

SOME REMARKS ON POISSON TYPE INTEGRALS IN THE POLYBALL

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Abstract. We consider some natural extensions of Poisson integral in the unit ball to polyballs and extend some known classical results to the case of product domains (polyballs). In particular we extend some known results in the unit ball on Poisson integrals related to BMO to the product domain case

1. Introduction

Let $B_n = \{z \mid |z| < 1\}$ be the unit ball in $\mathbb{C}^n$, $S^n = \partial B$ be the unit sphere in $\mathbb{C}^n$

Let $d(z, w) = |1 - \langle z, w \rangle|^{\frac{1}{2}}$, $z, w \in \bar{B}_n$, be the restriction of d on S_n . It is a non isotropic metric. (see [1,2]).

Let also $Q(\xi, r) = \left\{ \eta \in S_n : |1 - \langle \xi, \eta \rangle|^{\frac{1}{2}} < r \right\}$, $r > 0, \xi \in S_n$. We call Q d-ball, $S_n = \{|z| = 1\}$. We denote various constants in this paper by C, C_1, C_2 . We define Poisson kernel $P(z, w)$, $z, w \in \bar{B}_n$, $P(z, w) = \frac{(1 - |z|^2)^n}{|1 - \langle \bar{z}, w \rangle|^{2n}}$; and a Poisson integral of a positive Borel measure μ , $P(\mu)$ in a standard way. (see, for example, [1]). We refer to [1] and [3-6] for some new and classical results on these objects.

In this note we discuss some problems in open new research area of Poisson type integrals in product domains in $\mathbb{C}^n$. Some new objects in this note will be defined and some interesting new problems will be also posed and solved. We provide first Unit ball known results. (see [1]).

In [1,2] it was shown that

$$\int_{S_n} |f(\eta) - f(a)|^2 P(a, \eta) d\sigma(\eta) \geq \left(\frac{\tilde{c}}{\sigma(Q)} \int_Q |f - f(a)|^2 d\sigma \right); \quad (1)$$

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(these estimates were used in study of analytic BMO) one can also observe that (see [1,2] for definitions) for all $\frac{3}{4} < |a| < 1$

$$\int_{S_n} |f(\xi) - f(a)|^2 P(a, \xi) d\sigma(\xi) \leq c(\sup) \left(\frac{1}{\sigma(Q)} \right) \int_Q |f - f_Q|^2 d\sigma < \infty; \quad (2)$$

Next it was shown (see [1,2]) that

$$\left(\sup_{z \in B_n} \right) \int_{B_n} P(z, w) d\mu(w) \geq \left(\frac{\mu(Q_r(\xi))}{4^n r^{2n}} \right); \quad (3)$$

for a positive Borel measure μ in B .

And if $c = \left(\sup_{\xi, r} \right) \frac{\mu(Q_r(\xi))}{r^{2n}}$, for all positive r and all η on unit sphere then we have that

$$\left(\sup_{|z| > \frac{3}{4}} \right) \int_{B_n} (P(z, w)) d\mu(w) \leq (16^n c) \sum_{k=0}^{\infty} \left(\frac{1}{2^{nk}} \right); \quad (4)$$

(see [1,2]) for any positive μ Borel measure.

The question is how to extend these results for a positive μ Borel measure to more general situation if for example we consider more general Poisson type Kernels of the

following type $\tilde{P}(\vec{z}, \xi) = \frac{\prod_{j=1}^n (1 - |z_j|^2)^{\beta_j}}{\prod_{j=1}^n |1 - z_j \xi|^{\alpha_j}}$, where

$$\sum_{i=1}^m \beta_j = n; \quad \sum_{i=1}^m \alpha_j = 2n,$$

where $z_j \in B, j = 1, \dots, n, \xi \in S$ (we assume sometimes $|z_j| = |z|$) and to a group of positive Borel (μ_j) measures, where $j = 1, \dots, n$.

We add some remarks on how to extend (1)-(4) to this more general situation in this short paper. These results may also have various applications in function theory.

Note that we simply have to find some ways in this paper to modify old and known proofs to the product domain case.

Following the proof of $m = 1$ case (see [1,2]) we have for a positive Borel μ measure

$$M = \sup_{z_j \in B_n} \left(\int_{B_n} \dots \left(\int_{B_n} \frac{\left[\prod_{j=1}^m (1 - |z_j|)^{\alpha_j} \right]^{p_1}}{\prod_{j=1}^m |1 - \langle z_j, w_j \rangle|^{\beta_j p_1}} d\mu(w_1) \right)^{\frac{p_2}{p_1}} \dots \mu(w_m) \right)^{\frac{1}{p_m}} \geq (\tilde{A})$$

$$\geq c \frac{(\mu(Q_r(\xi)))^{\sum_{i=1}^m \frac{1}{p_i}}}{r^{2n}},$$

where $Q_r(\xi) = \{z \in B_n : d(z, \xi) < r\}; \xi \in S_n; r > 0; \sum_{j=1}^m \alpha_j = n;$

$\sum_{j=1}^m \beta_j = 2n; \beta_j > 0; \alpha_j > 0; j = 1, \dots, m; 0 < p_i < \infty; i = 1, \dots, m.$

The proof of $(\tilde{A})$.

Note first obviously we have

$$\left(\sup_{\substack{z_j \in B_n \\ j=1, \dots, m}} \right) |\Phi(z_1, \dots, z_m)| \geq \left(\sup_{z \in B_n} \right) |\Phi(z, \dots, z)|.$$

Indeed we have that even more generally

$$\begin{aligned} M &\geq \left(\int_{Q_r(\xi_1)} \frac{(1-|z|)^{\alpha_1 p_1} d\mu_1(w_1)}{|1-zw_1|^{\beta_1 p_1}} \right)^{\frac{1}{p_1}} \times \left(\int_{Q_r(\xi_2)} \frac{(1-|z|)^{\alpha_2 p_2} d\mu_2(w_2)}{|1-zw_2|^{\beta_2 p_2}} \right)^{\frac{1}{p_2}} \times \\ &\times \dots \left(\int_{Q_r(\xi_m)} \frac{(1-|z|)^{\alpha_m p_m} d\mu_m(w_m)}{|1-zw_m|^{\beta_m p_m}} \right)^{\frac{1}{p_m}} \geq \left(\frac{1}{(r^2)^{\sum_{j=1}^m \beta_j - \sum_{j=1}^m \alpha_j}} \right) \mu_1(Q_r(\xi_1))^{\frac{1}{p_1}} \dots \\ &\mu_m \left(Q_r(\xi_m)^{\frac{1}{p_m}} \right). \end{aligned}$$

The last estimate is based on one functional classical result (see, for example, [1,2] and references there also). Indeed to obtain these estimates we have to use that if

$$\xi \in S_n, r \in (0, 1), z = (1-r^2)\xi$$

,

$$1 - \langle z, w \rangle = r^2 + (1-r^2)(1 - \langle \xi, w \rangle)$$

then we have

$$|1 - \langle z, w \rangle| \leq (1-r^2)r^2 + r^2 \leq 2r^2, \text{ for } w \in Q_r(\xi)$$

and hence ,we have

$$\frac{(1-|z|^2)^n}{|1 - \langle z, w \rangle|^{2n}} \geq \frac{(1-|z|)^n}{|1 - \langle z, w \rangle|^{2n}} \geq \frac{r^{2n}}{(2r^2)^{2n}};$$

for $w \in Q_r(\xi), r \in (0, 1), \xi \in S_n, z \in B_n.$

So that we have the following estimate

$$\int_{Q_r(\xi)} \frac{(1-|z|^2)^n d\mu(w)}{|1 - \langle z, w \rangle|^{2n}} \geq \frac{\mu(Q_r(\xi))}{4^n r^{2n}}.$$

The proof of $(\tilde{A})$ is now complete.

Next we have the following known estimate (see [1])

$$\left(\sup_{|z| > \frac{3}{4}} \right) \int_{B_n} P(z, w) d\mu(w) \leq 16^n c \left(\sum_{k=0}^{\infty} \frac{1}{2^{nk}} \right), \text{ where} \quad (\tilde{A})$$

$$P(z, w) = \frac{(1 - |z|^2)^n}{|1 - \langle \bar{z}, w \rangle|^{2n}}; c = \left(\sup_{\xi, r} \right) \left(\frac{\mu(Q_r(\xi))}{r^{2n}} \right); n \in N, z, w \in \mathbb{B}_n.$$

The natural question to give an extension of this $(\tilde{A})$ estimate to

$$\widetilde{M} = \left(\sup_{|z_j| > \frac{3}{4}} \right) \left(\int_{B_n} \dots \int_{B_n} (\tilde{P}(\vec{z}, \vec{w})) d\mu_1(w_1) \dots d\mu_m(w_m) \right)^{\frac{1}{pm}}, \text{ where}$$

$$\tilde{P}(\vec{z}, \vec{w}) = \frac{\prod_{j=1}^m (1 - |z_j|)^{\alpha_j}}{\prod_{j=1}^m (1 - \langle z_j, w_j \rangle)^{\beta_j}};$$

where

$$\sum_{j=1}^m \alpha_j = n; \sum_{j=1}^m \beta_j = 2n; 0 < p_j < \infty; z_j, w_j \in B_n, j = 1, \dots, m$$

following carefully one functional known proof (see [1,2]).

Note that here μ_j are positive Borel measures on $B, i = 1, \dots, m$.

We have the following generalization

$$\left(\sup_{\substack{|z_j| > \tau \\ |z_j| = R}} \right) \int_{B_n} \frac{\prod_{j=1}^m (1 - |z_j|)^{\alpha_j} d\mu(w)}{\prod_{j=1}^m |(1 - z_j w)^{\beta_j}|} \leq (16^n) \mathbb{C} \left(\sum_{k=0}^{\infty} \frac{1}{2^{nk}} \right) \leq C_n \cdot C, \quad (B)$$

where $\sum_{j=1}^m \beta_j = 2n; \sum_{j=1}^m \alpha_j = n$; for some positive constant C_n .

Proof of (B)

We have for positive Borel measure μ , denoting the expression under the integral by M.

$$\begin{aligned} \sup_{\substack{z_j \in B \\ |z_j| = R}} \int_B \frac{\prod_{j=1}^m (1 - |z_j|)^{\alpha_j} d\mu(w)}{\prod_{j=1}^m |(1 - z_j w)^{\beta_j}|} &= \left[\sup_{\substack{z_j \in B \\ |z_j| = R \\ |z_j| > \tau}} + \sup_{\substack{|z_j| < \tau \\ |z_j| = R}} \right] \int_B (M) d\mu + \sup_{\substack{|z_j| > \tau \\ |z_j| < \tau}} \int_B (M) d\mu \leq \\ &\leq \left(\sup_{\substack{|z_j| > \tau \\ |z_j| = R}} \int_B (M) d\mu \right) + \sup_{\substack{|z_j| > \tau \\ |z_j| < \tau}} \int_B (M) d\mu = I_1 + I_2; \end{aligned}$$

Since it is easy to see $\left(\sup_{|z_j| < \tau} \right) \int_{B_n} (M) d\mu < \infty$; and since μ is a finite positive Borel measure on B .

We estimate I_1 and I_2 following known proof where only one function was involved. (see[1]). We have namely the following chain of estimates.

Let

$$r_k = \sqrt{2^{k+1}(1 - |z|)}, \xi = \frac{z}{|z|}; |z| > \tau, E_0 = Q_{r_0}(\xi), E_k = Q_{r_k}(\xi) - Q_{r_{k-1}}(\xi), k \geq 0.$$

z is fixed,

$$z \in B_n; \mu(E_k) \leq 2^{n(k+1)}(1 - |z|)^n C, |1 - zw| \geq 2^{k-1}(1 - |z|), k \geq 0.$$

And

$$\begin{aligned} \int_{E_k} \frac{\prod_{j=1}^m (1 - |z_j|)^{\alpha_j} d\mu(w)}{\prod_{j=1}^m |(1 - z_j w_j)|^{\beta_j}} &\leq (2^{n(k+1)} \cdot (1 - |z|)^n C) \times \frac{(1 - |z|)^{\sum \alpha_j}}{\prod_{j=1}^m (2^{(k-1)} \cdot (1 - |z|))^{\beta_j}} \leq \\ &\leq C(1 - |z|)^{n + \sum \alpha_j - \sum \beta_j} \times \left(\frac{2^{n(k+1)}}{2^{(k-1) \sum \beta_j}} \right) = \\ &= \frac{c(1 - |z|)^{n + \sum \alpha_j - \sum \beta_j}}{2^{k(\sum \beta_j - n)}} \cdot (2^{n + \sum \beta_j}); k \geq 0. \end{aligned}$$

This gives an estimate for I_1 .

We estimate I_2 omitting factors, we have

$$\left(\sup_{|z_j| < \tau} \right) (1 - |z_j|)^{-\tau} \leq c_0 \frac{(1 - |z_j|)^{\alpha_j}}{(1 - |z_j||w_j|)^{\beta_j}} \leq c_0 (1 - |z_j|)^{\alpha_j - \beta_j}; j = 1, \dots, m.$$

So if $\beta_j - n > 0$; $\sum_{j=1}^m \alpha_j - \sum_{j=1}^m \beta_j = -n$; $j = 1, \dots, m$.

We have

$$\left(\sup_{\substack{|z_j|=R \\ z_j \in B}} \right) \int_{B_n} \frac{\prod_{j=1}^m (1 - |z_j|)^{\alpha_j}}{\prod_{j=1}^m |(1 - z_j w_j)|^{\beta_j}} d\mu(w) \leq \tilde{c} \sup_{\xi, r} \left(\frac{\mu(Q_r(\xi))}{r^{2n}} \right).$$

And also

$$\left(\sup_{\substack{|z_j|=R \\ |z_j| > \tau}} \right) \int_B \frac{\prod_{j=1}^m (1 - |z_j|)^{\alpha_j} d\mu(w)}{\prod_{j=1}^m |(1 - z_j w_j)|^{\beta_j}} \geq (\tilde{c}) \left(\sup_{\xi, r} \right) \left(\frac{\mu(Q_r(\xi))}{r^{2n}} \right)$$

if $\sum_{j=1}^m \alpha_j - \sum_{j=1}^m \beta_j = (-n)$.

The proof is complete. We mention another known estimate and show similar extension to polyballs.

Note (see [1,2]) for $Q = Q(\frac{a}{|a|}, \sqrt{1-|a|^2})$; $a \in B_n$; $a \neq 0$,

$$I_a = \int_{S_n} |f(\xi) - f(a)|^2 \frac{(1-|a|^2)^n}{|1-\bar{\xi}a|^{2n}} d\sigma(\xi) \geq \frac{1}{4^n(1-|a|^2)^n}$$

and $\int_Q |f - f(a)|^2 d\sigma \geq \frac{Const}{\sigma(Q_1)} \int_{Q_1} |f - f(a)|^2 d\sigma$, where $Q = Q(\frac{a}{|a|}, \sqrt{1-|a|^2})$ as our parameter runs over $B_n/0$ the above Q_1 runs over all d balls of radius less than 1 (see[1,2]). We provide some generalizations of such estimates.

Let now $f \in L^{p_1}(S_n \times \dots \times S_n)$. Then

$$I_{\vec{a}} = \left(\int_{S_n} \dots \left(\int_{S_n} |f(\xi_1, \dots, \xi_n) - f(a_1, \dots, a_m)|^{p_1} \times \frac{\prod_{i=1}^m (1-|a_i|^2)^{\alpha_i}}{\prod_{i=1}^m |1-\bar{\xi}_i a_i|^{\beta_i}} d\sigma(\xi_1) \right) \dots d\sigma(\xi_m) \right)^{\frac{1}{pm}};$$

$$0 < p_i < \infty; i = 1, \dots, m, \sum_{i=1}^m \alpha_i = n; \sum_{i=1}^m \beta_i = 2n; \alpha_j, \beta_j > 0; j = 1, \dots, m.$$

Then following carefully again the known proof where only function was involved we have

$$I_{\vec{a}} \geq \frac{4^{-\frac{n}{p_1}}}{\prod_{i=1}^m \frac{(2\beta_i - \alpha_i)}{(1-|a_i|^2)^{p_1}}} \left(\int_Q \dots \left(\int_Q |f(\xi_1, \dots, \xi_n) - f(a_1, \dots, a_m)|^{p_1} d\sigma(\xi_1) \right) \dots d\sigma(\xi_m) \right)^{\frac{1}{pm}} \quad (C)$$

$$Q = Q(\frac{a}{|a|}, \sqrt{1-|a|^2}); a \in B_n; a \neq 0; 2\beta_i - \alpha_i > 0, i = 1, \dots, m.$$

And let for same restrictions on parameters

$$\tilde{I}_{\vec{a}} = \left(\int_{S_n} \dots \left(\int_{S_n} \prod_{i=1}^m |f_i(\xi_i) - f_i(a_i)|^{p_1} \times \frac{\prod_{i=1}^m (1-|a_i|)^{\alpha_i}}{\prod_{i=1}^m |1-\bar{\xi}_i a_i|^{\beta_i}} d\sigma(\xi_1) \right) \dots d\sigma(\xi_m) \right)^{\frac{1}{pm}}$$

then again following the known proof with one function we have

$$\tilde{I}_{\vec{a}} \geq \frac{4^{-\frac{n}{p_1}}}{\prod_{i=1}^m \frac{(2\beta_i - \alpha_i)}{(1-|a_i|^2)^{p_1}}} \prod_{i=1}^m \left(\int_Q |f_i(\xi_i) - f_i(a_i)|^{p_i} \right)^{\frac{1}{p_i}}, \quad (\tilde{C})$$

where $f_i \in L^{p_1}(B)$, $i = 1, \dots, m$; the proof is based on estimate $(|1 - (\bar{a}, \xi)|) = 1 - (\frac{a}{|a|}, \xi) + (1 - |a|)(\frac{a}{|a|}, \xi)$ and hence $|1 - (a\bar{\xi})| \leq 2(1 - |a|^2)$ (see[1,2]), where $\xi \in Q$. and is similar to one domain proof. (see [1,2])

These our estimates may have various applications. We omit some details of proofs of last estimates referying to [1,2].

We have also the following known estimate (see [1,2])

$$\int_{S_n} |f(\xi) - f(a)|^P \frac{(1 - |a|^2)}{|1 - a\xi|^{2n}} d\sigma \leq c C_*^{t,p}; \quad (A)$$

$r_0 < |a| < 1$, for some constant C , where

$$C_*^{t,p} = (\sup) \left(\frac{1}{\sigma(Q)^t} \right) \int_Q |f - f_Q|^P d\sigma < \infty; f \in H^P(B); (f_Q) = \frac{1}{\sigma(Q)} \left(\int_Q f d\sigma \right)$$

We refer to [1,2] for $t = 1, p = 2$ case of (A).

We wish to extend (A) again using an extension of classical Poisson Kernel $P(z, \xi)$ and carefully studying the classical known proof in [1,2]. We have the following result, the main result of this note.

Theorem Let $f \in L^P(S)$, $p \geq 1$. Then we have that

$$\left(\sup_{\substack{z_0 < |a_j| < 1 \\ j=1, \dots, m}} \right) \int_{S^n} |f(\xi) - f_{(Q_0)}|^P \times \prod_{j=1}^m \frac{(1 - |a_j|)^{\alpha_j}}{|1 - \xi a_j|^{\beta_j}} d\sigma(\xi) \leq \tilde{C} C_*^{1,p}; \quad (1.1)$$

for some positive constant C ; where $\alpha_j, \beta_j > 0, j = 1, \dots, m; Q_0 = Q\left(\frac{a_1}{|\tilde{a}|}, \sqrt{1 - |\tilde{a}|}\right)$,

$$a_1 \in B; |\tilde{a}| > r_0, \sum_{j=1}^m \beta_j = \sum_{j=1}^m \alpha_j + n;$$

And also

$$\left(\sup_j \right) \int_{S_n} |f(\xi) - f(a_j)|^P \cdot \tilde{P}(\vec{a}, \xi) d\sigma(\xi) \leq C_\sigma (C_*^{1,p}); \quad (1.2)$$

for all $|a_j| \in [r_0, 1); |a_j| = |\tilde{a}|; a_j = \tilde{a} \varphi_j; \tilde{P}(\vec{a}, \xi) = \tilde{P} = \left(\prod_{j=1}^m \frac{(1 - |a_j|)^{\alpha_j}}{|1 - \xi a_j|^{\beta_j}} \right)$; where

$$\sum_{j \geq 1}^m \beta_j = \sum_{j \geq 1}^m \alpha_j + n; f \in H^P, 1 \leq p < \infty, j = 1, \dots, m.$$

The proof of theorem. We fix $a_1 \in B, a_1 = |\tilde{a}| \varphi_1, |\tilde{a}| > r_0, Q_k = Q\left(\frac{a_1}{|\tilde{a}|}, 2^k \sqrt{1 - |\tilde{a}|}\right)$, $k = 0, 1, \dots, N$; N is a smallest integer, so that $Q_N = S_n$. We have

$$\begin{aligned} \int_{S^n} |f(\xi) - f_{Q_0}|^P \tilde{P}(\vec{a}, \xi) d\sigma(\xi) &= \int_{Q_0} |f(\xi) - f_{Q_0}|^P \tilde{P}(\vec{a}, \xi) d\xi + \\ &+ \sum_{k=1}^N \int_{\frac{Q_k}{Q_{k-1}}} |f(\xi) - f_{Q_0}|^P \left(\tilde{P}(\vec{a}, \xi) \right) d\sigma(\xi) \quad (G) \end{aligned}$$

Since $\tilde{P}(\vec{a}, \xi) \leq \frac{1}{\sigma(Q_0)^t}; \xi \in Q_0; t = \frac{\left(\sum_{j=1} \beta_j - \sum_{j=1} \alpha_j\right)}{n}$;

$$\int_{Q_0} |f(\xi) - f_{Q_0}|^P \tilde{P}(\vec{a}, \xi) d\sigma(\xi) \leq c(\mathbb{C}_*^{t,p}); \quad (G_1)$$

We then have (see [1,2]); $|1 - a\bar{\xi}| \sim (4^k)(1 - |a|); k \geq 1, \xi \in Q_k \setminus Q_{k-1}$.

Hence obviously $\tilde{P}(\vec{a}, \xi) \sim (4^{-\tau k})(1 - |a|)^{-s}$; and also $\sigma(Q_k) \asymp 4^{nk}(1 - |a|)^n$;
 $S = \sum_{j=1} \beta_j - \sum_{j=1} \alpha_j; \tau = \sum_{j=1} \beta_j; \xi \in (Q_k \setminus Q_{k-1})$;

This gives

$$\int_{Q_k \setminus Q_{k-1}} |f(\xi) - f_{Q_k}|^P \tilde{P}(a, \xi) d\sigma(\xi) \leq \frac{c_1}{4^{(\tau_1)k} \sigma^t(Q_k)} \int_{Q_k} |f - f_{Q_k}|^P d\sigma \leq (c_1) \frac{\mathbb{C}_*^{p,t}}{4^{(\tau_1)k}}$$

$$1 \leq k \leq N; \tau_1 + nt = \tau; \tau_1 = \sum_{j=1} \alpha_j; t = \frac{\left(\sum_{j=1} \beta_j - \sum_{j=1} \alpha_j\right)}{n}.$$

Also $f_{Q_k} - f_{Q_{k-1}} = \frac{1}{\sigma(Q_{k-1})} \int_{Q_{k-1}} (f_{Q_k} - f) d\sigma$; by Holder's inequality

$$|f_{Q_k} - f_{Q_{k-1}}|^P \leq \frac{1}{\sigma(Q_{k-1})} \int_{Q_k} |f - f_{Q_k}|^P d\sigma.$$

Now $\sigma(Q_k) \sim \sigma(Q_{k-1})$ (see [1,2]); so

$$|f_{Q_k} - f_{Q_{k-1}}|^P \leq \frac{\mathbb{C}}{\sigma(Q_k)} \int_{Q_k} |f - f_{Q_k}|^P d\sigma \leq c \mathbb{C}_*^{p,t};$$

since $\sigma(Q_k)^t \leq \sigma(Q_k)$; $1 \leq k \leq N$;

Combining this with

$$f_{Q_k} - f_{Q_0} = \sum_{i=1}^k (f_{Q_i} - f_{Q_{i-1}});$$

we get $|f_{Q_k} - f_{Q_0}|^P \leq c(\mathbb{C}_*^{t,p}) k^p$.

Since also

$$|f(\xi) - f_{Q_0}|^P \leq C_p (|f(\xi) - f_{Q_k}|^P + |f_{Q_k} - f_{Q_0}|^P); C_p = 2^{p-1};$$

We have

$$\int_{Q_k \setminus Q_{k-1}} |f_{Q_k} - f_{Q_0}|^P \tilde{P}(a, \xi) d\sigma(\xi) \leq \frac{c_1 \mathbb{C}_*^{t,p} k^P}{4^{\tau_2 k}}$$

,

$$\tau_2 = \sum_{j=1} \beta_j - n = \sum_{j=1} \alpha_j.$$

Also

$$\int_{Q_k \setminus Q_{k-1}} |f(\xi) - f_{Q_0}|^P \tilde{P}(a, \xi) d\sigma(\xi) \leq \frac{\tilde{c}(\mathbb{C}_*^{t,p})(k^P + 1)}{4^{\tau_1 k}},$$

$$1 \leq k \leq N.$$

And using (G), (G_1) we have finally

$$\int_{S_n} |f(\xi) - f_{Q_0}|^P \left(\tilde{P}(a, \xi) \right) d\sigma(\xi) \leq \tilde{c}(\mathbb{C}_*^{p,t}); |a_j| \in (r_0, 1), j = 1, \dots, m.$$

From $\int_{S_n} f(\xi) \tilde{P}(a_j, \xi) d\sigma(\xi) = f(a_j); j = 1, \dots, m$; and

$$|f(\xi) - f(a_j)|^P \leq (2^{p-1}) (|f(\xi) - f_{Q_0}|^P + |f_{Q_0} - f(a_j)|^P),$$

we have $\left(\sup_j \int_{S_n} |f(\xi) - f(a_j)|^P \left(\tilde{P}(a_j, \xi) \right) d\sigma(\xi) \leq c \left(\mathbb{C}_*^{p,1} \right); \text{ for all } a_j = |\tilde{a}| \varphi_j; |a_j| \in [r_0, 1); j = 1, \dots, m \text{ under (S) restriction on } (\alpha_j, \beta_j). \text{ Theorem is proved.} \right.$

Putting $f_Q^S = \frac{1}{\sigma(Q)^S} \left(\int_Q f d\sigma \right); S > 0$ we can similarly provide also another version of our theorem with other restrictions on $\alpha_j, \beta_j, j = 1, \dots, \bar{m}$, extending classical results.

It will be interesting also to consider $f_{\tilde{Q}}^S = \frac{1}{\sigma(\tilde{Q})^S} \left(\int_{\tilde{Q}} f d\sigma \right); \tilde{Q} = Q \times Q,$

$S > 0$ on product domains and to prove similar type results for them also.

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