

ψ - $\mathcal{I}$ -CLOSED SET, WEAKLY ψ - $\mathcal{I}$ -CLOSED SET AND CONTRA ψ - $\mathcal{I}$ -CONTINUOUS MAPPING IN IDEAL TOPOLOGICAL SPACES

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Abstract. In this paper, we introduce and study some basic properties of new classes of sets called ψ - $\mathcal{I}$ -closed sets and weakly ψ - $\mathcal{I}$ -closed sets. Moreover, we offer a new type of contra-continuous function in the context of ideal topology called contra- ψ - $\mathcal{I}$ continuous function and present its fundamental properties.

1. Introduction

Without doubt, the research works of Hamlett and Janković in the application of topological ideals to generalize the most rudimentary general topological properties evinced its usefulness and richness as a promising research area (see [22], [23], [24] and [28]). Since then many researchers have contributed to this field, among others, Abd El-Monsef, Arenas, Dontchev, Ergun, Ganster, Gupta, Jafari, Jayasudha, Lashien, Maki, Nasef, Noiri, Parimala, Rajesh, Rose, Umehara, Viswanathan and Ekici. In 1992, Janković and Hamlett [28] introduced the notion of $\mathcal{I}$ -open sets in an ideal topological space which is a topological space and an ideal $\mathcal{I}$ on it. Abd El-Monsef et al. [1] further investigated $\mathcal{I}$ -open sets and $\mathcal{I}$ -continuous functions. Throughout this paper, $int(A)$ and $cl(A)$ denote the interior and closure of A , respectively. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X , then the set operator $(.)^* : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, called the local function of A with respect to τ and $\mathcal{I}$, is defined as follows: For $A \subset X$, $A^*(\tau, \mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I}\}$, for every open set U of X containing x . A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(\tau, \mathcal{I})$ called the $*$ -topology, finer

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than τ is defined by $cl^*(A) = A \cup A^*(\tau, \mathcal{I})$ when there is no chance of confusion. $A^*(\mathcal{I})$ is denoted by A^* . If $\mathcal{I}$ is an ideal on X , then $(X, \tau, \mathcal{I})$ is called an ideal topological space.

The purpose of this paper is to give a new class of ψ - $\mathcal{I}$ -closed and weakly ψ - $\mathcal{I}$ -closed sets in an ideal topological space and derive some characterizations.

2. Preliminaries

N. Levine [29] and M. E. Abd El-Monsef et al.[2] introduced semi-open sets and β -open sets, respectively. Levine [30] generalised the notion of closed sets. Bhattacharaya and Lahiri [6] generalized the notion of closed sets to semi-generalized closed sets via semi-open sets. The complement of a semi-open (resp. semi-generalized closed) set is called semi-closed [7] (resp. semi-generalized open [6]) set. M. K. R. S. Veera Kumar [37] introduced ψ -closed sets. Later S. Jafari and T. Noiri [26] introduced and investigated the notion of contra- α -continuity between topological spaces.

Here we will recall some notions which will be used in the sequel.

Definition 2.1. A subset A of a topological space (X, τ) is said to be

- (i) semi-generalized closed set (briefly sg-closed) [6] if $scl(A) \subseteq U$ holds whenever $A \subseteq U$ and U is semi-open in (X, τ) .
- (ii) ψ -closed [37] if $scl(A) \subseteq U$, whenever $A \subseteq U$ and U is semi-generalised open in (X, τ) .

The complements of a sg-closed set (resp. ψ -closed set) are called a semi-generalized open set (briefly sg-open) (resp. ψ -open set).

Definition 2.2. A subset A of a topological space (X, τ) is said to be

- (i) semi-closed [7] if $\text{int}(cl(A)) \subseteq A$. The semi-closure of A is denoted by $scl(A)$.
- (ii) pre-closed [31] if $cl(\text{int}(A)) \subseteq A$;
- (iii) α -closed [34] if $cl(\text{int}(cl(A))) \subseteq A$;
- (iv) β -closed [2] if $\text{int}(cl(\text{int}(A))) \subseteq A$.

The complement of semi-closed (resp. pre-closed, α -closed and β -closed) are called semiopen (resp. pre-open, α -open and β -open).

Definition 2.3. A subset A of an ideal topological space (X, τ, I) is said to be

- (i) $\mathcal{I}$ -closed [1] if $cl(A^*) \subseteq A$;
- (ii) pre- $\mathcal{I}$ -closed [12] if $cl^*(\text{int}(A)) \subseteq A$;
- (iii) semi- $\mathcal{I}$ -closed [25] if $\text{int}(cl^*(A)) \subseteq A$;
- (iv) α - $\mathcal{I}$ -closed [24] if $cl^*(\text{int}(cl^*(A))) \subseteq A$;
- (v) β - $\mathcal{I}$ -closed [21] if $\text{int}(cl^*(\text{int}(A))) \subseteq A$.

The complements of semi- $\mathcal{I}$ -closed, α - $\mathcal{I}$ -closed and β - $\mathcal{I}$ -closed are called semi- $\mathcal{I}$ -open, α - $\mathcal{I}$ -open and β - $\mathcal{I}$ -open, respectively.

Definition 2.4. A subset A of an ideal topological space (X, τ, I) is said to be

- (i) $\mathcal{I}_g$ -closed [13] if $A^* \subseteq U$, whenever $A \subseteq U$ and U is open in (X, τ, I) .
- (ii) $\alpha\mathcal{I}_g$ -closed [32] if $A^* \subseteq U$, whenever $A \subseteq U$ and U is α -open in (X, τ, I) .
- (iii) $\mathcal{I}_g$ -closed [4] if $A^* \subseteq U$, whenever $A \subseteq U$ and U is semi-open in (X, τ, I) .

Definition 2.5. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be

- (i) contra-continuous [10] if for every $V \in \sigma$, $f^{-1}(V)$ is open in (X, τ) ;
- (ii) contra-semi-continuous [14] if for every open set $V \in \sigma$, $f^{-1}(V)$ is semi-closed;
- (iii) contra- α -continuous [26] if for every open set $V \in \sigma$, $f^{-1}(V)$ is α -closed;
- (iv) contra- β -continuous [9] if for every open set $V \in \sigma$, $f^{-1}(V)$ is β -closed.

Definition 2.6. A function $f : (X, \tau, I) \rightarrow (Y, \sigma)$ is said to be

- (i) semi- I -continuous [25] if for every closed set $V \in \sigma$, $f^{-1}(V)$ is semi- I -open;
- (ii) α - I -continuous [24] if for every closed set $V \in \sigma$, $f^{-1}(V)$ is α - I -open;
- (iii) contra strongly- α - $\mathcal{I}$ -continuous [35] if $f^{-1}(V)$ is strongly α - $\mathcal{I}$ -open in (X, τ, I) as well as B_I set for every closed set V of Y ;
- (iv) contra semi- I -continuous [33] if $f^{-1}(V)$ is semi- $\mathcal{I}$ -open set in (X, τ, I) for every closed set V of Y ;
- (v) contra α - I -continuous [38] if $f^{-1}(V)$ is α - $\mathcal{I}$ -open set in (X, τ, I) for every closed set V of Y ;
- (vi) contra β - I -continuous [8] if $f^{-1}(V)$ is β - $\mathcal{I}$ -open set in (X, τ, I) for every closed set V of Y ;

Lemma 2.7. If (X, τ, I) is any ideal topological space and $A \subseteq X$ then the following are equivalent. [32], Theorem 3.4]

- (i) A is $\alpha\mathcal{I}_g$ -closed
- (ii) $cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is α -open in X

Lemma 2.8. If (X, τ, I) is any ideal topological space and $A \subseteq X$ then the following are equivalent. [4], Theorem 2.5]

- (i) A is $\mathcal{I}_g$ -closed
- (ii) $cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X

Definition 2.9. Let (X, τ, I) be an ideal topological space. A subset G of (X, τ, I) is said to be a weakly $I_r g$ -closed set if $(int(G))^* \subset H$ whenever $G \subset H$ and H is a regular open set in X .

Definition 2.10. Let (X, τ, I) be an ideal topological space and $G \subset X$. Then G is said to be a weakly $I_r g$ -open set if X/G is a weakly $I_r g$ -closed set.

3. ψ - $\mathcal{I}$ -closed set

Definition 3.1. A subset A of an ideal topological space (X, τ, I) is said to be ψ - $\mathcal{I}$ -closed if $cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is ψ -open in (X, τ, I) . Complement of a ψ - $\mathcal{I}$ -closed set is called ψ - $\mathcal{I}$ -open.

Theorem 3.2. Every closed set is ψ - $\mathcal{I}$ -closed but not conversely.

Proof. Let $A \subseteq U$ and U is ψ -open in (X, τ, I) . Since $cl^*(A) \subseteq cl(A)$ and A is closed, $cl^*(A) \subseteq cl(A) = A \subseteq U$. Therefore A is ψ - $\mathcal{I}$ -closed set in (X, τ, I) . $\square$

Example 3.3. Let $X = \{a, b, c\}$, $\tau = \{X, \varphi, \{a, b\}\}$ and $\mathcal{I} = \{\varphi, \{a\}\}$. Then ψ - $\mathcal{I}$ -closed sets are $\{X, \varphi, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$ and closed sets are $\{X, \varphi, \{a, b\}, \{a\}, \{b\}\}$. It is clear that $\{a\}$ is ψ - $\mathcal{I}$ -closed set but it is not closed.

Theorem 3.4. Let $(X, \tau, \mathcal{I})$ be a ideal topological space. Then the following statements holds

- (i) Every ψ - $\mathcal{J}$ -closed set is $\mathcal{J}_g$ -closed.
- (ii) Every ψ - $\mathcal{J}$ -closed set is $\alpha\mathcal{J}_g$ -closed.
- (iii) Every ψ - $\mathcal{J}$ -closed set is semi-closed.
- (iv) Every ψ - $\mathcal{J}$ -closed set is semi- $\mathcal{J}$ -closed.
- (v) Every ψ - $\mathcal{J}$ -closed set is $\mathcal{J}_g$ -closed.
- (vi) Every ψ - $\mathcal{J}$ -closed set is ψ -closed.
- (vii) Every ψ - $\mathcal{J}$ -closed set is β - $\mathcal{J}$ -closed.

Proof. (i) Let $A \subseteq U$ and U is ψ -open in (X, τ, I) . By hypothesis $cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X . Since every ψ -open is open, $cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X . Hence A is $\mathcal{J}_g$ closed in X .

(ii) Let $A \subseteq U$ and U is ψ -open in (X, τ, I) . By hypothesis $cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is α -open in X . Since every ψ -open is α -open, $cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is α -open in X . Hence A is $\alpha\mathcal{J}_g$ -closed in X .

(iii) Let $A \subseteq U$ and U is ψ -open in (X, τ, I) . Since $int(cl(A)) \subseteq cl^*(A)$ and A is ψ - $\mathcal{J}$ -closed, $int(cl(A)) \subseteq cl^*(A) = A \subseteq U$. Therefore A is semi-closed set in X .

(iv) Let $A \subseteq U$ and U is ψ -open in (X, τ, I) . Since $int(cl^*(A)) \subseteq cl^*(A)$ and A is ψ - $\mathcal{J}$ -closed, $int(cl^*(A)) \subseteq cl^*(A) = A \subseteq U$. Therefore A is semi- $\mathcal{J}$ -closed set in (X, τ, I) .

(v) Let $A \subseteq U$ and U is ψ -open in (X, τ, I) . By hypothesis $cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X . Since every ψ -open is semi-open, $cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X . Hence A is $\mathcal{J}_g$ -closed in (X, τ, I) .

(vi) Let $A \subseteq U$ and U is ψ -open in (X, τ, I) . Since $int(cl(A)) \subseteq cl^*(A)$ and A is ψ - $\mathcal{J}$ -closed, $int(cl(A)) \subseteq cl^*(A) = A \subseteq U$. Therefore A is ψ -closed set in (X, τ, I) .

(vii) Let $A \subseteq U$ and U is ψ -open in (X, τ, I) . Since $int(cl^*(int(A))) \subseteq cl^*(A)$ and A is ψ - $\mathcal{J}$ -closed, $int(cl^*(int(A))) \subseteq cl^*(A) = A \subseteq U$. Therefore A is β - $\mathcal{J}$ -closed set in X . $\square$

But the converse of the above theorem need not be true as shown in the following examples.

Example 3.5. (i) Let $X = \{a, b, c, d\}$, $\tau = \{X, \varphi, \{a\}, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{J} = \{\varphi, \{c\}, \{d\}, \{c, d\}\}$. Then $\mathcal{J}_g$ -closed sets are $\{X, \varphi, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$. and ψ - $\mathcal{J}$ -closed sets are $\{X, \varphi, \{c\}, \{d\}, \{c, d\}, \{b, c, d\}\}$. It is clear that $\{a, d\}$ is $\mathcal{J}_g$ closed set but it is not ψ - $\mathcal{J}$ -closed.

(ii) Let $X = \{a, b, c, d\}$, $\tau = \{X, \varphi, \{a\}, \{c\}, \{a, c, d\}\}$ and $\mathcal{J} = \{\varphi, \{a\}\}$. Then $\alpha\mathcal{J}_g$ -closed sets are $\{X, \varphi, \{b\}, \{a, b\}, \{b, c\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}$ and ψ - $\mathcal{J}$ -closed sets are $\{X, \varphi, \{b\}, \{b, c, d\}, \{b, d\}, \{a, b, d\}\}$. It is clear that $\{a, b\}$ is $\alpha\mathcal{J}_g$ closed set but it is not ψ - $\mathcal{J}$ -closed.

(iii) Let $X = \{a, b, c, d\}$, $\tau = \{X, \varphi, \{a\}, \{c\}, \{a, c, d\}\}$ and $\mathcal{J} = \{\varphi, \{a\}\}$. Then semi-closed sets are $\{X, \varphi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{b, c, d\}\}$ and ψ - $\mathcal{J}$ -closed sets are $\{X, \varphi, \{b\}, \{b, c, d\}, \{b, d\}, \{a, b, d\}\}$. It is clear that $\{a, b\}$ is semi-closed set but it is not ψ - $\mathcal{J}$ closed.

(iv) Let $X = \{a, b, c, d\}$, $\tau = \{X, \varphi, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, b, d\}\}$ and $\mathcal{J} = \{\varphi, \{c\}, \{d\}, \{c, d\}\}$. Then semi- $\mathcal{J}$ -closed sets are $\{X, \varphi, \{b\}, \{c\}, \{d\}, \{b, c\}, \{c, d\}, \{b, d\}, \{b, c, d\}\}$ and ψ - $\mathcal{J}$ -closed sets are

- $\{X, \varphi, \{c\}, \{d\}, \{c, d\}, \{b, c, d\}\}$. It is clear that $\{b, d\}$ is semi- $\mathcal{J}$ -closed set but it is not ψ - $\mathcal{J}$ -closed.
- (v) Let $X = \{a, b, c, d\}$, $\tau = \{X, \varphi, \{a\}, \{b, c\}\}$ and $\mathcal{J} = \{\varphi, \{c\}\}$. Then $\mathcal{J}_g$ -closed sets are $\{X, \varphi, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$ and ψ - $\mathcal{J}$ -closed sets are $\{X, \varphi, \{d\}, \{a, d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$. It is clear that $\{b, d\}$ is $\mathcal{J}_g$ -closed but it is not ψ - $\mathcal{J}$ -closed.
- (vi) Let $X = \{a, b, c, d\}$, $\tau = \{X, \varphi, \{a\}, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{J} = \{\varphi, \{c\}, \{d\}, \{c, d\}\}$. Then ψ -closed sets are $\{X, \varphi, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}\}$ and ψ - $\mathcal{J}$ -closed sets are $\{X, \varphi, \{c\}, \{d\}, \{c, d\}, \{b, c, d\}\}$. It is clear that $\{a, d\}$ is $\mathcal{J}_g$ closed set but it is not ψ - $\mathcal{J}$ -closed.
- (vii) Let $X = \{a, b, c, d\}$, $\tau = \{X, \varphi, \{a\}, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{J} = \{\varphi, \{c\}, \{d\}, \{c, d\}\}$. Then β - $\mathcal{J}$ -closed sets are $\{X, \varphi, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}\}$. and ψ - $\mathcal{J}$ -closed sets are $\{X, \varphi, \{c\}, \{d\}, \{c, d\}, \{b, c, d\}\}$. It is clear that $\{a, d\}$ is β - $\mathcal{J}$ closed set but it is not ψ - $\mathcal{J}$ -closed.

Theorem 3.6. (i) Every pre-closed set and ψ - $\mathcal{J}$ -closed set in (X, τ, I) are independent to each other

(ii) Every $\mathcal{J}$ -closed set and ψ - $\mathcal{J}$ -closed set in (X, τ, I) are independent to each other

Proof. Follows from the following examples. $\square$

- Example 3.7.** (i) Let $X = \{a, b, c, d\}$, $\tau = \{X, \varphi, \{a\}, \{c\}, \{a, c, d\}\}$ and $\mathcal{J} = \{\varphi, \{b\}\}$. Here pre- $\mathcal{J}$ -closed sets are $\{X, \varphi, \{b\}, \{b, d\}, \{b, c, d\}\}$ and ψ - $\mathcal{J}$ -closed sets are $\{X, \varphi, \{d\}, \{a, d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$. It is clear that $\{b\}$ is pre-closed but it is not ψ - $\mathcal{J}$ -closed and also $\{a, c, d\}$ is ψ - $\mathcal{J}$ -closed but not pre-closed.
- (ii) Let $X = \{a, b, c, d\}$,
 $\tau = \{X, \varphi, \{a\}, \{b\}, \{c, d\}, \{a, b\}, \{a, c, d\}, \{a, c, d\}, \{b, c, d\}\}$ and $\mathcal{J} = \{\varphi, \{c\}\}$.
 Here $\mathcal{J}$ -closed sets are
 $\{X, \varphi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$
 and ψ - $\mathcal{J}$ -closed sets are
 $\{X, \varphi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\},$
 $\{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. It is clear that $\{d\}$ is $\mathcal{J}$ -closed but it is not ψ - $\mathcal{J}$ -closed
 and also $\{a, c, d\}$ is ψ - $\mathcal{J}$ -closed but not $\mathcal{J}$ -closed.

Remark 3.8. The union of two ψ - $\mathcal{J}$ -closed sets need not be a ψ - $\mathcal{J}$ -closed set.

Example 3.9. Consider the ideal topological space (X, τ, I) , where Let $X = \{a, b, c, d\}$, $\tau = \{X, \varphi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ and $\mathcal{J} = \{\varphi, \{a\}\}$. In this ideal space the sets $\{a\}$ and $\{b, c\}$ are ψ - $\mathcal{J}$ -closed sets but their union $\{a, b, c\}$ is not a ψ - $\mathcal{J}$ -closed set.

Remark 3.10. The intersection of two ψ - $\mathcal{J}$ -closed sets need not be a ψ - $\mathcal{J}$ -closed set.

Example 3.11. Consider the ideal topological space (X, τ, I) , where Let $X = \{a, b, c\}$, $\tau = \{X, \varphi, \{a\}, \{a, b\}\}$ and $\mathcal{J} = \{\varphi, \{a\}\}$. In this ideal space the sets $\{a, b\}$ and $\{b, c\}$ are ψ - $\mathcal{J}$ -closed sets but their intersection $\{b\}$ is not a ψ - $\mathcal{J}$ -closed set.

Remark 3.12. In the following diagram we denote by arrows the implications between ψ - $\mathcal{J}$ -closed sets and other sets.

4. Weakly ψ - $\mathcal{J}$ -closed set

Definition 4.1. A subset A of an ideal topological space (X, τ, I) is said to be weakly ψ - $\mathcal{J}$ -closed if $cl^*(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is ψ open in (X, τ, I) .

Theorem 4.2. Every closed set is weakly ψ - $\mathcal{J}$ -closed but not conversely.

Proof. Let $A \subseteq U$ and U is ψ -open in (X, τ, I) . Since $cl^*(int(A)) \subseteq cl(A)$ and A is closed, $cl^*(int(A)) \subseteq cl(A) = A \subseteq U$. Therefore A is weakly ψ - $\mathcal{J}$ -closed set in (X, τ, I) . $\square$

Example 4.3. Let $X = \{a, b, c, d\}$, $\tau = \{X, \varphi, \{a\}, \{b\}, \{b, c\}, \{a, b, c\}\}$ and $\mathcal{J} = \{\varphi, \{a\}\}$. Here closed sets are $\{X, \varphi, \{b, c, d\}, \{a, c, d\}, \{c, d\}, \{a, d\}, \{d\}\}$ and weakly ψ - $\mathcal{J}$ -closed sets are $\{X, \varphi, \{b, c, d\}, \{a, c, d\}, \{c, d\}, \{a, d\}, \{a\}, \{d\}\}$. It is clear that $\{a\}$ is weakly ψ - $\mathcal{J}$ -closed but it is not closed.

Theorem 4.4. Every pre- $\mathcal{J}$ -closed set is weakly ψ - $\mathcal{J}$ -closed but not conversely

Proof. Let $A \subseteq U$ and U is ψ -open in (X, τ, I) . By hypothesis $cl^*(int(A)) \subseteq U$ and A is weakly ψ - $\mathcal{J}$ -closed in (X, τ, I) . $\square$

Example 4.5. Let $X = \{a, b, c, d\}$, $\tau = \{X, \varphi, \{a\}, \{b, c\}\}$ and $\mathcal{J} = \{\varphi, \{c\}\}$. Here pre- $\mathcal{J}$ -closed sets are $\{X, \varphi, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$ and weakly ψ - $\mathcal{J}$ -closed sets are $\{X, \varphi, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$. It is clear that $\{b\}$ is weakly ψ - $\mathcal{J}$ -closed but it is not pre- $\mathcal{J}$ -closed.

Theorem 4.6. Every α - $\mathcal{J}$ -closed set is weakly ψ - $\mathcal{J}$ -Closed but not conversely.

Proof. Let $A \subseteq U$ and U is ψ -open in (X, τ, I) . By hypothesis $cl^*(int(A)) \subseteq cl^*(int(cl^*(A)))$ and A is weakly ψ - $\mathcal{J}$ -closed, $cl^*(int(A)) \subseteq cl^*(int(cl^*(A))) = A \subseteq U$. Therefore A is weakly ψ - $\mathcal{J}$ -closed set in (X, τ, I) . $\square$

Example 4.7. Let $X = \{a, b, c, d\}$, $\tau = \{X, \varphi, \{a\}, \{b, c\}\}$ and $\mathcal{J} = \{\varphi, \{c\}\}$. Here α - $\mathcal{J}$ -closed sets are $\{X, \varphi, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$ and weakly ψ - $\mathcal{J}$ -closed sets are $\{X, \varphi, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$. It is clear that $\{b\}$ is weakly ψ - $\mathcal{J}$ -closed but it is not α - $\mathcal{J}$ -closed.

Theorem 4.8. Every weakly ψ - $\mathcal{J}$ -closed set is $\mathcal{J}$ -closed set but not conversely

Proof. Let $A \subseteq U$ and U is ψ -open in (X, τ, I) . Since $cl(A^*) \subseteq cl^*(int(A))$ and A is weakly ψ - $\mathcal{J}$ -closed, $cl(A^*) \subseteq cl^*(int(A)) = A \subseteq U$. Therefore A is weakly ψ -closed set in (X, τ, I) . $\square$

Example 4.9. Let $X = \{a, b, c, d\}$, $\tau = \{X, \varphi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ and $\mathcal{J} = \{\varphi, \{a\}\}$. Here $\mathcal{J}$ -closed sets are $\{X, \varphi, \{a\}, \{d\}, \{a, d\}, \{b, c, d\}, \{b, c\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$ and weakly ψ - $\mathcal{J}$ -closed sets are $\{X, \varphi, \{a\}, \{d\}, \{a, d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$. It is clear that $\{b, c\}$ is weakly ψ - $\mathcal{J}$ -closed but it is not $\mathcal{J}$ -closed.

Theorem 4.10. Every b -closed set is weakly ψ - $\mathcal{J}$ -closed set but not conversely

Proof. Let $A \subseteq U$ and U is ψ -open in (X, τ, I) . Since $int(cl(int(A))) \subseteq cl^*(int(A))$ and A is weakly ψ - $\mathcal{J}$ -closed, $int(cl(int(A))) \subseteq cl^*(int(A)) = A \subseteq U$. Therefore A is b -closed set in X . $\square$

Example 4.11. Let $X = \{a, b, c, d\}$, $\tau = \{X, \varphi, \{a\}, \{b, c\}\}$ and $\mathcal{I} = \{\varphi, \{a\}\}$. Here b -closed sets are $\{X, \varphi, \{a\}, \{b, c\}\}$ and weakly ψ - $\mathcal{I}$ -closed sets are $\{X, \varphi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. It is clear that $\{b\}$ is weakly ψ - $\mathcal{I}$ -closed but it is not $\mathcal{I}$ -closed.

Theorem 4.12. *Every b - $\mathcal{I}$ -closed set is weakly ψ - $\mathcal{I}$ -closed set but not conversely*

Proof. Let $A \subseteq U$ and U is ψ -open in (X, τ, I) . Since $\text{int}(cl^*(\text{int}(A))) \subseteq cl^*(\text{int}(A))$ and A is weakly ψ - $\mathcal{I}$ -closed, $\text{int}(cl^*(\text{int}(A))) \subseteq cl^*(\text{int}(A)) = A \subseteq U$. Therefore A is b - $\mathcal{I}$ -closed set in X . $\square$

Example 4.13. Let $X = \{a, b, c, d\}$, $\tau = \{X, \varphi, \{a\}, \{b, c\}\}$ and $\mathcal{I} = \{\varphi, \{a\}\}$. Here b - $\mathcal{I}$ -closed sets are $\{X, \varphi, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$ and weakly ψ - $\mathcal{I}$ -closed sets are $\{X, \varphi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. It is clear that $\{b\}$ is weakly ψ - $\mathcal{I}$ -closed but it is not b - $\mathcal{I}$ -closed.

Theorem 4.14. *Every α -closed set is weakly ψ - $\mathcal{I}$ -closed set but not conversely*

Proof. Let $A \subseteq U$ and U is ψ -open in (X, τ, I) . Since $cl^*(\text{int}(A)) \subseteq cl(\text{int}(cl(A)))$ and A is α -closed, $cl^*(\text{int}(A)) \subseteq cl(\text{int}(cl(A))) = A \subseteq U$. Therefore A is weakly ψ -closed set in (X, τ, I) . $\square$

Example 4.15. Let $X = \{a, b, c, d\}$, $\tau = \{X, \varphi, \{c\}\}$ and $\mathcal{I} = \{\varphi\}$. Here α -closed sets are $\{X, \varphi, \{a\}, \{b\}, \{a, b\}\}$ and weakly ψ - $\mathcal{I}$ -closed sets are $\{X, \varphi, \{a, b\}\}$. It is clear that $\{b\}$ is weakly ψ - $\mathcal{I}$ -closed but it is not $\mathcal{I}$ -closed.

Remark 4.16. In the following diagram we denote by arrows the implications between weakly ψ - $\mathcal{I}$ -closed sets and other sets.

5. Contra ψ - $\mathcal{I}$ -continuous mapping

Definition 5.1. A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is called contra ψ - $\mathcal{I}$ -continuous if for every $V \in \sigma$, $f^{-1}(V)$ is ψ - $\mathcal{I}$ -open in (X, τ, I) for every closed set V of Y .

Theorem 5.2. *Let $(X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ be any mapping. Then the following statements holds*

- (i) *Every contra α - $\mathcal{I}$ -continuous function is contra ψ - $\mathcal{I}$ -continuous.*
- (ii) *Every contra α -continuous function is contra ψ - $\mathcal{I}$ -continuous.*
- (iii) *Every contra-continuous function is contra ψ - $\mathcal{I}$ -continuous.*
- (iv) *Every contra semi- $\mathcal{I}$ -continuous function is contra ψ - $\mathcal{I}$ -continuous*
- (v) *Every contra semi-continuous function is contra ψ - $\mathcal{I}$ -continuous.*
- (vi) *Every contra strongly α -continuous function is contra ψ - $\mathcal{I}$ -continuous.*
- (vii) *Every contra β - $\mathcal{I}$ -continuous function is contra ψ - $\mathcal{I}$ -continuous.*
- (viii) *Every contra β -continuous function is contra ψ - $\mathcal{I}$ -continuous.*

Proof. (i) Suppose that $x \in X$ and V is any closed set of Y containing $f(x)$. Since f is α - $\mathcal{I}$ continuous and $cl(V)$ is closed in Y , $f^{-1}(V)$ is α - $\mathcal{I}$ -open and $f^{-1}(cl(V))$ is α - $\mathcal{I}$ closed. Now put $U = f^{-1}(V)$. Then we have $U \in \psi\mathcal{I}O(x, X)$. This shows that f is contra ψ - $\mathcal{I}$ -continuous functions.

- (ii) Suppose that $x \in X$ and V is any closed set of Y containing $f(x)$. Since f is contra α -continuous and let A be a α -open set. Then $A = U \cap V$, where $U \in \tau$. Then $\text{int}(\text{cl}(\text{int}(V))) \supset A$. This shows that f is contra- ψ - $\mathcal{J}$ continuous function.
- (iii) Suppose that $x \in X$ and V is any closed set of Y containing $f(x)$. Since f is contra-continuous. $U \in \psi\mathcal{J}O(x, X)$. Therefore f is contra- ψ - $\mathcal{J}$ -continuous function.
- (iv) Suppose that $x \in X$ and V is any closed set of Y containing $f(x)$. Since f is contra semi- $\mathcal{J}$ continuous and every ψ - $\mathcal{J}$ set is semi- $\mathcal{J}$ -open. Therefore f is contra ψ - $\mathcal{J}$ -continuous function.
- (v) Suppose that $x \in X$ and V is any closed set of Y containing $f(x)$. Since f is contra semi-continuous and every ψ - $\mathcal{J}$ -open set is semi- $\mathcal{J}$ -open. Therefore f is contra ψ - $\mathcal{J}$ -continuous function.
- (vi) Suppose that $x \in X$ and V is any closed set of Y containing $f(x)$. Since f is contra strongly α - $\mathcal{J}$ -continuous and every ψ - $\mathcal{J}$ -open set is α - $\mathcal{J}$ -open as well as B_I set for every closed set V of Y . Therefore f is contra ψ - $\mathcal{J}$ -continuous function.
- (vii) Suppose that $x \in X$ and V is any closed set of Y containing $f(x)$. Since f is contra β - $\mathcal{J}$ continuous and every β - $\mathcal{J}$ -open set is ψ - $\mathcal{J}$ -open. Therefore f is contra ψ - $\mathcal{J}$ -continuous function.
- (viii) Suppose that $x \in X$ and V is any closed set of Y containing $f(x)$. Since f is contra β -continuous and every ψ - $\mathcal{J}$ set is α - $\mathcal{J}$ -open. Therefore f is contra ψ - $\mathcal{J}$ -continuous function. $\square$

The converse of Theorem 5.2 are need not be true as shown in the following examples.

- Examples.**
- (i). Let $X = \{a, b, c\}$, $\tau_1 = \{X, \varphi, \{a\}, \{a, b\}\}$, $\tau_2 = \{X, \{\varphi\}, \{a\}, \{c\}, \{b, c\}\}$ and $\mathcal{J} = \{\varphi\}, \{a\}$ Define the identity mapping $f: (X, \tau_1, \mathcal{J}) \rightarrow (X, \tau_2)$ is contra α - $\mathcal{J}$ -continuous but not contra ψ - $\mathcal{J}$ -continuous.
 - (ii). Let $X = \{a, b, c\}$, $\tau_1 = \{X, \varphi, \{a\}, \{a, b\}\}$, $\tau_2 = \{X, \{\varphi\}, \{a\}, \{c\}, \{b, c\}\}$ and $\mathcal{J} = \{\varphi\}, \{a\}$ Define the identity mapping $f: (X, \tau_1, \mathcal{J}) \rightarrow (X, \tau_2)$ is contra α -continuous but not contra ψ - $\mathcal{J}$ -continuous.
 - (iii). Let $X = \{a, b, c\}$, $\tau_1 = \{X, \varphi, \{a\}, \{a, b\}\}$, $\tau_2 = \{X, \{\varphi\}, \{a\}, \{c\}, \{b, c\}\}$ and $\mathcal{J} = \{\varphi\}, \{a\}$ Define the identity mapping $f: (X, \tau_1, \mathcal{J}) \rightarrow (X, \tau_2)$ is contra-continuous but not contra ψ - $\mathcal{J}$ -continuous.
 - (iv). Let $X = \{a, b, c\}$, $\tau_1 = \{X, \varphi, \{a\}, \{a, b\}\}$, $\tau_2 = \{X, \{\varphi\}, \{a\}, \{c\}, \{b, c\}\}$ and $\mathcal{J} = \{\varphi\}, \{a\}$ Define the identity mapping $f: (X, \tau_1, \mathcal{J}) \rightarrow (X, \tau_2)$ is contra semi- $\mathcal{J}$ -continuous but not contra ψ - $\mathcal{J}$ -continuous.
 - (v). Let $X = \{a, b, c\}$, $\tau_1 = \{X, \varphi, \{a\}, \{a, b\}\}$, $\tau_2 = \{X, \{\varphi\}, \{a\}, \{c\}, \{b, c\}\}$ and $\mathcal{J} = \{\varphi\}, \{a\}$ Define the identity mapping $f: (X, \tau_1, \mathcal{J}) \rightarrow (X, \tau_2)$ is contra semi-continuous but not contra ψ - $\mathcal{J}$ -continuous.
 - (vi). Let $X = \{a, b, c\}$, $\tau_1 = \{X, \varphi, \{a\}, \{a, b\}\}$, $\tau_2 = \{X, \{\varphi\}, \{a\}, \{c\}, \{b, c\}\}$ and $\mathcal{J} = \{\varphi\}, \{a\}$ Define the identity mapping $f: (X, \tau_1, \mathcal{J}) \rightarrow (X, \tau_2)$ is contra strongly α - $\mathcal{J}$ -continuous but not contra ψ - $\mathcal{J}$ -continuous.
 - (vii). Let $X = \{a, b, c\}$, $\tau_1 = \{X, \varphi, \{a\}, \{a, b\}\}$, $\tau_2 = \{X, \{\varphi\}, \{a\}, \{c\}, \{b, c\}\}$ and $\mathcal{J} = \{\varphi\}, \{a\}$. Define the identity mapping $f: (X, \tau_1, \mathcal{J}) \rightarrow (X, \tau_2)$ is contra β - $\mathcal{J}$ -continuous but not contra ψ - $\mathcal{J}$ -continuous.

- (viii). Let $X = \{a, b, c\}$, $\tau_1 = \{X, \varphi, \{a\}, \{a, b\}\}$, $\tau_2 = \{X, \{\varphi\}, \{a\}, \{c\}, \{b, c\}\}$ and $\mathcal{J} = \{\varphi\}, \{a\}$. Define the identity mapping $f: (X, \tau_1, \mathcal{J}) \rightarrow (X, \tau_2)$ is contra β -continuous but not contra ψ - $\mathcal{J}$ -continuous.

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