

SOME PROPERTIES OF CERTAIN CLASS OF MEROMORPHIC FUNCTIONS DEFINED BY LINEAR OPERATOR

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Abstract. Making use of certain linear operator, we define a new subclass of meromorphically uniformly convex functions with positive coefficients and obtain coefficient estimates, growth and distortion theorems, extreme points, closure theorems and radius of starlikeness and convexity for the new subclass $\Sigma_w^*(\alpha, \beta, k)$.

1. Introduction

Let A denote the class of all functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

in the open unit disc $U = \{z \in \mathcal{C} : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions and satisfy the following usual normalization condition $f(0) = f'(0) - 1 = 0$. We denote by S the subclass of A consisting of functions f which are all univalent in U . A function $f \in A$ is a starlike function by the order $\alpha, 0 \leq \alpha < 1$, if it satisfy

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, z \in U. \quad (1.2)$$

We denote this class with $S^*(\alpha)$.

A function $f \in A$ is a convex function by the order $\alpha, 0 \leq \alpha < 1$, if it satisfy

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, z \in U. \quad (1.3)$$

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We denote this class with $K(\alpha)$.

A function $f \in A$ is said to be in the class of uniformly convex functions of order γ and type β , denoted by $UCV(\beta, \gamma)$, if

$$Re \left\{ 1 + \frac{zf''(z)}{f'(z)} - \gamma \right\} > \beta \left| \frac{zf''(z)}{f'(z)} \right|, \quad (1.4)$$

where $\beta \geq 0, \gamma \in [-1, 1)$ and $\beta + \gamma \geq 0$ and it is said to be in the class corresponding class denoted by $SP(\beta, \gamma)$, if

$$Re \left\{ \frac{zf'(z)}{f(z)} - \gamma \right\} > \beta \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad (1.5)$$

where $\beta \geq 0, \gamma \in [-1, 1)$ and $\beta + \gamma \geq 0$. Indeed it follows from (1.4) and (1.5) that

$$f \in UCV(\gamma, \beta) \Leftrightarrow zf' \in SP(\gamma, \beta). \quad (1.6)$$

For $\beta = 0$, we get respectively, the classes $K(\gamma)$ and $S^*(\gamma)$. The function of the class $UCV(1, 0) \equiv UCV$ are called uniformly convex functions, were introduced and studied by Goodman with geometric interpretation in [7] and uniformly starlike functions in [8]. The class $SP(1, 0) \equiv SP$ is defined by Ronning [17]. The classes $UCV(1, \gamma) \equiv UCV(\gamma)$ and $SP(1, \gamma) \equiv SP(\gamma)$ are investigated by Ronning in [16]. For $\gamma = 0$, the classes $UCV(\beta, 0) \equiv \beta - UCV$ and $SP(\beta, 0) \equiv \beta - SP$ are defined respectively, by Kanas and Wisniowska in [12, 13].

Let w be fixed point in U and $A(w) = \{f \in H(U) : f(w) = f'(w) - 1 = 0\}$. In [11] Kanas and Ronning introduced the following classes

$$\begin{aligned} S_w &= \{f \in A(w) : f \text{ is univalent in } U\} \\ ST_w &= \left\{ f \in A(w) : Re \left(\frac{(z-w)f'(z)}{f(z)} \right) > 0, z \in U \right\} \\ CV_w &= \left\{ f \in A(w) : 1 + Re \left(\frac{(z-w)f''(z)}{f'(z)} \right) > 0, z \in U \right\}. \end{aligned} \quad (1.7)$$

Later Acu and Owa [1] studied the classes extensively.

The class ST_w is defined by geometric property that the image of any circular arc centered at w is starlike with respect to $f(w)$ and the corresponding class CV_w is defined by the property that the image of any circular arc centered at w is convex. We observed that the definitions are somewhat similar to the ones introduced by Goodman in [7, 8] for uniformly starlike and convex function except that, in this case, the point w is fixed.

Let Σ denote the class of the functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (1.8)$$

which are regular in domain $U^* = \{z \in \mathcal{C} : 0 < |z| < 1\} = U \setminus \{0\}$ with a simple pole at the origin with residue 1 there. Let Σ_s and $\Sigma^*(\alpha)$ and $\Sigma_k(\alpha), 0 \leq \alpha < 1$, denote the subclasses of Σ that are univalent, meromorphically starlike of order α and meromorphically convex of order α respectively. Analytically f of the form (1.8) is in $\Sigma^*(\alpha)$ if and only if

$$Re \left\{ -\frac{zf'(z)}{f(z)} \right\} > \alpha, \quad z \in U. \quad (1.9)$$

Similarly, $f \in \Sigma_k(\alpha)$ if and only if $f(z)$ is of the form (1.8) and satisfies

$$\operatorname{Re} \left\{ - \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \alpha, \quad z \in U. \quad (1.10)$$

It being understood that if $\alpha = 1$ then $f(z) = \frac{1}{z}$ is the only function which is $\Sigma^*(1)$ and $\Sigma_k(1)$. The classes $\Sigma^*(\alpha)$ and $\Sigma_k(\alpha)$ have been extensively studied by Pommerenke [15], Clunie [3], Royster [18] and others.

Let Σ_w denote the subclass of $A(w)$ consisting of the function of the form

$$f(z) = \frac{1}{z-w} + \sum_{n=1}^{\infty} a_n(z-w)^n. \quad (1.11)$$

The functions f in Σ_w are said to be starlike functions of order β if and only if

$$\operatorname{Re} \left\{ - \frac{(z-w)f'(z)}{f(z)} \right\} > \beta, \quad (z-w) \in U. \quad (1.12)$$

for some $\beta (0 \leq \beta < 1)$. We denote by $ST_w^*(\beta)$ the class of all starlike function of order β .

Similarly, a function f in Σ_w is said to be convex of order β if and only if

$$\operatorname{Re} \left\{ -1 - \frac{(z-w)f''(z)}{f'(z)} \right\} > \beta, \quad (z-w) \in U. \quad (1.13)$$

for some $\beta (0 \leq \beta < 1)$. We denote by $C_w(\beta)$ the class of all convex functions of order β .

Let us define the function $\bar{\varphi}(a, c; z)$ by

$$\bar{\varphi}(a, c; z) = \frac{1}{z-w} + \sum_{n=0}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_n(z-w)^n \quad (1.14)$$

for $c \neq 0, -1, -2, \dots$ and $a \in \mathbb{C}/\{0\}$, where $(\lambda)_n = \lambda(\lambda+1)_{n-1}$ is the Pochhammer symbol. We note that

$$\bar{\varphi}(a, c; z) = \frac{1}{z-w} {}_2F_1(1, a, c; z) \quad (1.15)$$

$$\text{where } {}_2F_1(b, a, c; z) = \sum_{n=0}^{\infty} \frac{(b)_n (a)_n}{(c)_n} \frac{(z-w)^n}{n!} \quad (1.16)$$

is the well-known Gaussian hypergeometric function. Corresponding to the function $\bar{\varphi}(a, c; z)$ and using the Hadamard product for $f \in \Sigma$, we define a new linear operator $L_w^*(a, c)$ on Σ by

$$L_w^*(a, c)f(z) = \bar{\varphi}(a, c; z) * f(z) = \frac{1}{z-w} + \sum_{n=1}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_n(z-w)^n. \quad (1.17)$$

The meromorphic functions with the generalized hypergeometric functions were considered recently by Dziok and Srivastava [4, 5], Cho and Kim [2].

For a function $f \in L_w^*(a, c)f(z)$, we define

$$\begin{aligned} I^0(L_w^*(a, c)f(z)) &= L_w^*(a, c)f(z), \text{ and for } k = 1, 2, \dots \\ I^k(L_w^*(a, c)f(z)) &= z(I^{k-1}(L_w^*(a, c)f(z)))' + \frac{2}{z-w} \\ &= \frac{1}{z-w} + \sum_{n=1}^{\infty} n^k \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_n (z-w)^n. \end{aligned} \quad (1.18)$$

We note that $I^k(L_w^*(a, c)f(z))$ studied by Ghanim and Darus [9, 10].

Motivated by the works of Madhavi et al. [14] and Venkateswarlu et al. [19, 20], we define the following new subclass $\Sigma_w^*(\alpha, \beta, k)$ of meromorphically uniformly convex functions with positive coefficients in Σ_w by making use of linear operator.

Definition 1.1. For $\alpha (-1 \leq \alpha < 1)$ and $\beta (\beta \geq 1)$, we let $\Sigma_w^*(\alpha, \beta, k)$ be the subclass of Σ_w consisting of the form (1.11) and satisfying the analytic criterion

$$Re \left\{ \frac{I^{k+1}L_w^*(a, c)f(z)}{I^kL_w^*(a, c)f(z)} - \alpha \right\} > \beta \left| \frac{I^{k+1}L_w^*(a, c)f(z)}{I^kL_w^*(a, c)f(z)} - 1 \right|, \quad z \in U \quad (1.19)$$

where $L_w^*(a, c)f(z)$ is given by (1.17).

The main objective of this paper is to study some usual properties of the geometric function theory such as coefficient bounds, growth and distortion properties, radius of starlikeness and convexity. Furthermore, we obtain convex linear combinations for the class $\Sigma_w^*(\alpha, \beta, k)$.

2. Coefficient Estimates

In this section we obtain the coefficient bounds of functions f for the class $\Sigma_w^*(\alpha, \beta, k)$.

Theorem 2.1. A function f of the form (1.11) is in $\Sigma_w^*(\alpha, \beta, k)$ if

$$\sum_{n=1}^{\infty} n^k [n(1 + \beta) - (\beta + \alpha)] \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| |a_n| \leq 1 - \alpha, \quad -1 \leq \alpha < 1 \text{ and } \beta \geq 1. \quad (2.1)$$

Proof. It suffices to show that

$$\beta \left| \frac{I^{k+1}L_w^*(a, c)f(z)}{I^kL_w^*(a, c)f(z)} - 1 \right| - Re \left\{ \frac{I^{k+1}L_w^*(a, c)f(z)}{I^kL_w^*(a, c)f(z)} - 1 \right\} \leq 1 - \alpha.$$

We have

$$\begin{aligned} &\beta \left| \frac{I^{k+1}L_w^*(a, c)f(z)}{I^kL_w^*(a, c)f(z)} - 1 \right| - Re \left\{ \frac{I^{k+1}L_w^*(a, c)f(z)}{I^kL_w^*(a, c)f(z)} - 1 \right\} \\ &\leq (1 + \beta) \left| \frac{I^{k+1}L_w^*(a, c)f(z)}{I^kL_w^*(a, c)f(z)} - 1 \right| \leq \frac{(1 + \beta) \sum_{n=1}^{\infty} n^k (n-1) \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| |a_n| |z-w|^n}{\frac{1}{|z-w|} - \sum_{n=1}^{\infty} n^k \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| |a_n| |z-w|^n}. \end{aligned}$$

Letting $(z - w) \rightarrow 1$ along the real axis, we obtain

$$\leq \frac{(1 + \beta) \sum_{n=1}^{\infty} n^k (n - 1) \frac{|(a)_{n+1}|}{|(c)_{n+1}|} |a_n|}{1 - \sum_{n=1}^{\infty} n^k \frac{|(a)_{n+1}|}{|(c)_{n+1}|} |a_n|}.$$

This last expression is bounded above by $(1 - \alpha)$ if

$$\sum_{n=1}^{\infty} n^k [n(1 + \beta) - (\beta + \alpha)] \frac{|(a)_{n+1}|}{|(c)_{n+1}|} |a_n| \leq 1 - \alpha.$$

Hence the theorem. $\square$

Our assertion in Theorem 2.1 is sharp for functions of the form

$$f_n(z) = \frac{1}{z - w} + \sum_{n=1}^{\infty} \frac{(1 - \alpha) |(c)_{n+1}|}{n^k [n(1 + \beta) - (\beta + \alpha)] |(a)_{n+1}|} (z - w)^n \quad (n \geq 1; k \in N_0) \quad (2.2)$$

Corollary 2.2. *Let the functions f be defined by (1.11) and $f \in A$. Then*

$$a_n \leq \frac{(1 - \alpha) |(c)_{n+1}|}{n^k [n(1 + \beta) - (\beta + \alpha)] |(a)_{n+1}|} \quad (n \geq 1; k \in N_0). \quad (2.3)$$

Theorem 2.3. *Let f define by (1.11) and $g(z) = \frac{1}{z - w} + \sum_{n=1}^{\infty} b_n (z - w)^n$ be in the class $\sum_w^{\infty}(\alpha, \beta, k)$. Then the function h defined by*

$$h(z) = (1 - \lambda)f(z) + \lambda g(z) = \frac{1}{z - w} + \sum_{n=1}^{\infty} q_n (z - w)^n \quad (2.4)$$

where $q_n = (1 - \lambda)a_n + \lambda b_n$, $0 \leq \lambda < 1$ is also in the class $\Sigma_w^*(\alpha, \beta, k)$.

3. Growth and Distortion Theorem

In this, section we study growth and distortion properties for the class

Theorem 3.1. *Let the function f defined by (1.11) be in the class $\Sigma_w^*(\alpha, \beta, k)$. Then*

$$\frac{1}{r} - r \leq |f(z)| \leq \frac{1}{r} + r. \quad (3.1)$$

$$\text{Equality holds for the function } f(z) = \frac{1}{(z - w)} + (z - w) \quad (3.2)$$

Proof. Since $f \in \Sigma_w^*(\alpha, \beta, k)$, by Theorem 2.1,

$$\sum_{n=1}^{\infty} n^k [n(1 + \beta) - (\beta + \alpha)] \frac{|(a)_{n+1}|}{|(c)_{n+1}|} |a_n| \leq 1 - \alpha.$$

$$\begin{aligned}
\text{Now } (1 - \alpha) \sum_{n=1}^{\infty} \frac{|(a)_{n+1}|}{|(c)_{n+1}|} a_n &= \sum_{n=1}^{\infty} (1 - \alpha) \frac{|(a)_{n+1}|}{|(c)_{n+1}|} a_n \\
&\leq \sum_{n=1}^{\infty} n^k [n(1 + \beta) - (\beta + a)] \frac{|(a)_{n+1}|}{|(c)_{n+1}|} |a_n| \\
&\leq 1 - a \\
&\Rightarrow \sum_{n=1}^{\infty} \frac{|(a)_{n+1}|}{|(c)_{n+1}|} a_n \leq 1.
\end{aligned}$$

$$\begin{aligned}
\text{Since } f(z) &= \frac{1}{z - w} + \sum_{n=1}^{\infty} \frac{|(a)_{n+1}|}{|(c)_{n+1}|} a_n (z - w)^n \\
\Rightarrow |f(z)| &= \left| \frac{1}{z - w} + \sum_{n=1}^{\infty} \frac{|(a)_{n+1}|}{|(c)_{n+1}|} a_n (z - w)^n \right| \\
&\leq \frac{1}{|z - w|} + \sum_{n=1}^{\infty} \frac{|(a)_{n+1}|}{|(c)_{n+1}|} a_n |z - w|^n \\
&\leq \frac{1}{r} + r \sum_{n=1}^{\infty} \frac{|(a)_{n+1}|}{|(c)_{n+1}|} a_n \leq \frac{1}{r} + r \\
\text{and } |f(z)| &= \left| \frac{1}{z - w} + \sum_{n=1}^{\infty} \frac{|(a)_{n+1}|}{|(c)_{n+1}|} a_n (z - w)^n \right| \\
&\geq \frac{1}{|z - w|} - \sum_{n=1}^{\infty} \frac{|(a)_{n+1}|}{|(c)_{n+1}|} a_n |z - w|^n \\
&\geq \frac{1}{r} - r \sum_{n=1}^{\infty} \frac{|(a)_{n+1}|}{|(c)_{n+1}|} a_n \geq \frac{1}{r} - r
\end{aligned}$$

which yields the theorem. $\square$

Theorem 3.2. *Let the function f be defined by (1.11) be in the class $\Sigma_w^*(\alpha, \beta, k)$. Then*

$$\frac{1}{r^2} - 1 \leq |f'(z)| \leq \frac{1}{r^2} + 1. \quad (3.3)$$

Equality holds for the function $f(z) = \frac{1}{z-w} + z - w$.

Proof. We have

$$\begin{aligned}
|f'(z)| &= \left| \frac{-1}{(z - w)^2} + \sum_{n=1}^{\infty} n \frac{(a)_{n+1}}{(c)_{n+1}} a_n (z - w)^{n-1} \right| \\
&\leq \frac{1}{|z|^2} + \sum_{n=1}^{\infty} n \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_n |z - w|^{n-1} \\
&\leq \frac{1}{r^2} + \sum_{n=1}^{\infty} \frac{|(a)_{n+1}|}{|(c)_{n+1}|} n a_n. \quad (3.4)
\end{aligned}$$

Since $f(z) \in \Sigma_w^*(\alpha, \beta, k)$, we have

$$\begin{aligned} (1 - \alpha) \sum_{n=1}^{\infty} \frac{|(a)_{n+1}|}{|(c)_{n+1}|} n a_n &\leq \sum_{n=1}^{\infty} n^{k-1} [n(1 + \beta) - (\beta + \alpha)] \frac{|(a)_{n+1}|}{|(c)_{n+1}|} n a_n \\ &\leq 1 - \alpha. \\ \Rightarrow \sum_{n=1}^{\infty} \frac{|(a)_{n+1}|}{|(c)_{n+1}|} n a_n &\leq 1. \end{aligned} \quad (3.5)$$

Substituting (3.5) in (3.4), we get $|f'(z)| \leq \frac{1}{r^2} + 1$.

And $|f'(z)| \geq \frac{1}{r^2} - 1$. This completes the proof. $\square$

4. Radii of Starlikeness and Convexity

The radii of starlikeness and convexity for the class for the class $\Sigma_w^*(\alpha, \beta, k)$ is given by the following theorems.

Theorem 4.1. *If the function f is defined by (1.11) is in the class $\Sigma_w^*(\alpha, \beta, k)$ then f is meromorphically starlike of order δ ($0 \leq \delta < 1$) in $(z - w) < r_1$, where*

$$r_1 = r_1(\alpha, \beta, k) = \inf_{n \geq 1} \left\{ \frac{n^k(1 - \delta)[n(1 + \beta) - (\beta + \alpha)]}{(n + 2 - \delta)(1 - \alpha)} \right\}^{\frac{1}{n+1}}. \quad (4.1)$$

The result is sharp for the function f given by (2.2).

Proof. It suffices to prove that

$$\left| \frac{(z - w)(I^k(L_w^*(a, c)f(z)))'}{I^k(L_w^*(a, c)f(z))} + 1 \right| < 1 - \delta. \quad (4.2)$$

For $(z - w) < r_1$ the left hand side, we have

$$\begin{aligned} \left| \frac{(z - w)(I^k(L_w^*(a, c)f(z)))'}{I^k(L_w^*(a, c)f(z))} + 1 \right| &= \left| \frac{\sum_{n=1}^{\infty} n^k(n + 1) \frac{|(a)_{n+1}|}{|(c)_{n+1}|} a_n |z - w|^n}{\frac{1}{z - w} + \sum_{n=1}^{\infty} n^k \frac{|(a)_{n+1}|}{|(c)_{n+1}|} a_n (z - w)^n} \right| \\ &\leq \frac{\sum_{n=1}^{\infty} n^k(n + 1) \frac{|(a)_{n+1}|}{|(c)_{n+1}|} a_n |z - w|^n}{\left| \frac{1}{z - w} \right| - \sum_{n=1}^{\infty} n^k \frac{|(a)_{n+1}|}{|(c)_{n+1}|} a_n |z - w|^n}. \end{aligned} \quad (4.3)$$

The last expression is less than $1 - \delta$ if

$$\begin{aligned} \sum_{n=1}^{\infty} n^k(n + 1) \frac{|(a)_{n+1}|}{|(c)_{n+1}|} a_n |z - w|^n &\leq (1 - \delta) \left(\frac{1}{|z - w|} - \sum_{n=1}^{\infty} n^k \frac{|(a)_{n+1}|}{|(c)_{n+1}|} a_n |z - w|^n \right) \\ \text{or } \sum_{n=1}^{\infty} n^k \frac{(n + 2 - \delta)}{1 - \delta} \frac{|(a)_{n+1}|}{|(c)_{n+1}|} |z - w|^{n+1} &\leq 1. \end{aligned} \quad (4.4)$$

With the aid of (2.1) and (4.4) is true if

$$\sum_{n=1}^{\infty} n^k \frac{(n + 2 - \delta)}{(1 - \delta)} |z - w|^{n+1} \leq \frac{n^k[n(1 + \beta) - (\beta + \alpha)]}{(1 - \alpha)}, \quad n \geq 1. \quad (4.5)$$

Solving (4.5) for $|z - w|$, we obtain

$$|z - w| < \left\{ \frac{n^k(1 - \delta)[n(1 + \beta) - (\beta + \alpha)]}{(n + 2 - \delta)(1 - \alpha)} \frac{(a)_{n+1}}{(c)_{n+1}} \right\}^{\frac{1}{n+1}}.$$

This completes the proof of Theorem 4.1. $\square$

Theorem 4.2. *If the function f be defined by (1.11) is in the class $\Sigma_w^*(\alpha, \beta, k)$ then $f(z)$ is meromorphically convex of order δ ($0 \leq \delta < 1$) in $(z - w) < r_2$, where*

$$r_2 = r_2(\alpha, \beta, k) = \inf_{n \geq 1} \left\{ \frac{n^k(1 - \delta)[n(1 + \beta) - (\beta + \alpha)]}{(n + 2 - \delta)(1 - \alpha)} \right\}^{\frac{1}{n+1}}. \quad (4.6)$$

The result is sharp for the function f given by (2.2).

Proof. By using the technique employed in the proof of Theorem 4.1, we can show that

$$\left| \frac{(z - w)(I^k(L_w^*(a, c)f(z)))''}{(I^k(L_w^*(a, c)f(z)))'} + 2 \right| \leq (1 - \delta) \quad (4.7)$$

for $|z - w| < r_2$, with the aid of Theorem 2.1. Thus we have the assertion of Theorem 4.2. $\square$

5. Convex Linear Combinations

In this section, we obtain convex linear combinations for the class $\Sigma_w^*(\alpha, \beta, k)$.

Theorem 5.1. *Let $f_0(z) = \frac{1}{z - w}$* (5.1)

$$\text{and } f_0(z) = \frac{1}{z - w} + \frac{(1 - \alpha)|(c)_{n+1}|}{n^k[n(1 + \beta) - (\beta + \alpha)]|(a)_{n+1}|} (z - w)^n, \quad (5.2)$$

$n \geq 1, -1 < \alpha \leq 1, \beta \geq 0$ and $k \geq 0$. Then $f(z) \in \Sigma_w^*(\alpha, \beta, k)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z) \text{ where } \lambda_n \geq 0 \text{ and } \sum_{n=0}^{\infty} \lambda_n = 1. \quad (5.3)$$

Proof. From (5.1), (5.2) and (5.3), it is easily seen that

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z) = \frac{\lambda_0}{z - w} + \sum_{n=1}^{\infty} \frac{\lambda_n(1 - \alpha)|(c)_{n+1}|}{n^k[n(1 + \beta) - (\beta + \alpha)]|(a)_{n+1}|} (z - w)^n$$

Since

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n^k[n(1 + \beta) - (\beta + \alpha)]|(a)_{n+1}|}{(1 - \alpha)|(c)_{n+1}|} \frac{\lambda_n(1 - \alpha)|(c)_{n+1}|}{n^k[n(1 + \beta) - (\beta + \alpha)]|(a)_{n+1}|} &= \sum_{n=1}^{\infty} \lambda_n \\ &= 1 - \lambda_0 \leq 1. \end{aligned}$$

It follows from Theorem 2.1 that $f \in \Sigma_w^*(\alpha, \beta, k)$.

Conversely, let us suppose that $f \in \Sigma_w^*(\alpha, \beta, k)$. Since

$$a_n \leq \frac{(1 - \alpha)|(c)_{n+1}|}{n^k[n(1 + \beta) - (\beta + \alpha)]|(a)_{n+1}|}, \quad n \geq 1, -1 < \alpha \leq 1, \beta \geq 0 \text{ and } k \geq 0.$$

Setting $\lambda_n = \frac{n^k[n(1+\beta) - (\beta+\alpha)]|(a)_{n+1}|}{(1-\alpha)|(c)_{n+1}|}$, $n \geq 1, k \geq 0$ and $\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n$.

It follows that $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$. This completes the proof of the theorem. $\square$

Finally, we prove the following.

Theorem 5.2. *The class $\Sigma_w^*(\alpha, \beta, k)$ is closed under convex linear combinations.*

Proof. Suppose that the function $f_1(z)$ and $f_2(z)$ defined by

$$f_j(z) = \frac{1}{z-w} + \sum_{n=1}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_{n,j} (z-w)^n \quad (j = 1, 2; \quad z-w \in U) \quad (5.4)$$

are in the class $\Sigma_w^*(\alpha, \beta, k)$. Setting

$$f(z) = \mu f_1(z) + (1-\mu) f_2(z), \quad (0 \leq \mu < 1) \quad (5.5)$$

We find from (5.4) that

$$f(z) = \frac{1}{z-w} + \sum_{n=1}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| \{ \mu a_{n,1} + (1-\mu) a_{n,2} \} (z-w)^n, \quad 0 \leq \mu < 1, \quad (z-w) \in U. \quad (5.6)$$

In view of Theorem 2.1, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} n^k [n(1+\beta) - (\beta+\alpha)] \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| \{ \mu a_{n,1} + (1-\mu) a_{n,2} \} \\ &= \mu \sum_{n=0}^{\infty} n^k [n(1+\beta) - (\beta+\alpha)] \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_{n,1} + (1-\mu) \sum_{n=0}^{\infty} n^k [n(1+\beta) - (\beta+\alpha)] \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_{n,2} \\ &\leq \mu(1-\alpha) + (1-\mu)(1-\alpha) = (1-\alpha). \end{aligned}$$

which shows that $f(z) \in \Sigma_w^*(\alpha, \beta, k)$. Hence the Theorem. $\square$

6. Conclusion

This research has introduced a new linear operator related to meromorphic functions and studied some properties were studied. Accordingly, some results related to closure theorems have also been considered, inviting future research for this field of study.

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