08	2.17 e-03	3.302 e-04	8.3902 e-06	7.11527 e-07	2.1809 e-03
0.09	2.763 e-03	4.054 e-04	-	3.70798 e-06	2.7772 e-03
0.1	3.431 e-03	4.847 e-04	2.0441 e-05	8.44346 e-06	3.4489 e-03

TABLE 6. Comparison of absolute errors w.r.t. HWM and others methods for Test Problem 3.

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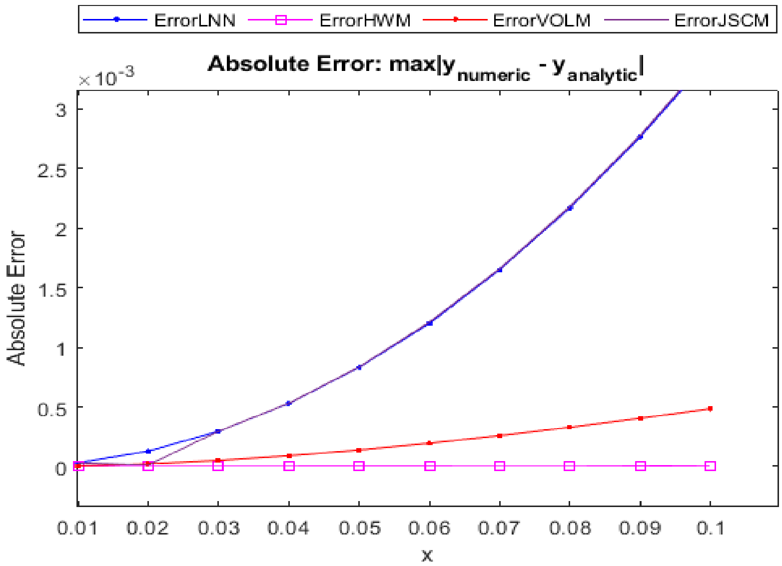

FIGURE 6. Comparison of Absolute errors for the Test Problem 3.

Similar nature of Figure 5 and 6 are observed in case of Test Problem 3 as of Test Problem 1 , 2 and Table 3.

5. Error Analysis

We present the error analysis of the numerical solution obtained using HWM for all the problems. We find numerical solution $y(x)_{numeric}$ with the help of proposed scheme and find the Absolute error function $E(x)$ by

$$E(x) = Abs(y(x)_{exact} - y(x)_{numeric}) \tag{5.1}$$

where $y(x)_{exact}$ is an exact solution of the problem. It is noted that the error function converges to 0 as the number of collocation points increases.
In the Test Problem 1, we considered $N = 3$. From Table 2 and Figure 2, one can observe that the maximum absolute error is 10^{-3} and the minimum absolute error is 10^{-6}, indicating better convergence compared to other existing methods.
For Test Problem 2, we examined the case where $N = 3$. Referring to Table 4 and Figure 4, it is evident that the maximum absolute error is 10^{-6} and the minimum absolute error is 10^{-8}, indicating a better convergence compared to other existing methods.
For $N = 3$, Table 6 and Figure 6 illustrate a maximum absolute error of 10^{-6} and a minimum absolute error of 10^{-7} for Test Problem 3, indicating better convergence.

6. Conclusion

In this study, the Haar wavelet-based collocation method has been effectively utilized to derive approximate Haar wavelet solutions for various cases of the Liénard equation. The effectiveness and precision of the Haar wavelet method have been demonstrated through numerical experiments, showcasing its performance in comparison to other established techniques. The desired solution accuracy has been attained with a small number of basis points. Finally, it is concluded that the proposed Haar wavelet solution serves as a valuable instrument for addressing both linear and nonlinear Lienard equations.

Acknowledgements

The authors would like to extend their sincere thanks to the anonymous referee for his/her valuable comments and suggestions which immensely improved the presentation of the paper.

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