

NOVEL FRACTALS OF HUTCHINSON BARNSELEY OPERATOR IN HAUSDORFF G -METRIC SPACES

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Abstract. In this article, we construct fractals (attractors) with the help of iterated function systems (IFS) which is a finite family of contraction mappings defined on Hausdorff generalized metric (G - metric) space. In this proposed space, we prove some results to obtain attractors for Hutchinson Barnsley (HB) operator which satisfy different contractive conditions. We derive Collage theorem which is used to find out suitable iterated function systems (IFS) for constructing fractals. Also, we have presented some practical examples of classical fractals to validate our results. One can further generalize and extend our results and construct fractals or multifractals by using more general contractive conditions.

1. Introduction

In today's scenario of research, fractal theory has been applied in many diverse branches of science due to its extensive applications. The term fractal was firstly used by Benoit Mandelbrot in his book "The Fractal Geometry of Nature" (1982) and defined it as "rough or fragmented geometric shapes that can be split into congruent pieces, each of which is a reduced size copy of the origin one". Various techniques like Iterated Function Systems, Escape Time Fractals, Strange Attractors and Random Fractals have been adopted to generate Fractals.

This paper is devoted to Iterated Function Systems (IFS) technique to generate fractals. Hutchinson [20] defined iterated function systems (IFS) and Barnsley [9] enriched the theory of IFS and named it as Hutchinson Barnsley (HB) theory. Hutchinson introduced IFS as a finite collection of contractive self mappings and defined HB operator on hyperspace of nonempty compact sets. He showed that the unique fixed point of HB operator is a fractal (attractor). Thus, fractals can be constructed by using fixed point theory. For years, IFS has become emerging technique for researchers

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to generate and analyse new fractal objects. In this sequel, extensions of IFS and numerous developments have been made to construct fractals and similar sets (see, [3, 4, 5, 11, 15, 16, 25, 27, 37, 38, 39, 40, 41, 42] and references therein).

In fixed point theory, Banach Contraction Principle [8] contributed a lot. The enhancements of Banach Contraction Principle have been made either by considering more general contractive conditions on mappings [10, 14, 26, 28] or by generalizing the domain [1, 6, 19, 21, 22, 23, 43]. Further, several fixed point results were obtained by generalizing the concept of metric space [12, 13, 17, 18].

The authors [31, 34] introduced the fact of G - metric, which is more general and suitable concept than that of D - metric considered by Dhage [12, 13]. In [33], Mustafa et al. (2004) claimed that some fundamental topological properties of D - metric are wrong. This motivate them to define a new more general metric called G - metric. The elementary concepts about Generalized metric (G - metric) can be found in [33, 31, 34]. They showed that G - metric spaces are a generalization of regular metric spaces. The interested readers can refer ([2, 29, 30, 32, 35, 36, 7, 24] and references therein) to know about leading results of fixed point theory in G - metric spaces.

In this paper, firstly we discuss some basic definitions and results that are used in our further study. In Section 3, we prove some results to obtain fractals (attractors) of Hutchinson Barnsley operator defined on a Hausdorff G - metric space by adopting different contractive conditions. Section 4, deals with the derivation of collage theorem which helps us to choose appropriate IFS to generate fractals. Also, some practical examples of general fractals demonstrate the applications of our results. Finally, Section 5 summaries our findings and applications and generalization of our results have also been discussed.

2. Preliminaries

Definition 2.1. [31] Consider a non empty set U and set of real numbers $\mathbb{R}$. A function $G: U \times U \times U \rightarrow \mathbb{R}^+$ is called G - metric on U if for all $l, m, n, b \in U$, it satisfy the following:

- (G1) $G(l, m, n) = 0$ if $l = m = n$.
- (G2) $0 < G(l, l, m)$; with $l \neq m$.
- (G3) $G(l, l, m) \leq G(l, m, n)$; with $n \neq m$.
- (G4) $G(l, m, n) = G(l, n, m) = G(m, n, l) = \dots$, (i.e., symmetry in all variables).
- (G5) $G(l, m, n) \leq G(l, b, b) + G(b, m, n)$ (rectangle inequality).

Here, $G(l, m, n)$ can be taken as perimeter of a triangle having vertices l , m and n . The pair (U, G) is said to be G - metric space.

Notice that the metric d_G on U is induced by every G - metric defined by

$$d_G(l, m) = G(l, m, m) + G(l, l, m)$$

Example 2.2. [7] Suppose that (U, d) is a metric space. Then, a function $G: U \times U \times U \rightarrow [0, \infty)$ defined by

$$G(p, q, r) = \max\{d(p, q), d(q, r), d(r, p)\}, \forall p, q, r \in U$$

is a G - metric on U .

Kaewcharoen et al. (2011) introduced Hausdorff G - distance as follows:

Definition 2.3. (see[24]) Let $\mathcal{H}(D)$ be the collection of all nonempty compact subsets of U . For $L, M, N \in \mathcal{H}(D)$, a function $H_G: \mathcal{H}(D) \times \mathcal{H}(D) \times \mathcal{H}(D) \rightarrow \mathbb{R}^+$ defined by

$$H_G(L, M, N) = \max \left\{ \sup_{l \in L} G(l, M, N), \sup_{m \in M} G(m, N, L), \sup_{n \in N} G(n, L, M) \right\}, \quad (2.1)$$

where

$$\begin{aligned} G(l, M, N) &= d_G(l, M) + d_G(M, N) + d_G(l, N) \\ d_G(l, M) &= \inf \{d_G(l, m) : m \in M\} \\ d_G(L, M) &= \inf \{d_G(l, m) : l \in L, m \in M\}. \end{aligned}$$

Also,

$$G(l, m, N) = \inf \{G(l, m, n) : n \in N\}.$$

is said to be a Hausdorff G - metric. The pair $(\mathcal{H}(D), H_G)$ is said to be Hausdorff G - metric space.

For more details about the elementary topological properties such as convergence of a sequence, cauchy sequence, closed, bounded, compact set and completeness of a G - metric space one can refer [31, 35].

Now, we consider following basic concepts which we use in our further study.

Definition 2.4. Consider a Hausdorff G - metric space $(\mathcal{H}(D), H_G)$. A sequence $\{A_n\}$ of compact subsets of $\mathcal{H}(D)$, is Hausdorff G convergent to $A \in \mathcal{H}(D)$ if for any $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that $H_G(A, A_n, A_m) < \epsilon$ for all $n, m \geq M$. i.e.,

$$\lim_{n, m \rightarrow +\infty} H_G(A, A_n, A_m) = 0.$$

Here, A is said to be limit of the sequence $\{A_n\}$ and can be written as $A_n \rightarrow A$ as $n \rightarrow +\infty$ or $\lim_{n \rightarrow +\infty} A_n = A$.

Proposition 2.5. Let $(\mathcal{H}(D), H_G)$ be a Hausdorff G -metric space. Then following conditions are equivalent :

- (1) $\{A_n\}$ is Hausdorff G - convergent to A .
- (2) $H_G(A_n, A_n, A) \rightarrow 0$ as $n \rightarrow +\infty$.
- (3) $H_G(A_n, A, A) \rightarrow 0$ as $n \rightarrow +\infty$.
- (4) $H_G(A_n, A_m, A) \rightarrow 0$ as $n, m \rightarrow +\infty$.

Definition 2.6. Let $(\mathcal{H}(D), H_G)$ be a Hausdorff G -metric space, sequence $\{A_n\}$ is called Hausdorff G - Cauchy if for any $\epsilon > 0$, there is a positive integer M for which $H_G(A_n, A_m, A_l) < \epsilon$ for all $n, m, l \geq M$ i.e., $H_G(A_n, A_m, A_l) \rightarrow 0$ as $n, m, l \rightarrow +\infty$.

Proposition 2.7. The conditions given below are equivalent for a Hausdorff G -metric space $(\mathcal{H}(D), H_G)$:

- (1) $\{A_n\}$ is a Hausdorff G - Cauchy sequence.
- (2) For given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $H_G(A_n, A_m, A_m) < \epsilon$, $\forall m, n \geq N$.

Remark 2.8. A Hausdorff G - metric space $(\mathcal{H}(D), H_G)$ is always complete.

Lemma 2.9. Let $(\mathcal{H}(D), H_G)$ be a Hausdorff G - metric space and $P, Q, R, S, T, U \in \mathcal{H}(D)$ then

- (i) If $R \subseteq S$, then $\sup_{p \in P} G(p, Q, S) \leq \sup_{p \in P} G(p, Q, R)$

$$(ii) \sup_{x \in P \cup Q} G(x, R, S) = \max \left\{ \sup_{p \in P} G(p, R, S), \sup_{q \in Q} G(q, R, S) \right\}$$

$$(iii) H_G(P \cup Q, R \cup S, T \cup U) \leq \max \{H_G(P, R, T), H_G(Q, S, U)\}$$

The topological properties of G - metric space were studied by Mustafa et al. [35]. In 1922, Stefan Banach (1892 – 1945) stated the concept of Banach Contraction Principle to find the unique fixed point of a self map of a complete metric space.

Definition 2.10. [28] Consider a metric space (U, d) . A mapping $g: U \rightarrow U$ is contraction if there is a constant $\lambda \in [0, 1)$ for which

$$d(g(a), g(b)) \leq \lambda d(a, b), \forall a, b \in U.$$

Banach [8] proved that in complete metric space every self contraction mapping has a unique fixed point. Mustafa [34] defined the contraction mapping and extended Banach Contraction Principle in the setting of G - metric space as given below:

Definition 2.11. [34] Consider a G - metric space (U, G) . A contraction is a self mapping g on U for which

$$G(g(u), g(v), g(w)) \leq \lambda G(u, v, w), \forall u, v, w \in U \text{ and } \lambda \in [0, 1).$$

Theorem 2.12. [34] Consider a complete G - metric space (U, G) and a mapping $g: U \rightarrow U$ satisfying

$$G(g(u), g(v), g(w)) \leq \lambda G(u, v, w),$$

where $\lambda \in [0, 1)$ and for all $u, v, w \in U$. Then, g has a unique fixed point.

Theorem 2.13. Consider a complete G - metric space (U, G) and a contraction $g: U \rightarrow U$. Then the following hold:

- (1) contraction g maps elements of $\mathcal{H}(D)$ to elements of $\mathcal{H}(D)$, where $\mathcal{H}(D)$ is a collection of nonempty compact subsets of U .
- (2) for any $L \in \mathcal{H}(D)$, if $g(L) = \{g(l) : l \in L\}$ then $g: \mathcal{H}(D) \rightarrow \mathcal{H}(D)$ is a contraction mapping on $\mathcal{H}(D)$.

Proof. (1) g being a contraction mapping is continuous and under a continuous mapping image of a compact set is compact, therefore, we have

$$Q \in \mathcal{H}(D) \Rightarrow g(Q) \in \mathcal{H}(D).$$

(2) Let $L, M, N \in \mathcal{H}(D)$ with $H_G(g(L), g(M), g(N)) \neq 0$.

Then, by Definition 2.3, we have

$$H_G(g(L), g(M), g(N)) = \max \left\{ \sup_{l \in L} G(g(l), g(M), g(N)), \sup_{m \in M} G(g(m), g(N), g(L)), \sup_{n \in N} G(g(n), g(L), g(M)) \right\} \quad (2.2)$$

Since g is a contraction mapping on U , then by Definition 2.11, we have

$$G(g(l), g(M), g(N)) \leq \lambda_1 G(l, M, N), \quad G(g(m), g(N), g(L)) \leq \lambda_2 G(m, N, L) \text{ and}$$

$G(g(n), g(L), g(M)) \leq \lambda_3 G(n, L, M)$ for some constants $\lambda_1, \lambda_2, \lambda_3 \in [0, 1)$. Using these relations in (2.2), we obtain

$$\begin{aligned} H_G(g(L), g(M), g(N)) &\leq \max \left\{ \lambda_1 \sup_{l \in L} G(l, M, N), \lambda_2 \sup_{m \in M} G(m, N, L), \right. \\ &\quad \left. \lambda_3 \sup_{n \in N} G(n, L, M) \right\} \\ &= \lambda \max \left\{ \sup_{l \in L} G(l, M, N), \sup_{m \in M} G(m, N, L), \sup_{n \in N} G(n, L, M) \right\} \end{aligned}$$

This gives

$$H_G(g(L), g(M), g(N)) \leq \lambda H_G(L, M, N), \quad (2.3)$$

where $\lambda = \max\{\lambda_1, \lambda_2, \lambda_3\}$; $\lambda_i \in [0, 1)$ for $i = 1, 2, 3$ and also $\lambda \in [0, 1)$. Thus, g is a contraction on $\mathcal{H}(D)$, where λ is contractivity constant. $\square$

Theorem 2.14. Let (U, G) be a G -metric space and $\{g_i: i = 1, 2, 3, \dots, N\}$ be a finite family of contraction mappings defined on U with contractivity factors as $\lambda_i, i = 1, 2, \dots, N$. Define a mapping $F: \mathcal{H}(D) \rightarrow \mathcal{H}(D)$ as

$$\begin{aligned} F(R) &= g_1(R) \cup g_2(R) \cup \dots \cup g_N(R). \text{ i.e.,} \\ F(R) &= \bigcup_{i=1}^N g_i(R), \forall R \in \mathcal{H}(D). \end{aligned} \quad (2.4)$$

Then, F is a contraction on $\mathcal{H}(D)$ having contractivity factor λ , where $\lambda = \max\{\lambda_i\}$, $i = 1, 2, \dots, N$.

Proof. We prove the result for $N = 2$. Let us suppose g_1, g_2 be two contraction mappings on U . Choose $R, S, T \in \mathcal{H}(D)$ with $H_G(F(R), F(S), F(T)) \neq 0$. With the help of Lemma 2.9 (iii) and Theorem 2.13 (2), we have

$$\begin{aligned} H_G(F(R), F(S), F(T)) &= H_G((g_1(R) \cup g_2(R)), (g_1(S) \cup g_2(S)), (g_1(T) \cup g_2(T))) \\ &\leq \max\{H_G(g_1(R), g_1(S), g_1(T)), H_G(g_2(R), g_2(S), g_2(T))\} \\ &\leq \max\{\lambda_1 H_G(R, S, T), \lambda_2 H_G(R, S, T)\} \\ &\leq \lambda H_G(R, S, T) \end{aligned}$$

This gives,

$$H_G(F(R), F(S), F(T)) \leq \lambda H_G(R, S, T), \quad (2.5)$$

where $\lambda = \max\{\lambda_1, \lambda_2\}$ for $\lambda_1, \lambda_2 \in [0, 1)$. Thus, by induction method, the result is true for any N . $\square$

Definition 2.15. Let (U, G) be a complete G -metric space. If $g_i: U \rightarrow U$, $i = 1, 2, \dots, N$ are contractions on U , then $\{U; g_1, g_2, \dots, g_N\}$ is said to be iterated function systems (IFS).

The operator $F: \mathcal{H}(D) \rightarrow \mathcal{H}(D)$ defined in above Theorem 2.14 by (2.4) is said to be Hutchinson Barnsley (HB) operator or iterated function systems (IFS) of Hausdorff G -metric space $(\mathcal{H}(D), H_G)$.

Definition 2.16. Let $(\mathcal{H}(D), H_G)$ be a Hausdorff G - metric space. An attractor R of Hutchinson Barnsley operator F , is a nonempty subset of $\mathcal{H}(D)$ for which

- (i) $F(R) = R = \bigcup_{i=1}^N g_i(R)$.
- (ii) There exists an open set $V \subset U$ such that for any $X \subset V$, $\lim_{n \rightarrow \infty} F^n(Y) = X$ holds, where Y is a compact proper subset of V and limit is taken w.r.t. Hausdorff G - metric.

3. Main Results

In this section, we obtain fractals (attractors) for self mappings satisfying certain contractive conditions defined on Hausdorff G - metric space.

Theorem 3.1. Consider a complete G - metric space (U, G) and $\{U; g_1, g_2, \dots, g_l\}$ is an IFS. Then the following holds:

- (i) The mapping $F: \mathcal{H}(D) \rightarrow \mathcal{H}(D)$ defined by

$$F(P) = \bigcup_{i=1}^l g_i(P), \quad \forall P \in \mathcal{H}(D),$$

is a contraction Hutchinson operator.

- (ii) The sequence of compact sets $\{P_0, F(P_0), F^2(P_0), \dots\}$ converges to an attractor of F , where $P_0 \in \mathcal{H}(D)$ is any initial set.
- (iii) F has a unique attractor $P \in \mathcal{H}(D)$, i.e.,

$$P = F(P) = \bigcup_{i=1}^l g_i(P)$$

Proof. (i) The proof is obvious from Theorem 2.14.

(ii) Let $P_0 \in \mathcal{H}(D)$ be any set. If $P_0 = F(P_0)$, then result holds. Therefore, we suppose that $P_0 \neq F(P_0)$. Define

$$P_1 = F(P_0), P_2 = F(P_1), \dots, P_{n+1} = F(P_n)$$

for $n \in \mathbb{N}$. If we take $P_l = P_{l+1}$ for some $l \in \mathbb{N}$ then, we have $P_l = F(P_l)$ and thus the theorem holds. So, we assume that $P_n \neq P_{n+1}$ for all $n \in \mathbb{N}$. Then, by Theorem 2.14, using contractive condition, we have

$$\begin{aligned} H_G(P_n, P_{n+1}, P_{n+1}) &\leq \lambda H_G(P_{n-1}, P_n, P_n) \\ &\leq \lambda \{ \lambda H_G(P_{n-2}, P_{n-1}, P_{n-1}) \} \\ &\quad \dots \\ &\quad \dots \\ &\quad \dots \\ &\leq \lambda^n H_G(P_0, P_1, P_1) \} \end{aligned}$$

Further, for all $n, m \in \mathbb{N}; n < m$, using rectangle inequality of G - metric space and contractive condition of H_G , we obtain

$$\begin{aligned} H_G(P_n, P_m, P_m) &\leq H_G(P_n, P_{n+1}, P_{n+1}) + H_G(P_{n+1}, P_{n+2}, P_{n+2}) \\ &\quad + H_G(P_{n+2}, P_{n+3}, P_{n+3}) + \dots + H_G(P_{m-1}, P_m, P_m) \\ &\leq (\lambda^n + \lambda^{n+1} + \lambda^{n+2} + \dots + \lambda^{m-1}) H_G(P_0, P_1, P_1) \\ &= \frac{\lambda^n(1 - \lambda^{m-n-1})}{1 - \lambda} H_G(P_0, P_1, P_1). \end{aligned}$$

This gives $\lim_{n, m \rightarrow \infty} H_G(P_n, P_m, P_m) = 0$, since $\lambda \in [0, 1)$. Thus, the sequence $\{P_n\}$ is a Hausdorff G - Cauchy sequence. But, $(\mathcal{H}(D), H_G)$ is complete being compact, then there exists $P \in \mathcal{H}(D)$ such that $\{P_n\}$ is Hausdorff G - convergent to P . Let us assume that $F(P) \neq P$, then

$$H_G(P_n, F(P), F(P)) \leq \lambda H_G(P_{n-1}, P, P).$$

Now, taking the limit as $n \rightarrow \infty$ and using the continuity of H_G , we have

$$H_G(P, F(P), F(P)) \leq \lambda H_G(P, P, P),$$

which is not possible. Thus, our assumption is wrong. This gives $F(P) = P$, where P is an attractor of F . Hence, the sequence of compact sets $\{P_0, F(P_0), F^2(P_0), \dots\}$ converges to an attractor P of F .

(iii) To prove uniqueness of above obtained attractor P , let us suppose that $P \neq Q$, where Q is another attractor of F . i.e., $F(Q) = Q$. Consider,

$$H_G(P, P, Q) = H_G(F(P), F(P), F(Q)) \leq \lambda H_G(P, P, Q),$$

which is a contradiction, as $\lambda \in [0, 1)$. This yields that $P = Q$. Thus, the attractor P is unique. $\square$

Corollary 3.2. Let $(\mathcal{H}(D), H_G)$ be a Hausdorff G - metric space and $F: \mathcal{H}(D) \rightarrow \mathcal{H}(D)$ be Hutchinson Barnsley operator satisfying

$$H_G(F(P), F(Q), F(R)) \leq \lambda H_G(P, F(P), Q),$$

for all $P, Q \in \mathcal{H}(D)$ and $\lambda \in [0, 1)$. Then, F has a unique attractor.

Corollary 3.3. Let $(\mathcal{H}(D), H_G)$ be a Hausdorff G - metric space and $F: \mathcal{H}(D) \rightarrow \mathcal{H}(D)$ be Hutchinson Barnsley operator satisfying the condition given by

$$H_G(F(P), F(Q), F(R)) \leq \lambda H_G(P, F(P), R) + \mu H_G(P, F(P), Q), \forall P, Q, R \in \mathcal{H}(D).$$

where $0 \leq \lambda + \mu < 1$. Then, F has a unique attractor.

Theorem 3.4. Let $(\mathcal{H}(D), H_G)$ be a Hausdorff G - metric space and $F: \mathcal{H}(D) \rightarrow \mathcal{H}(D)$ be Hutchinson Barnsley operator satisfying the following condition for all $P, Q \in \mathcal{H}(D)$;

$$\begin{aligned} H_G(F(P), F(Q), F^2(Q)) &\leq a H_G(P, F(P), F^2(P)) + b H_G(Q, F(Q), F^2(Q)) \\ &\quad + c H_G(P, F(P), F(Q)) + d H_G(Q, F(Q), F^3(P)), \end{aligned} \quad (3.6)$$

where $0 \leq a + b + c + d < 1$. Then, F has a unique attractor.

Proof. Choose $P_0 \in \mathcal{H}(D)$. Now, we construct a sequence $\{P_n\}$ of sets in $\mathcal{H}(D)$ as follows:

$$P_{n+1} = F(P_n), \forall n = 0, 1, 2, \dots$$

If $P_{l+1} = F(P_l)$ for some $l \in \mathbb{N}$, then Hutchinson Barnsley operator F has an attractor. Therefore, suppose that

$$P_n \neq P_{n+1}, \text{ for all } n \in \mathbb{N}.$$

Thus, we have that $H_G(P_n, P_{n+1}, P_{n+2}) > 0$. Using given condition (3.6), we have

$$\begin{aligned} H_G(F(P_{n-1}), F(P_n), F^2(P_n)) &\leq aH_G(P_{n-1}, F(P_{n-1}), F^2(P_{n-1})) \\ &\quad + bH_G(P_n, F(P_n), F^2(P_n)) \\ &\quad + cH_G(P_{n-1}, F(P_{n-1}), F(P_n)) \\ &\quad + dH_G(P_n, F(P_n), F^3(P_{n-1})), \end{aligned}$$

which gives

$$\begin{aligned} H_G(P_n, P_{n+1}, P_{n+2}) &\leq aH_G(P_{n-1}, P_n, P_{n+1}) + bH_G(P_n, P_{n+1}, P_{n+2}) \\ &\quad + cH_G(P_{n-1}, P_n, P_{n+1}) + dH_G(P_n, P_{n+1}, P_{n+2}), \end{aligned}$$

Therefore,

$$H_G(P_n, P_{n+1}, P_{n+2}) \leq kH_G(P_{n-1}, P_n, P_{n+1}),$$

where $k = \frac{a+c}{1-b-d} < 1$. Then, repeating this process n times, we have

$$H_G(P_n, P_{n+1}, P_{n+2}) \leq k^n H_G(P_0, P_1, P_2), \forall n \in \mathbb{N}. \quad (3.7)$$

Using (G4) and (G5), we have

$$\begin{aligned} H_G(P_{n+1}, P_{n+1}, P_n) &\leq H_G(P_{n+1}, P_n, P_n) + H_G(P_n, P_{n+1}, P_n) \\ &= 2H_G(P_n, P_n, P_{n+1}) \\ \Rightarrow H_G(P_{n+1}, P_{n+1}, P_n) &\leq 2H_G(P_n, P_n, P_{n+1}). \end{aligned}$$

Thus, we get

$$H_G(P_{n+1}, P_{n+1}, P_n) \leq 2k^n H_G(P_1, P_1, P_2) \leq 2k^n H_G(P_0, P_1, P_2).$$

Also, for all $m, n \in \mathbb{N}$; $m < n$, using rectangle inequality, we obtain

$$\begin{aligned} H_G(P_n, P_m, P_m) &\leq H_G(P_n, P_{n+1}, P_{n+1}) + H_G(P_{n+1}, P_{n+2}, P_{n+2}) \\ &\quad + \dots + H_G(P_{m-1}, P_m, P_m) \\ &\leq 2(k^n + k^{n+1} + \dots + k^{m-1})H_G(P_0, P_1, P_1) \\ &= \frac{2k^n(1 - k^{m-n-1})}{1 - k} H_G(P_0, P_1, P_1) \\ H_G(P_n, P_m, P_m) &\leq \frac{2k^n(1 - k^{m-n-1})}{1 - k} H_G(P_0, P_1, P_1), \end{aligned} \quad (3.8)$$

and therefore, $\lim_{m, n \rightarrow \infty} H_G(P_n, P_m, P_m) = 0$, since $k < 1$. Thus, the sequence P_n is a Hausdorff Cauchy sequence in $(\mathcal{H}(D), H_G)$. But $(\mathcal{H}(D), H_G)$ is complete Hausdorff G -metric space, the sequence P_n converges to some $P \in \mathcal{H}(D)$. Further, consider

$$\begin{aligned} H_G(F(P_n), F(P), F^2(P)) &\leq aH_G(P_n, F(P_n), F^2(P_n)) + bH_G(P, F(P), F^2(P)) \\ &\quad + cH_G(P_n, F(P_n), F(P)) + dH_G(P, F(P), F^3(P_n)). \end{aligned}$$

Then

$$H_G(P_{n+1}, F(P), F^2(P)) \leq aH_G(P_n, P_{n+1}, P_{n+2}) + bH_G(P, F(P), F^2(P)) \\ + cH_G(P_n, P_{n+1}, F(P)) + dH_G(P, F(P), P_{n+3}).$$

Now, taking limit as $n \rightarrow \infty$, we have

$$H_G(P, F(P), F^2(P)) \leq \frac{c+d}{1-b} H_G(P, P, F(P)).$$

It is clear, that F has an attractor if $F(P) = F^2(P)$. Hence, suppose that $F(P) \neq F^2(P)$. Then, using (G3), we have

$$H_G(P, F(P), F^2(P)) \leq \frac{c+d}{1-b} H_G(P, P, F(P)) \leq \frac{c+d}{1-b} H_G(P, F(P), F^2(P)),$$

this is possible only if $H_G(P, F(P), F^2(P)) = 0$,
i.e.,

$$P = F(P) = F^2(P).$$

Hence, Hutchinson Barnsley operator F has a unique attractor. $\square$

Before moving on the next theorem, we consider the following classes of auxiliary functions

$$\Psi = \{\psi: [0, \infty) \rightarrow [0, \infty) \text{ such that } \psi \text{ is non-decreasing and continuous}\}$$

and

$$\Phi = \{\phi: [0, \infty) \rightarrow [0, \infty) \text{ such that } \phi \text{ is lower semicontinuous}\},$$

where $\psi(u) = \phi(u)$ iff $u = 0$ for all $u \in [0, \infty)$.

Theorem 3.5. Let $(\mathcal{H}(D), H_G)$ be a Hausdorff G -metric space and $F: \mathcal{H}(D) \rightarrow \mathcal{H}(D)$ be Hutchinson Barnsley operator satisfying the given condition for all $P, Q \in \mathcal{H}(D)$

$$\psi(H_G(F(P), F^2(P), F(Q))) \leq \psi(H_G(P, F(P), Q)) - \phi(H_G(P, F(P), Q)), \quad (3.9)$$

where $\psi \in \Psi$ and $\phi \in \Phi$. Then, F has a unique attractor.

Proof. Choose $P_0 \in \mathcal{H}(D)$. Now, we construct a sequence $\{P_n\}$ of sets in $\mathcal{H}(D)$ as follows:

$$P_{n+1} = F(P_n), \forall n = 0, 1, 2, \dots$$

If $P_\alpha = P_{\alpha+1} = F(P_\alpha)$ for some $\alpha \in \mathbb{N}$, then Hutchinson Barnsley operator F has an attractor. Therefore, suppose that

$$P_n \neq P_{n+1}$$

for all $n \in \mathbb{N}$. Using (G2), we have $H_G(P_n, P_{n+1}, P_{n+1}) > 0$.

Putting $P = P_{n-1}$ and $Q = P_n$ in given condition (3.9), we have

$$\psi(H_G(F(P_{n-1}), F^2(P_{n-1}), F(P_n))) \leq \psi(H_G(P_{n-1}, F(P_{n-1}), P_n)) \\ - \phi(H_G(P_{n-1}, F(P_{n-1}), P_n)) \\ \Rightarrow \psi(H_G(P_n, P_{n+1}, P_{n+1})) \leq \psi(H_G(P_{n-1}, P_n, P_n)) - \phi(H_G(P_{n-1}, P_n, P_n)). \quad (3.10)$$

This gives,

$$\psi(H_G(P_n, P_{n+1}, P_{n+1})) \leq \psi(H_G(P_{n-1}, P_n, P_n)). \quad (3.11)$$

Since, ψ is continuous. This gives $H_G(P_n, P_{n+1}, P_{n+1}) \leq H_G(P_{n-1}, P_n, P_n)$. Therefore, the sequence $\{H_G(P_n, P_{n+1}, P_{n+1})\}$ is a decreasing sequence in $[0, \infty)$ and thus converges

to some $P \in [0, \infty)$. We have to prove that $P = 0$, on the contrary, suppose that $P > 0$. Now, taking limit as $n \rightarrow \infty$ in (3.10), we obtain

$$\psi(P) \leq \psi(P) - \phi(P),$$

which gives that $\phi(P) = 0$, which is a contradiction. Hence, $P = 0$. i.e.,

$$\lim_{n \rightarrow \infty} H_G(P_n, P_{n+1}, P_{n+1}) = 0. \quad (3.12)$$

Now, we shall show that $\{P_n\}$ is a Hausdorff G - Cauchy sequence. On the contrary, suppose that there exists $\epsilon > 0$ and sequence $\{P_{n(\alpha)}\}$ of $\{P_n\}$ such that

$$H_G(P_{m(\alpha)}, F(P_{m(\alpha)}), P_{n(\alpha)}) \geq \epsilon, \quad (3.13)$$

with $n(\alpha) \geq m(\alpha) > \alpha$. Moreover, we take the smallest integer $n(\alpha)$ corresponding to $m(\alpha)$ such that $n(\alpha) > m(\alpha)$ and it satisfy (3.13). Hence,

$$H_G(P_{m(\alpha)}, F(P_{m(\alpha)}), P_{n(\alpha)-1}) < \epsilon. \quad (3.14)$$

By using (G4), (G5) and (3.13), we have

$$\begin{aligned} \epsilon &\leq H_G(P_{m(\alpha)}, F(P_{m(\alpha)}), P_{n(\alpha)}) = H_G(P_{n(\alpha)}, F(P_{m(\alpha)}), P_{m(\alpha)}) \\ &\leq H_G(P_{n(\alpha)}, P_{n(\alpha)-1}, P_{n(\alpha)-1}) + H_G(P_{n(\alpha)-1}, F(P_{m(\alpha)}), P_{m(\alpha)}) \\ &\leq H_G(P_{m(\alpha)}, F(P_{m(\alpha)}), P_{n(\alpha)-1}) + 2K_{n(\alpha)-1} \\ &\leq \epsilon + 2K_{n(\alpha)-1}, \end{aligned} \quad (3.15)$$

where $K_{n(\alpha)-1} = H_G(P_{n(\alpha)-1}, P_{n(\alpha)}, P_{n(\alpha)})$. Taking $\alpha \rightarrow \infty$ in above equation, we obtain

$$\lim_{\alpha \rightarrow \infty} H_G(P_{m(\alpha)}, F(P_{m(\alpha)}), P_{n(\alpha)}) = \epsilon. \quad (3.16)$$

Also, again using (G4) and (G5), we obtain

$$\begin{aligned} H_G(P_{m(\alpha)}, F(P_{m(\alpha)}), P_{n(\alpha)}) &\leq H_G(P_{m(\alpha)}, P_{m(\alpha)-1}, P_{m(\alpha)-1}) \\ &\quad + H_G(P_{m(\alpha)-1}, F(P_{m(\alpha)}), P_{n(\alpha)}) \\ &= H_G(P_{m(\alpha)}, P_{m(\alpha)-1}, P_{m(\alpha)-1}) + H_G(P_{n(\alpha)}, P_{m(\alpha)-1}, F(P_{m(\alpha)})) \\ &\leq H_G(P_{m(\alpha)}, P_{m(\alpha)-1}, P_{m(\alpha)-1}) + H_G(P_{n(\alpha)}, P_{n(\alpha)-1}, P_{n(\alpha)-1}) \\ &\quad + H_G(P_{n(\alpha)-1}, P_{m(\alpha)-1}, F(P_{m(\alpha)})) \\ &\leq 2K_{m(\alpha)} + 2K_{n(\alpha)} + H_G(P_{n(\alpha)-1}, P_{m(\alpha)-1}, F(P_{m(\alpha)})) \end{aligned}$$

This gives

$$H_G(P_{m(\alpha)}, F(P_{m(\alpha)}), P_{n(\alpha)}) \leq 2K_{m(\alpha)} + 2K_{n(\alpha)} + H_G(P_{n(\alpha)-1}, P_{m(\alpha)-1}, F(P_{m(\alpha)})), \quad (3.17)$$

and

$$\begin{aligned} H_G(P_{n(\alpha)-1}, P_{m(\alpha)-1}, F(P_{m(\alpha)})) &\leq H_G(P_{n(\alpha)-1}, P_{n(\alpha)}, P_{n(\alpha)}) \\ &\quad + H_G(P_{n(\alpha)}, P_{m(\alpha)-1}, F(P_{m(\alpha)})) \\ &= H_G(P_{n(\alpha)-1}, P_{n(\alpha)}, P_{n(\alpha)}) + H_G(P_{m(\alpha)-1}, F(P_{m(\alpha)}), P_{n(\alpha)}) \\ &\leq H_G(P_{n(\alpha)-1}, P_{n(\alpha)}, P_{n(\alpha)}) + H_G(P_{m(\alpha)-1}, P_{m(\alpha)}, P_{m(\alpha)}) \\ &\quad + H_G(P_{m(\alpha)}, F(P_{m(\alpha)}), P_{n(\alpha)}) \\ &\leq K_{n(\alpha)-1} + K_{m(\alpha)-1} + H_G(P_{m(\alpha)}, F(P_{m(\alpha)}), P_{n(\alpha)}) \end{aligned}$$

Then, we have

$$H_G(P_{n(\alpha)-1}, P_{m(\alpha)-1}, F(P_{m(\alpha)})) \leq K_{n(\alpha)-1} + K_{m(\alpha)-1} + H_G(P_{m(\alpha)}, F(P_{m(\alpha)}), P_{n(\alpha)}). \quad (3.18)$$

Now, letting $\alpha \rightarrow \infty$ in (3.17) and (3.18) and then using (3.16), we get

$$\lim_{\alpha \rightarrow \infty} H_G(P_{n(\alpha)-1}, P_{m(\alpha)-1}, F(P_{m(\alpha)})) = \epsilon. \quad (3.19)$$

Again, by using (G4) and (G5), we have

$$\begin{aligned} H_G(P_{n(\alpha)-1}, P_{m(\alpha)-1}, F(P_{m(\alpha)})) &= H_G(F(P_{m(\alpha)}), P_{m(\alpha)-1}, P_{n(\alpha)-1}) \\ &= H_G(P_{m(\alpha)+1}, P_{m(\alpha)-1}, P_{n(\alpha)-1}) \\ &\leq H_G(P_{m(\alpha)+1}, P_{m(\alpha)}, P_{m(\alpha)}) + H_G(P_{m(\alpha)}, P_{m(\alpha)-1}, P_{n(\alpha)-1}) \\ &= H_G(P_{m(\alpha)+1}, P_{m(\alpha)}, P_{m(\alpha)}) + H_G(P_{m(\alpha)-1}, P_{m(\alpha)}, P_{n(\alpha)-1}) \\ &\leq 2K_{m(\alpha)} + H_G(P_{m(\alpha)-1}, F(P_{m(\alpha)-1}), P_{n(\alpha)-1}) \end{aligned}$$

Thus, we obtain

$$H_G(P_{n(\alpha)-1}, P_{m(\alpha)-1}, F(P_{m(\alpha)})) \leq 2K_{m(\alpha)} + H_G(P_{m(\alpha)-1}, F(P_{m(\alpha)-1}), P_{n(\alpha)-1}), \quad (3.20)$$

and

$$\begin{aligned} H_G(P_{m(\alpha)-1}, F(P_{m(\alpha)-1}), P_{n(\alpha)-1}) &= H_G(P_{m(\alpha)-1}, P_{m(\alpha)}, P_{n(\alpha)-1}) \\ &\leq H_G(P_{m(\alpha)-1}, P_{m(\alpha)+1}, P_{m(\alpha)+1}) + H_G(P_{m(\alpha)+1}, P_{m(\alpha)}, P_{n(\alpha)-1}) \\ &\leq H_G(P_{m(\alpha)-1}, P_{m(\alpha)}, P_{m(\alpha)}) + H_G(P_{m(\alpha)}, P_{m(\alpha)+1}, P_{m(\alpha)+1}) \\ &\quad + H_G(P_{m(\alpha)+1}, P_{m(\alpha)}, P_{n(\alpha)-1}) \\ &\leq K_{m(\alpha)-1} + K_{m(\alpha)} + H_G(P_{m(\alpha)+1}, P_{m(\alpha)}, P_{n(\alpha)-1}) \\ &= K_{m(\alpha)-1} + K_{m(\alpha)} + H_G(P_{m(\alpha)}, F(P_{m(\alpha)}), P_{n(\alpha)-1}) \\ &< K_{m(\alpha)-1} + K_{m(\alpha)} + \epsilon \\ &\Rightarrow H_G(P_{m(\alpha)-1}, F(P_{m(\alpha)-1}), P_{n(\alpha)-1}) < K_{m(\alpha)-1} + K_{m(\alpha)} + \epsilon. \end{aligned} \quad (3.21)$$

Taking limit as $\alpha \rightarrow \infty$ in (3.20) and (3.21) and applying (3.19), we obtain

$$\lim_{\alpha \rightarrow \infty} H_G(P_{m(\alpha)-1}, F(P_{m(\alpha)-1}), P_{n(\alpha)-1}) = \epsilon. \quad (3.22)$$

Putting $P = P_{m(\alpha)-1}$ and $Q = P_{n(\alpha)-1}$ in given condition (3.9), we obtain

$$\begin{aligned} \psi(H_G(F(P_{m(\alpha)-1}), F^2(P_{m(\alpha)-1}), F(P_{n(\alpha)-1}))) &\leq \psi(H_G(P_{m(\alpha)-1}, F(P_{m(\alpha)-1}), \\ &\quad P_{n(\alpha)-1})) - \phi(H_G(P_{m(\alpha)-1}, F(P_{m(\alpha)-1}), P_{n(\alpha)-1})). \end{aligned} \quad (3.23)$$

Letting $\alpha \rightarrow \infty$ in above (3.23), we have

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon),$$

which gives $\epsilon = 0$, a contradiction. Thus,

$$\lim_{m, n \rightarrow \infty} H_G(P_m, F(P_m), P_n) = \lim_{m, n \rightarrow \infty} H_G(P_m, P_{m+1}, P_n) = 0.$$

This implies that $\{P_n\}_0^\infty$ is a Hausdorff G - Cauchy sequence. As $(\mathcal{H}(D), H_G)$ is a Hausdorff G - complete, then there exists an attractor $W \in \mathcal{H}(D)$ such that $P_n \rightarrow W$ as $n \rightarrow \infty$. Also, from condition (3.9),

$$\begin{aligned}\psi(H_G(P_{n+1}, P_{n+2}, F(W))) &= \psi(H_G(F(P_n), F^2(P_n), F(W))) \\ &\leq \psi(H_G(P_n, F(P_n), W)) - \phi(H_G(P_n, F(P_n), W)) \\ &= \psi(H_G(P_n, P_{n+1}, W)) - \phi(H_G(P_n, P_{n+1}, W)).\end{aligned}$$

Taking limit as $n \rightarrow \infty$, we have

$$\phi(H_G(W, W, F(W))) \leq \psi(0) - \psi(0) = 0.$$

Since, ϕ is continuous, implies $(H_G(W, W, F(W))) = 0$. i.e., $W = F(W)$.

For proving uniqueness, let us suppose that $X \in \mathcal{H}(D)$ is another attractor of Hutchinson Barnsley operator F such that $F(X) = X$. Then, by condition (3.9), we have

$$\psi(H_G(F(W), F^2(W), F(X))) \leq \psi(H_G(W, F(W), X)) - \phi(H_G(W, F(W), X)), \quad (3.24)$$

which gives that $\phi(H_G(W, F(W), X)) = 0$, i.e., $W = X$. Hence, there exists a unique attractor of F satisfying the condition given by (3.9). $\square$

Corollary 3.6. Let $(\mathcal{H}(D), H_G)$ be a Hausdorff G - metric space and $F: \mathcal{H}(D) \rightarrow \mathcal{H}(D)$ be Hutchinson Barnsley operator F satisfying the given condition

$$\begin{aligned}H_G(F(P), F^2(P), F(Q)) + H_G(F(P), F^2(P), F(R)) &\leq aH_G(P, F(P), Q) \\ &\quad + bH_G(P, F(P), R),\end{aligned}$$

for all $P, Q, R \in \mathcal{H}(D)$, where $0 \leq a + b < 2$ holds. Then, F has a unique attractor.

Proof. By using $Q = R$, we have

$$H_G(F(P), F^2(P), F(Q)) \leq \frac{(a+b)}{2} H_G(P, F(P), Q),$$

where $0 \leq \frac{(a+b)}{2} < 1$. i.e., condition (3.9) is satisfied and hence, F has a unique attractor. $\square$

4. Applications

Now, we shall discuss how fractal objects can be actually produced. The focus of this section is on Iterated function systems (IFS) which enables us to produce these fractals. Collage theorem enables us to approximate fractals by using IFS having a specific attractor that will construct the required image, no matter what initial set is to be taken. The inverse problem of reconstructing fractals is also solved by using Collage theorem. M .F. Barnsley investigated collage theorem for Hausdorff metric space, but here we generalize this concept to Hausdorff G - metric space.

Theorem 4.1. (*Collage theorem*). Let $(\mathcal{H}(D), H_G)$ be a complete Hausdorff generalized metric space. Let $g_i: U \rightarrow U$ are contractions with contractive factor as λ_i for $i = 1, 2, \dots, N$ and $\{U; g_1, g_2, \dots, g_N\}$ be iterated function systems (IFS) having contractive factor as λ , where $\lambda = \max\{\lambda_i, i = 1, 2, \dots, N\}$ and $0 \leq \lambda < 1$. If $F: \mathcal{H}(D) \rightarrow \mathcal{H}(D)$ is HB operator with contractive factor λ and $L \in \mathcal{H}(D)$ then,

$$H_G(L, P^*, P^*) \leq \frac{1}{1-\lambda} H_G(L, F(L), F(L)), \quad (4.25)$$

for some attractor $P^* \in \mathcal{H}(D)$.

Proof. Using rectangle inequality of G - metric space, we obtain

$$\begin{aligned} H_G(L, F^n(L), F^n(L)) &\leq H_G(L, F(L), F(L)) + H_G(F(L), F^n(L), F^n(L)) \\ &\leq H_G(L, F(L), F(L)) + H_G(F(L), F^2(L), F^2(L)) \\ &\quad + \cdots + H_G(F^{n-1}(L), F^n(L), F^n(L)). \end{aligned} \quad (4.26)$$

Since, $\mathcal{F}$ is a contractive operator with contractive factor λ , (4.26) reduces to

$$\begin{aligned} H_G(L, F^n(L), F^n(L)) &\leq H_G(L, F(L), F(L)) + \lambda H_G(L, F(L), F(L)) \\ &\quad + \lambda^2 H_G(L, F(L), F(L)) + \lambda^3 H_G(L, F(L), F(L)) \\ &\quad + \cdots + \lambda^{n-1} H_G(L, F(L), F(L)) \\ &= (1 + \lambda + \lambda^2 + \lambda^3 + \cdots + \lambda^{n-1}) H_G(L, F(L), F(L)) \\ &= \frac{(1 - \lambda^{n-1})}{1 - \lambda} H_G(L, F(L), F(L)), \text{ for } \lambda < 1. \end{aligned}$$

Letting limit as $n \rightarrow \infty$, we have

$$H_G(L, P^*, P^*) \leq \frac{1}{1 - \lambda} H_G(L, F(L), F(L)), \text{ for } \lambda < 1. \quad (4.27)$$

□

4.1. Numerical approach of our results: Now, we present some examples to demonstrate that with the help of collage theorem, we can select suitable IFS satisfying proposed contractive conditions to produce classical fractals. With the help of Matlab programming, we perform this construction.

Example 4.2. Let $D = [0, \infty)$ and $U = [0, 1]$ be a compact subset of D . Also, let $\mathcal{H}(D)$ be the collection of all compact subsets of D and

$$H_G(P, Q, R) = \begin{cases} 0, & \text{if } P = Q = R \\ |P - Q| + |Q - R| + |R - P|, & \text{otherwise} \end{cases}$$

be a Hausdorff G - metric space. Let $g_1: U \rightarrow U$ and $g_2: U \rightarrow U$ be two contractions defined by

$$g_1(u) = \frac{1}{3}u$$

and

$$g_2(u) = 1 - \frac{1}{3}u$$

Each of which is a contraction mapping with contractivity factor $\frac{1}{3}$.

Define $F: \mathcal{H}(D) \rightarrow \mathcal{H}(D)$ by

$$F = g_1 \cup g_2$$

i.e.,

$$F(P) = g_1(P) \cup g_2(P), \text{ for all } P \in \mathcal{H}(D)$$

Then, from Theorem 2.14, F is a contraction mapping with contractive factor $\frac{1}{3}$. Now, consider

$$H_G(F(P), F(Q), F(R)) = H_G\{g_1(P) \cup g_2(P), g_1(Q) \cup g_2(Q), g_1(R) \cup g_2(R)\}$$

$$\leq \frac{1}{3}H_G(P, Q, R)$$

Therefore,

$$H_G(F(P), F(Q), F(R)) \leq \frac{1}{3}H_G(P, Q, R)$$

Thus, Theorem 3.1 holds for this example. In this example, the attractor obtained is a Cantor Ternary set (see Fig. 1). To construct this, we iterate F five times which gives first five steps of Cantor Ternary set.

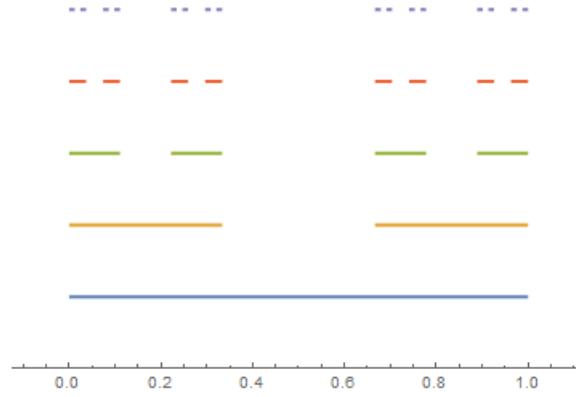


FIGURE 1. First five steps of Cantor Ternary set

Example 4.3. Let $D = [0, \infty)$ and $U = [0, 1]$ be a compact subset of D . Also, let $\mathcal{H}(D)$ be the collection of all compact subsets of D and the Hausdorff G - metric is same as defined in (2.1). Suppose that g_1, g_2 and g_3 are three self contractions on $U \times U$ defined by

$$\begin{aligned} g_1(p, q) &= \frac{1}{2}p, \frac{1}{2}q \\ g_2(p, q) &= \frac{1}{2}p + \frac{1}{2}, \frac{1}{2}q \\ g_3(p, q) &= \frac{1}{2}p + \frac{1}{4}, \frac{1}{2}q + \frac{\sqrt{3}}{4}. \end{aligned}$$

Each of which is a contraction mapping with contractivity factor as $\frac{1}{2}$.

Define $F: \mathcal{H}(D) \times \mathcal{H}(D) \rightarrow \mathcal{H}(D) \times \mathcal{H}(D)$ by

$$F = g_1 \cup g_2 \cup g_3$$

$$i.e., F(Q, R) = g_1(Q, R) \cup g_2(Q, R) \cup g_3(Q, R), \text{ for all } Q, R \in \mathcal{H}(D).$$

Let us suppose that $\psi(p) = p$ and $\phi(p) = \frac{1}{2}p$.

Then, from Theorem 2.14, F is a contraction mapping with contractive factor $\frac{1}{2}$. This

example satisfy all the axioms of Theorem 3.5. Therefore, HB operator F has a unique attractor, which is Sierpinski triangle. We show first three steps of constructing the fine structure of this classical fractal (see Figs. 2a - 2c).

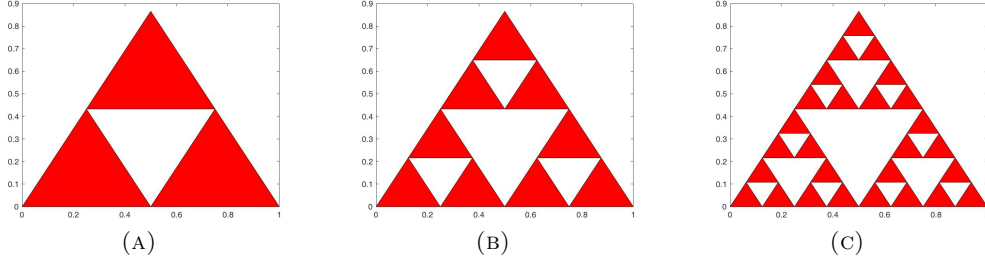


FIGURE 2. Generation of Sierpinski Triangle (A) first step (B) second step (C) third step.

Example 4.4. Let $D = [0, \infty)$ and $U = [0, 1]$ be a compact subset of D . Also, let $\mathcal{H}(D)$ be the collection of all compact subsets of D and the Hausdorff G -metric is same as defined in (2.1). Suppose that g_1, g_2, g_3, g_4 and g_5 are five self contractions on $U \times U$ defined by

$$\begin{aligned} g_1(p, q) &= \frac{1}{3}p, \frac{1}{3}q \\ g_2(p, q) &= \frac{1}{3}p + \frac{2}{3}, \frac{1}{3}q \\ g_3(p, q) &= \frac{1}{3}p, \frac{1}{3}q + \frac{2}{3} \\ g_4(p, q) &= \frac{1}{3}p + \frac{1}{3}, \frac{1}{3}q + \frac{1}{3} \\ g_5(p, q) &= \frac{1}{3}p + \frac{2}{3}, \frac{1}{3}q + \frac{2}{3}. \end{aligned}$$

Define $F: \mathcal{H}(D) \times \mathcal{H}(D) \rightarrow \mathcal{H}(D) \times \mathcal{H}(D)$ by

$$F = g_1 \cup g_2 \cup g_3 \cup g_4 \cup g_5$$

Then, from Theorem 2.14, F is a contraction mapping with contractive factor $\frac{1}{3}$. This example satisfy all the axioms of Theorem 3.4. Therefore, HB operator F has a unique attractor. First three steps of this attractor is shown in Figs. 3a - 3c, which takes the form of Sierpinski carpet.

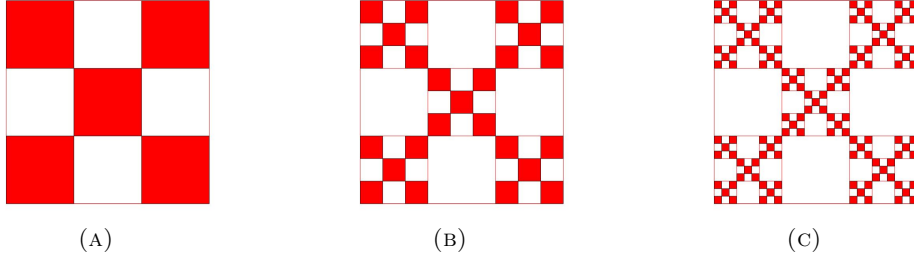


FIGURE 3. Generation of Sierpinski Carpet (A) first step (B) second step (C) third step.

Remark 4.5. If we take following contractions instead of contractions considered in Example 4.4, we generate a new type of Sierpinski carpet presented by Figs. 4a - 4c.

$$\begin{aligned}
 g_1(p, q) &= \frac{1}{3}p, \frac{1}{3}q \\
 g_2(p, q) &= \frac{1}{3}p + \frac{1}{3}, \frac{1}{3}q \\
 g_3(p, q) &= \frac{1}{3}p + \frac{2}{3}, \frac{1}{3}q \\
 g_4(p, q) &= \frac{1}{3}p, \frac{1}{3}q + \frac{1}{3} \\
 g_5(p, q) &= \frac{1}{3}p, \frac{1}{3}q + \frac{2}{3} \\
 g_6(p, q) &= \frac{1}{3}p + \frac{1}{3}, \frac{1}{3}q + \frac{2}{3} \\
 g_7(p, q) &= \frac{1}{3}p + \frac{2}{3}, \frac{1}{3}q + \frac{1}{3} \\
 g_8(p, q) &= \frac{1}{3}p + \frac{2}{3}, \frac{1}{3}q + \frac{2}{3}.
 \end{aligned}$$

Define $F: \mathcal{H}(D) \times \mathcal{H}(D) \rightarrow \mathcal{H}(D) \times \mathcal{H}(D)$ by

$$F = g_1 \cup g_2 \cup g_3 \cup g_4 \cup g_5 \cup g_6 \cup g_7 \cup g_8$$

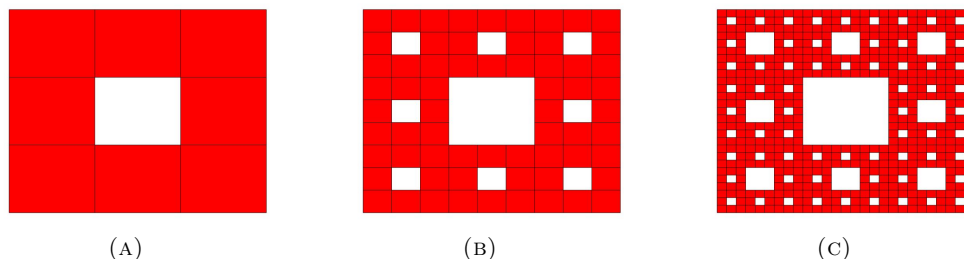


FIGURE 4. Generation of a new carpet (A) first step (B) second step (C) third step.

5. Conclusion

This article deals with the applications of IFS operator which is defined on Hausdorff G -metric spaces to construct fractals and these geometric constructions are made in environment of Matlab. We begin our article with a solid foundation of iterated function systems (IFS) in the setting of Hausdorff G -metric spaces and then some results to compute attractors (fractals) using different contractive conditions have been derived. In Section 4, we prove collage theorem which is used to find out the suitable IFS to construct desired fractal objects. Also, to validate our results, numerous practical examples are presented which demonstrate that different attractors (fractals) can be approximated by using appropriate IFS (see Figs. 1 - 4c). Our work arise interesting questions to generate new attractors by using generalized contractive conditions. Also, our next research is to construct multifractals by using multivalued IFS in this generalized metric space.

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