

A NOTE ON WEINSTEIN TRANSFORM ON PRODUCTS OF CENTRAL MORREY SPACES

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Abstract. We prove boundedness of convolution operators given by product domain versions of the Weinstein transform acting on products of central weighted Morrey spaces.

1. Introduction

The aim of this note is to obtain boundedness results on some function spaces of convolution operators induced by a product version of the kernel $\mathcal{K}_\alpha$, defined by

$$\mathcal{K}_\alpha(y, s) := \frac{\Gamma((\alpha + n + 1)/2)}{\Gamma((\alpha + 1)/2) \pi^{n/2}} \frac{s^{\alpha+1}}{(|y|^2 + s^2)^{(\alpha+n+1)/2}} \quad (1.1)$$

where $\alpha > -1$ and $(y, s) \in \mathbb{R}_+^{n+1}$.

This kernel is related to the elliptic partial differential equation

$$D_\alpha u := s^{-\alpha} \left(\frac{\partial^2 u}{\partial y_1^2} + \dots + \frac{\partial^2 u}{\partial y_n^2} + \frac{\partial^2 u}{\partial s^2} - \frac{\alpha}{s} \frac{\partial u}{\partial s} \right) = 0 \quad (1.2)$$

with $\alpha > -1$. Solutions to (1.2) are called generalized axially symmetric potentials (cf. [13]). Notice that, for $\alpha = 0$ we recover the Laplace equation.

Several authors have approached the study of boundary value problems related to this kernel, as well as modified versions of it (see, for example, [14], [2], [3], [6]).

As it happens in the case of the Poisson kernel for the upper half-space, it can also be proved that: (i) $\|\mathcal{K}_{\alpha,s}\|_{L^1} = 1$, where $\mathcal{K}_{\alpha,s}(y) := \mathcal{K}_\alpha(y, s)$, (ii) $\mathcal{K}_{\alpha,s} \rightarrow \delta_0$ in $\mathcal{S}'$ as $s \rightarrow 0^+$, and (iii) $\mathcal{K}_\alpha$ is a solution to the equation (1.2) in $\mathbb{R}_+^{n+1}$ (see [14]). In that article, generalizing techniques developed in [3] and [2], the author characterizes a

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class of weighted integrable distributions f convolvable in an appropriate sense with the kernel $\mathcal{K}_\alpha$, solving in this way the Dirichlet problem

$$\begin{cases} D_\alpha u = 0 & \text{in } \mathbb{R}_+^{n+1} \\ u = f & \text{in } \mathbb{R}^n. \end{cases}$$

In the case that we consider in this work, the first step will be to characterize the class of tempered distributions convolvable in an appropriate sense with our product version of $\mathcal{K}_\alpha$, for the domain $\mathbb{R}_+^{n+1} \times \dots \times \mathbb{R}_+^{n+1}$, consisting of m copies of the upper half-space $\mathbb{R}_+^{n+1}$. We will make use of the so called $\mathcal{S}'$ -convolution developed by Y. Hirata and H. Ogata in [7] and, R. Shiraishi in [11], to extend to appropriate pairs of tempered distributions the classical definition of convolution.

The reason for this choice of convolution is the fact that the Fourier transform of $\mathcal{K}_{\alpha,t}$ is given by (see [14])

$$\widehat{\mathcal{K}_{\alpha,t}}(\xi) = \frac{t|\xi|^{\alpha+1}}{\Gamma(\alpha+1)} \int_1^\infty e^{-t|\xi|s} (s^2 - 1)^{\alpha/2} ds,$$

which fails to be smooth at $\xi = 0$.

Once obtained the above referred characterization, we will proceed to show, for certain particular subspaces, some continuity results of the convolution operator induced by this product kernel, on product versions of L^p spaces and central Morrey spaces. All of this will be done in Sections 2 and 3.

We will use standard notation and, as usual, we will be denoting by the same letter C a constant that could be changing line by line.

2. The product kernel

The notion of convolution that we consider is the so called $\mathcal{S}'$ -convolution. We refer the reader to [11], [7], [10], [4] and [12] for more information about the spaces of distributions that we mention here.

Y. Hirata and H. Ogata ([7]) introduced the $\mathcal{S}'$ -convolution to extend the validity of the Fourier exchange formula $\widehat{T * S} = \widehat{T} \widehat{S}$, originally proved by L. Schwartz ([10]) for pairs of distributions in the Cartesian product $\mathcal{O}'_c \times \mathcal{S}'$. Later, R. Shiraishi proved in [11] an equivalent definition of $\mathcal{S}'$ -convolution, which is the one we shall use.

Definition 2.1. ([11]) Given $T, S \in \mathcal{S}'$, it is said that the $\mathcal{S}'$ -convolution of T and S exists if $T(\tilde{S} * \varphi) \in \mathcal{D}'_{L^1}$ for every $\varphi \in \mathcal{S}$.

When the $\mathcal{S}'$ -convolution exists, the map

$$\begin{aligned} \mathcal{S} &\rightarrow \mathbb{C} \\ \varphi &\rightarrow \left(T(\tilde{S} * \varphi), 1 \right)_{\mathcal{D}'_{L^1}, B_c} \end{aligned}$$

is linear and continuous. Thus, it defines a tempered distribution which will be denoted by $T * S$.

Here, $\mathcal{D}'_{L^1}$ is the space of integrable distributions, B_c is the space of C^∞ functions on $\mathbb{R}^n$ with bounded derivatives endowed with the topology of uniform convergence on compact sets, $T(\tilde{S} * \varphi)$ is the multiplicative product of the distribution T with the regularization $\tilde{S} * \varphi$, and $\tilde{S}$ denotes the reflection of S .

R. Shiraishi proved in [11] that $T * S$ exists if and only if $S * T$ exists, and they coincide. It is important to remark the fact that Definition 2.1 coincides with the classical definition in all the cases in which the latter makes sense.

Now, let us consider the product kernel $\overrightarrow{\mathcal{K}}_\alpha$ defined as follows:

$$\overrightarrow{\mathcal{K}}_\alpha(x, t) = \prod_{j=1}^m \mathcal{K}_{\alpha_j}(x_j, t_j) \quad (2.1)$$

where $(x, t) := ((x_1, t_1), \dots, (x_m, t_m)) \in \mathbb{R}_+^{n+1} \times \dots \times \mathbb{R}_+^{n+1}$, the cartesian product of m copies of the upper half-space, and $\alpha = (\alpha_1, \dots, \alpha_m)$ with $\alpha_j > -1$, for $j = 1, \dots, m$.

The kernel given by (2.1) is a generalization of the product kernel studied in [3] and [6]. In both of these papers the authors consider the following spaces of weighted distributions.

Definition 2.2. Let us consider $w_j = \left(1 + |x_j|^2\right)^{1/2}$, $x_j \in \mathbb{R}^n$, $j = 1, \dots, m$, $\mu_1, \dots, \mu_m \in \mathbb{R}$. We shall denote by

$$w_1^{\mu_1} \dots w_m^{\mu_m} \mathcal{D}'_{L^1} := \{T \in \mathcal{S}' : w_1^{-\mu_1} \dots w_m^{-\mu_m} T \in \mathcal{D}'_{L^1}\}$$

endowed with the topology induced by the map

$$\begin{aligned} w_1^{\mu_1} \dots w_m^{\mu_m} \mathcal{D}'_{L^1} &\rightarrow \mathcal{D}'_{L^1} \\ T &\rightarrow w_1^{-\mu_1} \dots w_m^{-\mu_m} T. \end{aligned}$$

We remark the fact that the space $\mathcal{D}'_{L^1}$ considered in the above definition is taken with respect to functions with variables in $\mathbb{R}^{nm}$.

Using the same proof as the one given in [3], we can show the following result.

Proposition 2.3. Given $T \in \mathcal{S}'$, and $\mu_1, \dots, \mu_m \in \mathbb{R}$, the following statements are equivalent:

- a): $T \in w_1^{\mu_1} \dots w_m^{\mu_m} \mathcal{D}'_{L^1}$.
- b): $T = T_0 + \sum |x_{j_1}|^{\mu_{j_1}} \dots |x_{j_k}|^{\mu_{j_k}} T_{j_1 \dots j_k}$, where T_0 is a compactly supported distribution, $T_{j_1 \dots j_k} \in \mathcal{D}'_{L^1}$, and the sum is taken over all the different k -tuples $(j_1, \dots, j_k)$ with $1 \leq j_1 < \dots < j_k \leq m$, $1 \leq k \leq m$.

Now, we are going to characterize the space of optimal distributions $\mathcal{S}'$ -convolvable with the kernel $\overrightarrow{\mathcal{K}}_\alpha(x, t)$ given in (2.1).

Proposition 2.4. For $T \in \mathcal{S}'$, and $\alpha_1, \dots, \alpha_n > -1$, the following statements are equivalent:

- (a): $T \in w_1^{\mu_1} \dots w_m^{\mu_m} \mathcal{D}'_{L^1}$.
- (b): T is $\mathcal{S}'$ -convolvable with the kernel $\overrightarrow{\mathcal{K}}_{\alpha, t}$ for every $t = (t_1, \dots, t_m)$.

Proof. All that we need to do is to use Proposition 2.3 and to follow the same scheme given in [6], Th. 3.2. We will not repeat such proof here. $\square$

3. Weinstein operator on product spaces.

In this section we will study the behavior of the kernels $\overrightarrow{\mathcal{K}}_{\alpha, t}$ when they act by convolution with product versions of L^p spaces and central Morrey spaces.

The central Morrey spaces $B^{p,\lambda}(\mathbb{R}^n)$ have been studied by different authors like [5], [8], [1], among others, and they are defined in the following way:

Given $1 < p < \infty$ and $\lambda \geq -1/p$, a function $f \in L^p_{loc}(\mathbb{R}^n)$ belongs to the central Morrey space $B^{p,\lambda}(\mathbb{R}^n)$ if $\|f\|_{B^{p,\lambda}} < \infty$, where

$$\|f\|_{B^{p,\lambda}} := \sup_{r>0} \left(\frac{1}{|B_r(0)|^{1+p\lambda}} \int_{B_r(0)} |f(y)|^p dy \right)^{1/p}.$$

Endowed with this norm, they are Banach spaces. The space $B^{p,\lambda}(\mathbb{R}^n)$ reduces to zero when $\lambda < -1/p$, and for $\lambda = -1/p$, we have $B^{p,-1/p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

An alternative way to describe the spaces $B^{p,\lambda}(\mathbb{R}^n)$ is the following ([1], [5]): $f \in B^{p,\lambda}(\mathbb{R}^n)$ if and only if

$$\sup_{k \in \mathbb{Z}} 2^{-nk/p(1+p\lambda)} \|f \chi_{C_k}\|_{L^p} < \infty, \quad (3.1)$$

where

$$C_k = \{y \in \mathbb{R}^n : 2^{k-1} < |y| \leq 2^k\}.$$

The quantity in (3.1) defines an equivalent norm in $B^{p,\lambda}(\mathbb{R}^n)$.

We will restrict to the case $\lambda = 0$ and we simply write $B^p(\mathbb{R}^n)$ instead of $B^{p,0}(\mathbb{R}^n)$.

We notice that for $s > n$ we have the inclusion $B^p(\mathbb{R}^n) \subset L^p \left((1 + |y|^2)^{-s/2} \right)$, (see [1]).

Hence, it is immediate that, for $1 < p_j < \infty$, $s_j > n$, and $j = 1, \dots, m$,

$$\begin{aligned} B^{p_1}(\mathbb{R}^n) \times \dots \times B^{p_m}(\mathbb{R}^n) &\subset L^{p_1} \left((1 + |x_1|^2)^{-s_1/2} \right) \times \dots \times L^{p_m} \left((1 + |x_m|^2)^{-s_m/2} \right) \\ &\subset (1 + |x_1|^2)^{s_1/2} \dots (1 + |x_m|^2)^{s_m/2} \mathcal{D}'_{L^1}(\mathbb{R}^{nm}). \end{aligned}$$

Let us denote by $\overrightarrow{\Lambda}_{\alpha,t}$ the family of operators formally defined on m -uples of functions with variable on $\mathbb{R}^n$:

$$\overrightarrow{\Lambda}_{\alpha,t}(f_1, \dots, f_m) := (\mathcal{K}_{\alpha,t_1} * f_1) \dots (\mathcal{K}_{\alpha,t_m} * f_m), \quad (3.2)$$

where $t = (t_1, \dots, t_m)$ and $t_j > 0$ for $j = 1, \dots, m$.

Now, we can state the following result.

Proposition 3.1. Let us consider $1 < p, p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, $(x, t) := ((x_1, t_1), \dots, (x_m, t_m)) \in \mathbb{R}_+^{n+1} \times \dots \times \mathbb{R}_+^{n+1}$. Then, the family of operators $\overrightarrow{\Lambda}_{\alpha,t}$ is bounded from $L^{p_1} \left((1 + |x_1|^2)^{-(\alpha+n+1)/2} \right) \times \dots \times L^{p_m} \left((1 + |x_m|^2)^{-(\alpha+n+1)/2} \right)$ into $L^p \left((1 + |x_1|^2)^{-(\alpha+n+1)/2} \dots (1 + |x_m|^2)^{-(\alpha+n+1)/2} \right)$.

Proof. Denote by $\omega_\alpha(x_j) := (1 + |x_j|^2)^{-(\alpha+n+1)/2}$, $j = 1, \dots, m$. J. Wittsten proved the estimate ([14], pp. 912-913)

$$(\mathcal{K}_{\alpha,t_j} * \omega_\alpha)(y_j) \leq C_{\alpha,n} \omega_\alpha(y_j), \quad (3.3)$$

with $C_{\alpha,n}$ a constant only depending on α and n .

Taking into account that $\|\mathcal{K}_{\alpha,t_j}\|_{L^1(\mathbb{R}^n)} = 1$, and the fact that the kernels $\mathcal{K}_{\alpha,t_j}$ are radial on the variable x_j , we can use Jensen's inequality and Tonelli's theorem to obtain

$$\begin{aligned} & \int_{\mathbb{R}^{nm}} |(\mathcal{K}_{\alpha,t_1} * f_1)|^p(x_1) \cdots |(\mathcal{K}_{\alpha,t_m} * f_m)|^p(x_m) \omega_\alpha(x_1) \cdots \omega_\alpha(x_m) dx_1 \cdots dx_m \\ & \leq C_{\alpha,n} \int_{\mathbb{R}^{nm}} |f_1(y_1)|^p \cdots |f_m(y_m)|^p (\mathcal{K}_{\alpha,t_1} * \omega_\alpha)(y_1) dy_1 \cdots (\mathcal{K}_{\alpha,t_m} * \omega_\alpha)(y_m) dy_m \\ & \leq C_{\alpha,n} \int_{\mathbb{R}^{nm}} |f_1(y_1)|^p \cdots |f_m(y_m)|^p \omega_\alpha(y_1) dy_1 \cdots \omega_\alpha(y_m) dy_m \\ & \leq C_{\alpha,n} \left[\int_{\mathbb{R}^{nm}} |f_1(y_1)|^{p_1} \omega_\alpha(y_1) dy_1 \cdots \omega_\alpha(y_m) dy_m \right]^{p/p_1} \times \cdots \\ & \times \left[\int_{\mathbb{R}^{nm}} |f_m(y_m)|^{p_m} \omega_\alpha(y_1) dy_1 \cdots \omega_\alpha(y_m) dy_m \right]^{p/p_m} \end{aligned}$$

where, in the two last inequalities we have used the generalized Hölder inequality and the estimate (3.3).

Writing $f = (f_1, \dots, f_m)$, we obtain

$$\left\| \overrightarrow{\Lambda_{\alpha,t}}(f) \right\|_{L^p\left(\prod_{j=1}^n \omega_\alpha(x_j) dx_j\right)} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}(\omega_\alpha(x_j) dx_j)}$$

where C is a constant only depending on $\alpha, n, p_1, \dots, p_m$. $\square$

In the next result, we prove the continuity of $\overrightarrow{\Lambda_{\alpha,t}}$ on the product of B^p spaces.

Theorem 3.2. *Let us consider $1 < p, p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, $(x, t) := ((x_1, t_1), \dots, (x_m, t_m)) \in \mathbb{R}_+^{n+1} \times \dots \times \mathbb{R}_+^{n+1}$. Then, the family of operators $\overrightarrow{\Lambda_{\alpha,t}}$ is bounded from $B^{p_1}(\mathbb{R}^n) \times \dots \times B^{p_m}(\mathbb{R}^n)$ into $B^p(\mathbb{R}^n) \times \dots \times B^p(\mathbb{R}^n)$.*

Proof. We proceed by using a standard dyadic decomposition on each variable $x_j \in \mathbb{R}^n$, $j = 1, \dots, n$.

Writing $f = (f_1, \dots, f_m) \in B^{p_1}(\mathbb{R}^n) \times \dots \times B^{p_m}(\mathbb{R}^n)$, we can find a constant $C_{n,\alpha}$ only depending on n , and α such that

$$\begin{aligned} \left| \overrightarrow{\Lambda_{\alpha,t}}(f) \right|(x) & \leq C_{n,\alpha} \int_{\mathbb{R}^{nm}} \prod_{j=1}^m \frac{t_j^{\alpha+1} |f_j(y_j)|}{\left(t_j^2 + |y_j - x_j|^2\right)^{(\alpha+n+1)/2}} dy_j \\ & = C_{n,\alpha} \prod_{j=1}^m \left\{ \int_{|y_j - x_j| \leq t_j} \frac{t_j^{\alpha+1} |f_j(y_j)|}{\left(t_j^2 + |y_j - x_j|^2\right)^{(\alpha+n+1)/2}} dy_j \right. \\ & \quad \left. + \sum_{k=0}^{\infty} \int_{2^k t_j < |y_j - x_j| \leq 2^{k+1} t_j} \frac{t_j^{\alpha+1} |f_j(y_j)|}{\left(t_j^2 + |y_j - x_j|^2\right)^{(\alpha+n+1)/2}} dy_j \right\} \end{aligned}$$

$$\begin{aligned}
&\leq C_{n,\alpha} \prod_{j=1}^m \left\{ \frac{1}{t_j^n} \int_{|y_j - x_j| \leq t_j} |f(y_j)| dy_j \right. \\
&\quad \left. + \sum_{k=0}^{\infty} \frac{1}{(2^k t_j)^n} \int_{|y_j - x_j| \leq 2^{k+1} t_j} |f(y_j)| dy_j \right\} \\
&\leq C_{n,\alpha} \prod_{j=1}^m M(f_j)(x_j), \tag{3.4}
\end{aligned}$$

where M is the Hardy-Littlewood maximal operator.

Thus, using the generalized Hölder inequality, we have for $r_1, \dots, r_m > 0$

$$\begin{aligned}
&\frac{1}{|B_{r_1}(0)| \cdots |B_{r_m}(0)|} \int_{B_{r_1}(0) \times \cdots \times B_{r_m}(0)} \left| \overrightarrow{\Lambda_{\alpha,t}}(f) \right|^p(x) dx \\
&\leq C_{n,\alpha} \prod_{j=1}^m \left(\frac{1}{|B_{r_j}(0)|} \int_{B_{r_j}(0)} |M(f_j)|^{p_j}(x_j) dx_j \right)^{p/p_j},
\end{aligned}$$

and, invoking the fact that the operator $M : B^{p_j}(\mathbb{R}^n) \rightarrow B^{p_j}(\mathbb{R}^n)$ is bounded for each $j = 1, \dots, m$, (see [5]), we obtain

$$\left\| \overrightarrow{\Lambda_{\alpha,t}}(f) \right\|_{B^p(\mathbb{R}^n) \times \cdots \times B^p(\mathbb{R}^n)} \leq C_{n,\alpha} \prod_{j=1}^m \|f_j\|_{B^{p_j}(\mathbb{R}^n)},$$

which shows the desired continuity. $\square$

Now, we are going to consider products of weighted B^p spaces.

First, let us remind some concepts. Let w be a weight on $\mathbb{R}^n$, that is, $w \in L^1_{loc}$ and, $0 < w < \infty$ a.e. on $\mathbb{R}^n$.

We denote by $B^p(w)$ the space

$$B^p(w) := \left\{ f \in L^p_{loc,w} : \|f\|_{B^p(w)} < \infty \right\},$$

where $L^p_{loc,w}$ is the space of functions locally in $L^p(w)$ and

$$\|f\|_{B^p(w)} = \sup_{k \in \mathbb{Z}} w(C_k)^{-1/p} \|f \chi_{C_k}\|_{L^p(w)}. \tag{3.5}$$

When w is doubling, the norm given in (3.5) is equivalent to the norm given by

$$\|f\|_{B^p(w)}^* = \sup_{r>0} \left(\frac{1}{w(B_r(0))} \int_{B_r(0)} |f(y)|^p dw(y) \right)^{1/p}.$$

In [9], the author proves:

Proposition 3.3. ([9]) Let us assume that $1 < p < \infty$ and $w \in A_p$. Then, the Hardy-Littlewood maximal operator M is bounded from $B^p(w)$ into $B^p(w)$.

Here, the set A_p denotes the classical family of weights w satisfying the condition

$$\left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C$$

for every ball B , and some positive constant C .

Now, let us consider the operator $\mathcal{M}$ defined for $x = (x_1, \dots, x_m) \in \mathbb{R}^{nm}$ as

$$\mathcal{M}(f_1, \dots, f_m)(x) = \prod_{j=1}^m M(f_j)(x_j).$$

In view of Proposition 3.3, we can easily obtain

Corollary 3.4. *Let us assume that $1 < p, p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, and $w \in A_p$. Then, the operator $\mathcal{M}$ is bounded from $B^{p_1}(w) \times \dots \times B^{p_m}(w)$ into $B^p(w) \times \dots \times B^p(w)$.*

Proof. Since $1 < p < p_j$ for each j , we have that $A_p \subset A_{p_j}$. Thus, the Hardy-Littlewood maximal operator M is bounded from $B^{p_j}(w)$ into $B^{p_j}(w)$ for every j . From this,

$$\begin{aligned} & \frac{1}{w(B_{r_1}(0)) \cdots w(B_{r_m}(0))} \int_{B_{r_1}(0) \times \dots \times B_{r_m}(0)} |\mathcal{M}(f, \dots, f_m)|^p(x) w(x_1) \cdots w(x_m) dx \\ & \leq \prod_{j=1}^m \left(\frac{1}{w(B_{r_j}(0))} \int_{B_{r_j}(0)} |M(f_j)|^{p_j}(x_j) w(x_j) dx_j \right)^{p/p_j} \\ & \leq \prod_{j=1}^m C_{p_j, w} \|f_j\|_{B^{p_j}(w)}^p, \end{aligned}$$

and this implies that

$$\|\mathcal{M}(f, \dots, f_m)\|_{B^p(w) \times \dots \times B^p(w)} \leq C_{p_1, \dots, p_m, w} \prod_{j=1}^m \|f_j\|_{B^{p_j}(w)}.$$

This concludes the proof. $\square$

Finally, from Corollary 3.4 and estimate (3.4) we can state the following result.

Corollary 3.5. *Let us assume that $1 < p, p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, and $w \in A_p$. Then, the family of operators $\overrightarrow{\Lambda_{\alpha, t}}$ is bounded from $B^{p_1}(w) \times \dots \times B^{p_m}(w)$ into $B^p(w) \times \dots \times B^p(w)$.*

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