

## SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS USING OPERATIONAL MATRIX OF BERNOULLI ORTHONORMAL POLYNOMIALS

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**Abstract.** In this paper, a new numerical method has been proposed, which is based on the normalized Bernoulli polynomials for solving linear differential equations. An orthonormal family of polynomials of degree  $n$  from the  $n^{th}$  degree Bernoulli polynomials has been constructed. The properties of Bernoulli polynomials have been employed to construct the operational matrices of integration. These polynomials are orthogonalized, normalized and then their operational matrix of integration is derived. To demonstrate the efficiency and accuracy of the proposed method, the solutions of two numerical examples have been illustrated.

### 1. Introduction

A real-life system or phenomenon whether physical, sociological, economic, etc., gives rise to what is known as a mathematical model when described mathematically. Usually these models involve assumptions about the rate of change of one or more variables and can be expressed by means of differential, integral or integro-differential equations, or systems of them [1, 2, 3]. It is often required to employ numerical techniques to obtain an approximate solution. In 2000, He [4] studied variational iteration method to find approximate solutions to autonomous ordinary differential systems. Razzaghi and Yousefi [5] introduced the Legendre wavelets operational matrix of integration to solve problems such as identification, analysis and optimal control in 2001. They also introduced an operational matrix of integration based on sine and cosine wavelets and found its applications in calculus of variations. For details, see [6]. In 2010, Yousefi and Behroozifar [7] formulated B-polynomials operational matrix method. Later in the same year, Saadatmandi and Dehghan [8] used Legendre series together with the Legendre operational matrix of fractional derivatives to obtain numerical integration of fractional

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differential equations. In 2015, Biazar et al.[9] modified the traditional Homotopy Perturbation Method (HPM) and introduced Rational Homotopy Perturbation Method (RHPM) to solve stiff systems of ODEs more accurately with less iterations with the validity of the solutions in larger intervals than HPM. Shah et al. [10] introduced an operational matrix method of fractional order integration based on Haar wavelets to solve several types of fractional differential equations.

A significant role in various expansions and approximation formulas useful in analytic theory of numbers and in classical and numerical analysis is played by Bernoulli polynomials. For further details, one may read[11, 12, 13, 14, 15, 16, 17].

In this paper, a new numerical method has been proposed. This method is based on the modification of Bernoulli polynomials. An operational matrix of integration for Bernoulli orthonormal polynomials has been used for the numerical solutions of linear differential equations.

## 2. Bernoulli Polynomials

The Bernoulli polynomials of  $n^{th}$  degree are defined on the closed interval  $[0, 1]$  as

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k}, \quad (2.1)$$

where  $B_k := B_k(0)$  is the Bernoulli number for each  $k = 0, 1, \dots, n$ .

The Bernoulli numbers  $B_n$  have been expressed in the following form [18]

$$\begin{cases} B_n &= - \sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} \sum_{j=1}^k j^n, \text{ for } n \geq 0, n \neq 1 \text{ and} \\ B_1 &= -\frac{1}{2} \end{cases} \quad (2.2)$$

Thus the first few Bernoulli numbers are given by

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, \text{ and } B_n = 0 \text{ for all odd } n \geq 3.$$

Moreover,  $(-1)^{n+1} B_{2n} > 0$ , if  $n \geq 1$ .

Substituting value of  $B_n$  from relation (2.2) in equation (2.1), the first ten Bernoulli polynomials are given by

$$\begin{aligned} B_0(x) &= 1, & B_1(x) &= x - \frac{1}{2}, \\ B_2(x) &= x^2 - x + \frac{1}{6}, & B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \\ B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30}, & B_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x, \\ B_6(x) &= x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}, & B_7(x) &= x^7 - \frac{7}{2}x^6 + \frac{7}{2}x^5 - \frac{7}{6}x^3 + \frac{1}{6}x, \\ B_8(x) &= x^8 - 4x^7 + \frac{14}{3}x^6 - \frac{7}{3}x^4 + \frac{2}{3}x^2 - \frac{1}{30}, \\ B_9(x) &= x^9 - \frac{9}{2}x^8 + 6x^7 - \frac{21}{5}x^5 + 2x^3 - \frac{3}{10}x. \end{aligned}$$

Bernoulli polynomials satisfy the following well-known properties

$$\frac{d}{dx} B_n(x) = n B_{n-1}(x), \quad (2.3)$$

$$\int_0^1 B_n(x) dx = 0, \text{ for all } n \geq 1. \quad (2.4)$$

The integral formula is obtained by integrating equation (2.3) and is given by

$$B_n(x) = n \int_0^x B_{n-1}(t) dt + B_n. \quad (2.5)$$

### 3. Bernoulli Orthonormal Polynomials

Using Gram-Schmidt orthonormalization process on Bernoulli polynomials  $B_m(x)$ , a class of orthonormal Bernoulli polynomials of order  $m$ , denoted by  $w_{0m}, w_{1m}, \dots, w_{mm}$ , has been obtained.

The first ten Bernoulli orthonormal polynomials thus obtained are given by

$$\left. \begin{aligned} w_{09}(x) &= 1, \\ w_{19}(x) &= \sqrt{3}(-1 + 2x), \\ w_{29}(x) &= \sqrt{5}(1 - 6x + 6x^2), \\ w_{39}(x) &= \sqrt{7}(-1 + 12x - 30x^2 + 20x^3), \\ w_{49}(x) &= 3(1 - 20x + 90x^2 - 140x^3 + 70x^4), \\ w_{59}(x) &= \sqrt{11}(-1 + 30x - 210x^2 + 560x^3 - 630x^4 + 252x^5), \\ w_{69}(x) &= \sqrt{13}(1 - 42x + 420x^2 - 1680x^3 + 3150x^4 - 2772x^5 + 924x^6), \\ w_{79}(x) &= \sqrt{15}(-1 + 56x - 756x^2 + 4200x^3 - 11550x^4 + 16632x^5 - 12012x^6 + 3432x^7), \\ w_{89}(x) &= \sqrt{17}(1 - 72x + 1260x^2 - 9240x^3 + 34650x^4 - 72072x^5 + 84084x^6 - 51480x^7 \\ &\quad + 12870x^8), \\ w_{99}(x) &= \sqrt{19}(-1 + 90x - 1980x^2 + 18480x^3 - 90090x^4 + 252252x^5 - 420420x^6 \\ &\quad + 411840x^7 - 218790x^8 + 48620x^9) \end{aligned} \right\}. \quad (3.1)$$

A function  $g \in L^2[0, 1]$  may be written as

$$g(t) = \lim_{n \rightarrow \infty} \sum_{i=0}^n c_{in} w_{in}(t), \quad (3.2)$$

where  $c_{in} = \langle g, w_{in} \rangle$  and  $\langle \cdot, \cdot \rangle$  is the standard inner product on  $L^2[0, 1]$ .

If the series (3.2) is truncated at  $n = m$ , we get an approximation  $\tilde{g}$  of  $g$  as,

$$g(t) \cong \tilde{g}(t) = \sum_{i=0}^m c_{im} w_{im}(t) = C^T B(t), \quad (3.3)$$

$$\text{where } C = [c_{0m}, c_{1m}, \dots, c_{mm}]^T, \quad (3.4)$$

and

$$B(t) = [w_{0m}(t), w_{1m}(t), \dots, w_{mm}(t)]^T. \quad (3.5)$$

#### 4. Operational Matrix of Integration for Bernoulli Orthonormal Polynomials

Integrating Bernoulli orthonormal polynomials given in equation (3.1) w.r.t.  $x$  from 0 to  $t$  can be written as

$$\begin{aligned} \int_0^t w_{im}(x) dx &= \zeta_i(t), \quad 0 \leq t < 1, \quad i = 0, 1, \dots, m. \\ &= \sum_{j=0}^m c_{jm}^i w_{jm}(t), \quad \text{where } c_{jm}^i = \langle \zeta_i, w_{jm} \rangle, \end{aligned}$$

$$\text{or } \int_0^t w_{im}(x) dx = [c_{0m}^i, c_{1m}^i, \dots, c_{mm}^i] B(t), \text{ for } 0 \leq i \leq m. \quad (4.1)$$

Equations (4.1) can be written as

$$\int_0^t B(x) dx = P_{m+1} B(t), \quad (4.2)$$

where,  $P_{m+1}$  is the tridiagonal operational matrix of integration of order  $(m+1) \times (m+1)$  associated with orthonormal Bernoulli polynomials and is given by

$$P_{m+1} := \frac{1}{2} \begin{bmatrix} 1 & \frac{1}{\sqrt{(1) \cdot (3)}} & & & \\ -\frac{1}{\sqrt{(1) \cdot (3)}} & 0 & & \frac{1}{\sqrt{(3) \cdot (5)}} & \\ & & \ddots & & \ddots \\ & -\frac{1}{\sqrt{(3) \cdot (5)}} & & & -\frac{1}{\sqrt{(2m-1) \cdot (2m+1)}} \\ & & \ddots & & \\ & & & \frac{1}{\sqrt{(2m-1) \cdot (2m+1)}} & 0 \end{bmatrix} \quad (m+1) \times (m+1), \quad (4.3)$$

for  $m \geq 1$  and  $P_1 = \left[\frac{1}{2}\right]$ .

#### 5. Numerical Examples

In this section, a method to find approximate solution to differential equations, using orthonormal Bernoulli polynomials, is proposed, and numerical examples are considered to show its efficiency and accuracy. Firstly, let us see how this method works for the second order non-homogeneous linear differential equation of the form

$$ay''(t) + y(t) = p(t) \quad (5.1)$$

with initial conditions  $y(0) = y'(0) = 0$ ;  $a$  is a constant. In order to obtain the approximate solution, we approximate the functions  $y''(t)$  and  $p(t)$  by Bernoulli polynomial  $B(t)$ , i.e.,

$$\tilde{y}''(t) = C^T B(t), \quad (5.2)$$

$$\tilde{p}(t) = d^T B(t). \quad (5.3)$$

On integrating (5.2) two times and using the initial conditions, we have

$$\tilde{y}'(t) = C^T P_{m+1} B(t), \quad (5.4)$$

$$\tilde{y}(t) = C^T P_{m+1}^2 B(t), \quad (5.5)$$

where  $C = [c_{0m}, c_{1m}, \dots, c_{mm}]^T$  is to be determined.

Substitution of equations (5.2)-(5.5) in equation (5.1) gives

$$aC^T B(t) + C^T P_{m+1}^2 B(t) = d^T B(t). \quad (5.6)$$

Since (5.6) holds for all  $t \in [0, 1]$ , it reduces to

$$C^T = d^T [(aI + P_{m+1}^2)]^{-1}, \quad (5.7)$$

where  $I$  is the unit matrix of order  $(m+1) \times (m+1)$ .

Approximate solution  $\tilde{y}(t)$  can be obtained from (5.5) using (3.5), (4.3) and (5.7). We illustrate this method in the following examples.

**Example 5.1.** Consider a linear time-varying system

$$a\dot{y}(t) + y(t) = u(t), \text{ with } y(0) = 0, \quad (5.8)$$

where  $u(t)$  is the unit step function and the dot over  $y$  represents its first time derivative. The exact solution of equation (5.8) is given by

$$y(t) = 1 - e^{-\frac{t}{a}}. \quad (5.9)$$

To find an approximate solution of equation (5.8), let

$$\dot{\tilde{y}}(t) = C^T B(t) \text{ and} \quad (5.10)$$

$$\tilde{u}(t) = d^T B(t). \quad (5.11)$$

integrating equation (5.10) and using the initial condition  $y(0) = 0$  yields to

$$\tilde{y}(t) = C^T P_{m+1} B(t), \quad (5.12)$$

where  $C$  is to be determined.

Substitution of equations (5.10)-(5.12) in equation (5.8) gives

$$aC^T B(t) + C^T P_{m+1} B(t) = d^T B(t). \quad (5.13)$$

As equation (5.13) holds for all  $t \in [0, 1]$  it reduces to

$$C^T = d^T (aI + P_{m+1})^{-1}, \quad (5.14)$$

where  $I$  is the unit matrix of order  $(m+1) \times (m+1)$ .

Taking  $m = 9$  and using relation (3.5) in equation (5.11) results in

$$d^T = [1, 0, 0, 0, 0, 0, 0, 0, 0, 0]. \quad (5.15)$$

Substituting the values of  $P_{m+1}$  and  $d^T$  in equation (5.14) gives

$$C^T = \begin{bmatrix} 0.864665, -0.468815, 0.117736, -0.0195931, 0.00244398, -0.000243932, \\ 0.0000202949, -1.44769 \times 10^{-6}, 9.03781 \times 10^{-8}, -5.02877 \times 10^{-9} \end{bmatrix}. \quad (5.16)$$

An approximation,  $y_a(t)$ , to the exact solution  $y_e(t)$  is obtained by using relations (3.5), (4.3) and (5.14) in equation (5.12) for  $m = 9$  and  $a = 0.5$  as

$$y_a(t) = \frac{1}{1733584106} \left( 1 + 3467168100t - 3467165130t^2 + 2311409100t^3 - 1155494340t^4 \right. \\ \left. + 461440980t^5 - 152131980t^6 + 41132520t^7 - 8314020t^8 + 923780t^9 \right) \quad (5.17)$$

To choose appropriate value of  $m$ , convergence study has been carried out for different values of  $m$  and  $t$ . The exact values and approximate values of  $y(t)$  for  $t = 0.0(0.1)1.0$  for  $m = 5, 7$  and  $9$  have been obtained and are presented in Table 1. It is observed that the value of the approximation converges to that of the exact solution by increasing value of  $m$ .

The value of  $m$  has been fixed as  $9$ , since there was no further improvement even in eighth place of decimal for all values of  $t \in [0, 1]$ . Calculations were carried out using Wolfram Mathematica 7.0.

TABLE 1. Numerical results corresponding to Example 1

| t   | Exact solution<br>$y(t) = 1 - e^{-2t}$ | Present Method |              |              |
|-----|----------------------------------------|----------------|--------------|--------------|
|     |                                        | m=5            | m=7          | m=9          |
| 0.0 | 0.0000000000                           | 0.0000370274   | 0.0000001876 | 0.0000000006 |
| 0.1 | 0.1812692469                           | 0.1812552938   | 0.1812692484 | 0.1812692471 |
| 0.2 | 0.3296799540                           | 0.3296875269   | 0.3296799895 | 0.3296799538 |
| 0.3 | 0.4511883639                           | 0.4511982064   | 0.4511883125 | 0.4511883640 |
| 0.4 | 0.5506710359                           | 0.5506664672   | 0.5506710344 | 0.5506710359 |
| 0.5 | 0.6321205588                           | 0.6321092680   | 0.6321206094 | 0.6321205587 |
| 0.6 | 0.6988057881                           | 0.6988045588   | 0.6988057749 | 0.6988057882 |
| 0.7 | 0.7534030361                           | 0.7534144496   | 0.7534029885 | 0.7534030361 |
| 0.8 | 0.7981034820                           | 0.7981083778   | 0.7981035256 | 0.7981034819 |
| 0.9 | 0.8347011118                           | 0.8346862769   | 0.8347011039 | 0.8347011119 |
| 1.0 | 0.8646647168                           | 0.8647017440   | 0.8646649044 | 0.8646647173 |

The graphs of exact and approximate numerical solutions of equation (5.8) given by equations (5.9) and (5.17) for  $a = 0.5$  using  $P_{10}$  are shown in Figure 1. It is observed that the approximate solutions are so close to the exact one that their graphs overlap.

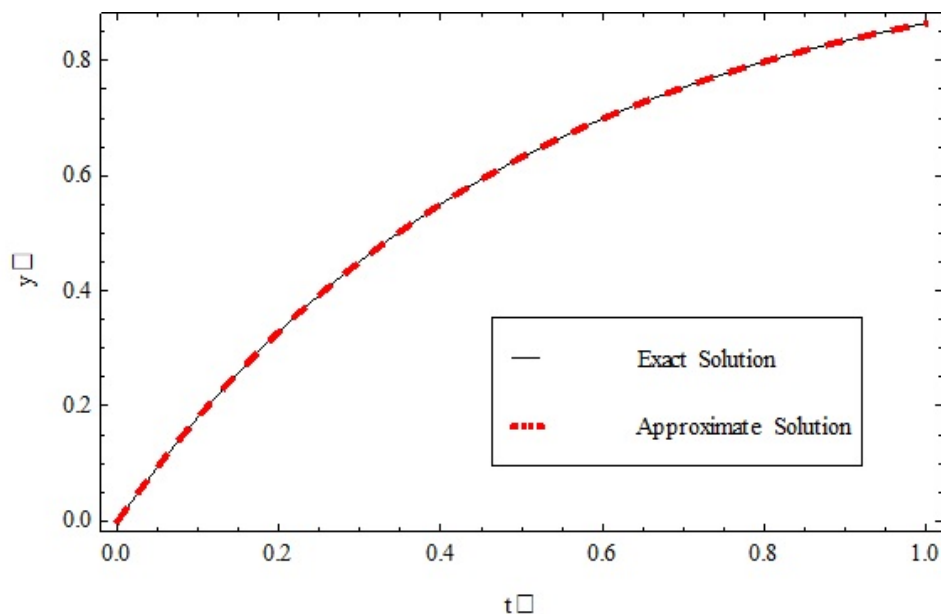


FIGURE 1. The exact and the approximate values of  $y$  for  $m = 9$  and  $a = 0.5$  in Example 1.

TABLE 2. Absolute errors  $\|y_e - y_a\|$  for different values of  $m$  Example 1.

| t   | m=5          | m=7          | m=9          |
|-----|--------------|--------------|--------------|
| 0.0 | 0.0000370274 | 0.0000001876 | 0.0000000006 |
| 0.1 | 0.0000139531 | 0.0000000015 | 0.0000000002 |
| 0.2 | 0.0000075729 | 0.0000000355 | 0.0000000002 |
| 0.3 | 0.0000098425 | 0.0000000514 | 0.0000000001 |
| 0.4 | 0.0000045687 | 0.0000000015 | 0.0000000000 |
| 0.5 | 0.0000112908 | 0.0000000506 | 0.0000000001 |
| 0.6 | 0.0000012293 | 0.0000000132 | 0.0000000001 |
| 0.7 | 0.0000114135 | 0.0000000476 | 0.0000000000 |
| 0.8 | 0.0000048958 | 0.0000000436 | 0.0000000001 |
| 0.9 | 0.0000148349 | 0.0000000079 | 0.0000000001 |
| 1.0 | 0.0000370272 | 0.0000001876 | 0.0000000005 |

In the next example, we examine the approximate solution to another linear differential equation with our proposed method and compare the results with those obtained in [19] by using Haar wavelets.

**Example 5.2.** Consider

$$y'' + y = e^t \quad (5.18)$$

with

$$\left. \begin{aligned} y(0) &= 0 \\ y'(0) &= 0 \end{aligned} \right\} \quad (5.19)$$

The exact solution of this problem is

$$y = \frac{1}{2}(e^t - \cos t - \sin t). \quad (5.20)$$

On following the steps of our proposed method, we have

$$C^T B(t) + C^T P_{m+1}^2 B(t) = d^T B(t). \quad (5.21)$$

As equation (5.21) holds for all  $t \in [0, 1]$ , it reduces to

$$C^T = d^T [(I + P_{m+1}^2)]^{-1}, \quad (5.22)$$

where  $I$  is the unit matrix of order  $(m+1) \times (m+1)$ .

Using equation (3.5) in equation (5.22) for  $m = 9$ , we have

$$d^T = \begin{bmatrix} 1.71828, 0.48795, 0.0625491, 0.00526552, 0.000330863, 0.0000165975 \\ 6.92573 \times 10^{-7}, 2.8856 \times 10^{-8}, 0, 0 \end{bmatrix}. \quad (5.23)$$

Substituting equations (4.3) and (5.23) in equation (5.22) for  $m = 9$ , we get

$$C^T = \begin{bmatrix} 1.50973, 0.3002, 0.00643661, 0.00201438, 0.000298533, 0.0000102647 \\ 6.5514 \times 10^{-8}, 1.35192 \times 10^{-8}, -7.35783 \times 10^{-11}, -1.17857 \times 10^{-11} \end{bmatrix}. \quad (5.24)$$

The approximation to the solution is obtained by using equations (3.5), (4.3) and (5.24) in equation (5.5) for  $m = 9$  as

$$y_a(t) = \begin{bmatrix} 2.34711 \times 10^{-13} - 2.44768 \times 10^{-11} t + 0.5 t^2 + 0.16666 t^3 \\ -1.33885 \times 10^{-6} t^4 + 4.7255 \times 10^{-6} t^5 + 0.00137948 t^6 + 0.000209397 t^7 \\ -7.33546 \times 10^{-6} t^8 + 2.49774 \times 10^{-6} t^9 \end{bmatrix}. \quad (5.25)$$

The graphs of exact and approximate numerical solutions of (5.18) given by equations (5.20) and (5.25) are shown in Figure 2. We also show in this graph an approximation obtained in [19] by using Haar wavelets. It is observed that the approximate solutions are very close to exact one.

TABLE 3. Comparison of Exact, Haar and Bernoulli Solution

| t     | Exact Solution | Haar Solution | Bernoulli Solution |
|-------|----------------|---------------|--------------------|
| 0.125 | 0.0081         | 0.0088        | 0.0081             |
| 0.375 | 0.0791         | 0.0810        | 0.0791             |
| 0.625 | 0.2361         | 0.2399        | 0.2361             |
| 0.875 | 0.4952         | 0.4992        | 0.4952             |

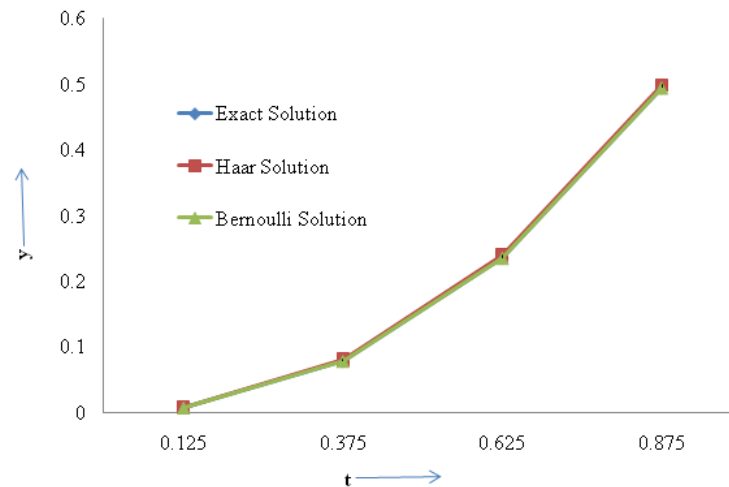


FIGURE 2. The exact and the approximate values of  $y$  for  $m = 9$  corresponding to Example 2.

### Conclusion

This paper proposes a method using the operational matrix of integration for Bernoulli orthonormal polynomials to obtain approximate solution of linear differential equation. The tridiagonal operational matrix of Bernoulli orthonormal polynomials has been derived to determine the approximate solution and the procedure to derive them has been explained. In Example 1, a numerical example of time-varying system has been solved to show the efficiency and accuracy of the proposed method. It is observed that the results obtained by taking  $m = 9$  are correct upto 8 decimal places. In Example 2, a second order linear differential equation has been solved; it had been previously considered and solved by Haar wavelet method in [19]. On considering the accuracy of approximate solution upto four decimal places, one may see that the solution obtained by using Bernoulli operational matrix of integration of small size is exactly same as the exact solution of the problem. Thus, in this example, our proposed method seems to produce a better approximation than the Haar wavelet method introduced in [19].

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