

THE NATURAL ORDERING IN STRICTLY REAL m -CONVEX Q -ALGEBRAS

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Abstract. J. B. Miller considered orders in strictly real Banach algebras. The aim of this paper is to extend the basic results about the natural ordering to strictly real Hausdorff m -convex ones. In this frame, one is led to put the additional condition that they are Q -algebras, which is, anyway, satisfied in the Banach case. The results about the associated order are also valid in this framework but they are not included here.

1. Introduction

In 1949, I. Kaplansky has introduced in [12] the class of strictly real Banach algebras. Since then, not much work has been done on them. In [16], J. B. Miller considers what he calls the natural ordering in a strictly real Banach algebra. The new idea is to use the principal component G_1 . He also introduced what he called the associated order. Here we work with unital strictly real m -convex Hausdorff Q -algebras. From section 5 onwards we intend, using the results of sections 3 and 4, to extend to this class of algebras the results of Miller concerning the natural order. Those relative to the associated order can also be extended to our framework, but we do not include them here because their proofs are the same, although some of these can be simplified using our results of Section 3.

In Theorem 3.9 we show that if E is a unital strictly real m -convex Hausdorff Q -algebra, then the spectrum of any $x \in E$ is the image of its Gelfand transform, as it happens in the case of unital m -convex commutative complex Hausdorff Q -algebras. In the aforementioned theorem and Corollary 4.4 we also prove that the same is true for other algebras related with E , such as its complexification $E_{\mathbb{C}}$ and E modulo its

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(Jacobson) radical. In M. Abel's terminology all these are algebras with functional spectrum. Besides this, in Section 4 it is shown that in any of them their radical coincides with the set of their quasi-nilpotent elements (Proposition 4.3). All these results are relevant for the rest of the work.

Section 5 concerns the principal component G_1 of the group $G(E)$ of invertible elements of E . It plays a crucial role in the definition of the natural ordering. Proposition 5.1 puts together facts that are used in the sequel.

The natural order is studied in Section 6. Its properties are looked at. We obtain the same results as in [16]. It appears (Proposition 6.4) that $E = G_1 - G_1$, a more precise statement than $E = G(E) - G(E)$. Also, the order has good interpolation properties (Proposition 6.5). It is even strongly archimedean according to Miller's terminology (Proposition 6.7). If it is full, then E is isomorphic to the real field $\mathbb{R}$ (Proposition 6.8).

In Section 7, we examine the order topology associated to the natural ordering. It first appears that it is Hausdorff (separated) if, and only if, the algebra is semi-simple (Proposition 7.4). We proceed also to a comparison of the topologies involved. It appears that the order topology τ_o and the topology τ_ρ , associated to the spectral radius function ρ , are in fact equivalent (Proposition 7.2). As a consequence, the cone G_1 , which is shown to be τ_ρ -normal, is also τ_o -normal. But its τ -normality is not granted. It actually reduces the algebra E to a particular Banach one (Remark 7.10).

The square root operation for positive elements of E is presented in Section 8, assuming that E is also complete. The unique positive square root belongs to the bicommutant of the element (Proposition 8.1).

Finally, in section 9, we study the dual cone of E and the related set of continuous linear functionals that are positive on every strictly positive element of E .

2. Preliminaries

An algebra E , over $\mathbb{R}$ or $\mathbb{C}$, with a linear topology τ , is a *topological algebra* if its multiplication is *separately continuous*, i.e., for any $x_0 \in E$, the mappings $x \rightarrow x_0x$ and $x \rightarrow xx_0$ are continuous. Its topological dual is denoted by E^* . If (E, τ) is also a locally convex space, then it is called a *locally convex algebra (l.c.a.)*. The multiplication is said to be *jointly continuous* if the map $E \times E \rightarrow E$, $(x, y) \rightarrow xy$ is continuous.

All topological algebras will be assumed to be Hausdorff unless otherwise specified.

A topological algebra E , with identity e , is a *Q-algebra* if the set $G(E)$ of all its invertible elements is open or, equivalently, if e has a neighborhood contained in $G(E)$.

To avoid confusion, the zero in the scalar field and the zero element in a vector space V will be written as 0 and 0_V , respectively.

Let E be a topological algebra with unit e . A net $(x_i)_{i \in I}$ in E is called *advertibly convergent* with respect to $x \in E$ if $xx_i \rightarrow e$ and $x_ix \rightarrow e$ in E .

The topological algebra E is called *advertibly complete* if every advertibly convergent Cauchy net in E converges there. Every *Q-algebra* is advertibly complete.

Let E be a *l.c.a.* whose topology τ is given by a family $(|\cdot|_\lambda)_{\lambda \in \Lambda}$ of seminorms.

For simplicity, and provided there is no risk of confusion, we shall write only $(|\cdot|_\lambda)$ instead of $(|\cdot|_\lambda)_{\lambda \in \Lambda}$, and, when appropriate, we shall also denote the algebra E by $(E, (|\cdot|_\lambda))$.

Recall that a $l.c.a.$ $(E, (|\cdot|_\lambda))$ has a jointly continuous multiplication if and only if for each λ there is a continuous seminorm $|\cdot|$ on E such that

$$|xy|_\lambda \leq |x| |y|; \forall x, y \in E.$$

A $l.c.a.$ $(E, (|\cdot|_\lambda))$ is said to be m -convex algebra ([15]) if

$$|xy|_\lambda \leq |x|_\lambda |x|_\lambda; \forall x, y \in E \text{ and } \forall \lambda.$$

A $l.c.a.$ $(E, (|\cdot|_\lambda))$ is said to be locally A -convex algebra (l - A - $c.a.$) ([8]) if

$$\forall x, \forall \lambda, \exists M = M(x, \lambda) > 0 : \max(|xy|_\lambda, |yx|_\lambda) \leq M |y|_\lambda; \forall y \in E.$$

When $M(x, \lambda)$ depends only on x , we write $M(x)$ and then E is called a locally uniformly A -convex algebra ($l.u$ - A - $c.a.$) ([7] and [8]). In this case

$$\forall x, \exists M(x) > 0 : \max(|xy|_\lambda, |yx|_\lambda) \leq M(x) |y|_\lambda; \forall y \in E, \forall \lambda.$$

As in the Banach case ([16]), a real $l.c.a.$ E , with unit e , is said to be *strictly real* if $Sp(x) \subset \mathbb{R}$, for every $x \in E$, where $Sp(x)$ is the usual spectrum $Sp_{E_{\mathbb{C}}}(x)$ of x in the complexification $E_{\mathbb{C}}$ of E . Actually, the notion of strict reality can be defined without any reference to a given topology. It is essentially algebraic. The fundamental property is the non voidness of $Sp(x)$, when E is a unital, m -convex algebra, since then the complex algebra $E_{\mathbb{C}}$ has these two properties.

Here are some general facts, already noticed in the Banach case ([16]).

Lemma 2.1. *Let E be a real algebra with unit e . Then the following are equivalent*

- (i) E is strictly real;
- (ii) $Sp(x^2) \subset \mathbb{R}_+^1$, for every $x \in E$;
- (iii) $e + x^2$ is invertible, for every $x \in E$.

From item (iii) of the above lemma, it is readily seen that the next are examples of strictly real algebras. All of them appear in [11], but we put them here for the sake of completeness.

Example 1. Let $E = C_b(\mathbb{R}, \mathbb{R})$ be the algebra of continuous bounded real valued functions on the real line $\mathbb{R}$. Endow it with the topology given by the family of seminorms $|\cdot|_\varphi$, $\varphi \in C_0(\mathbb{R}, \mathbb{R})$, where $C_0(\mathbb{R}, \mathbb{R})$ is the algebra of functions in $C_b(\mathbb{R}, \mathbb{R})$ vanishing at infinity and $|f|_\varphi = \sup\{|f(x)\varphi(x)| : x \in \mathbb{R}\}$. It is a $l.u$ - A - $c.a.$ which is not an m -convex algebra ([8, Example 3, p. 20]).

Example 2. Let $E = C(\mathbb{R}, \mathbb{R})$ be the algebra of continuous real valued functions on the real line $\mathbb{R}$. Endow it with the topology of uniform convergence on compacta. This is an m -convex algebra which is not a $l.u$ - A - $c.a.$

Example 3. Considering the standard cartesian product of $C_b(\mathbb{R}, \mathbb{R})$ and $C(\mathbb{R}, \mathbb{R})$, one obtains a l - A - $c.a.$ which is not m -convex nor locally uniformly A -convex algebra.

Example 4. This example of a $l.c.a.$, which is not locally A -convex, is first due to Warner ([18, p. 1028]). Let $E = C[0, 1]$ be the Banach algebra of continuous real functions on the interval $[0, 1]$, endowed with the uniform topology. If $w = \sigma(E, E^*)$ is the weak topology in E , associated to E^* , then the multiplication on (E, w) is not jointly continuous. Now, by a result of A. C. Cochran ([6, Theorem 3.4, p. 76]), it can not be a l - A - $c.a.$

¹ We denote the real intervals $[0, \infty)$ and $(-\infty, 0]$ by $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively.

Remark 2.2. To enlarge the class of strictly real *l.c.a.*'s, one notices that the property is kept by the following operations: cartesian product, projective limit, the image by an algebra morphism (epimorphic image), ...

3. The spectrum

In the sequel, every real or complex m -convex algebra E will be assumed to have a topology τ generated by a saturated family of submultiplicative seminorms $(|\cdot|_\lambda)_\lambda$. We shall use, when appropriate, E , $(E, (|\cdot|_\lambda)_\lambda)$ or (E, τ) to denote it. If E is also a Q -algebra with unit e , then it will be assumed that $|e|_\lambda = 1$ for all λ .

Let E be a real m -convex algebra, its complexification $E_\mathbb{C} = E + iE$ endowed with the product topology $\tau_\mathbb{C}$ is then a complex m -convex algebra ([19, p. 241]) and $E_\mathbb{C}$ is Hausdorff whenever E is Hausdorff.

Remark 3.1. The topological results about strictly real algebras will be stated assuming that they are also m -convex (Hausdorff) Q -algebras, but part of them are valid, and the given proofs work, if we replace this last condition by advertible completeness. We think that exceptions are, for example, Theorem 3.9 and Propositions 6.4 and 6.5.

Denote by $\mathcal{M}(E)$ the carrier space of E , i.e., the set of all non-zero multiplicative continuous linear functionals on algebra E , real or complex, according to whether E is assumed to be a real or complex topological algebra, respectively. In particular, if E is a real algebra, then $\mathcal{M}(E_\mathbb{C})$ is the set of all non-zero multiplicative continuous complex linear functionals on the complexification $E_\mathbb{C}$ of E .

For any unital m -convex algebra E , the Jacobson radical $Rad(E)$ is a bilateral ideal. For $x \in E$, we have $Sp_E(x) = Sp_B(s(x))$, where $B = E/Rad(E)$ and $s : E \rightarrow B$ is the canonical surjection. The algebra E is called *almost commutative* if it is commutative modulo its Jacobson radical $Rad(E)$.

Remark 3.2. If E is a Q -algebra or strictly real algebra and I is a closed bilateral ideal, then E/I has the corresponding property because open epimorphisms and epimorphisms preserve the respective property. If E is a real or complex topological Q -algebra, then $Rad(E)$ is a closed ideal. Thus, $E/Rad(E)$ is an m -convex algebra if E is an m -convex Q -algebra.

Wiener property: Let E be a complex commutative m -convex Q -algebra. Then $\mathcal{M}(E) \neq \emptyset$ and

$$Sp(x) = \widehat{x}(\mathcal{M}(E)) = \{\varphi(x) \mid \varphi \in \mathcal{M}(E)\} \quad (3.1)$$

for every $x \in E$, where $\widehat{x}$ is the Gelfand transform of x .

Any unital complex commutative complete m -convex algebra also satisfies (3.1) and the Wiener property is stated more often in relation to this type of algebras.

In M. Abel's terminology ([1]), a unital topological algebra in which the equality (3.1) is true for all its elements, is called a *topological algebra with functional spectrum*. We shall see that such is the case for any strictly real m -convex Q -algebra.

Remark 3.3. Suppose E is a topological real algebra. For a function $h : E \rightarrow \mathbb{R}$, define $\widetilde{h} : E_\mathbb{C} \rightarrow \mathbb{C}$ as $\widetilde{h}(x + iy) = h(x) + ih(y)$. If $\varphi \in \mathcal{M}(E)$, then $\widetilde{\varphi} \in \mathcal{M}(E_\mathbb{C})$. Conversely, $\Phi|_E \in \mathcal{M}(E)$, if $\Phi \in \mathcal{M}(E_\mathbb{C})$ and $\Phi(E) \subset \mathbb{R}$. The association $\varphi \rightarrow \widetilde{\varphi}$ is a bijection from $\mathcal{M}(E)$ onto $\widetilde{\mathcal{M}(E)} = \{\Phi \in \mathcal{M}(E_\mathbb{C}) \mid \Phi(E) \subset \mathbb{R}\}$.

If E is a strictly real topological algebra and $\mathcal{M}(E_{\mathbb{C}}) \neq \emptyset$, then $\Phi(x) \in Sp(x) \subset \mathbb{R}$ for every $\Phi \in \mathcal{M}(E_{\mathbb{C}})$ and $x \in E$. Thus,

$$\begin{aligned}\mathcal{M}(E_{\mathbb{C}}) &= \{\tilde{\varphi} \mid \varphi \in \mathcal{M}(E)\} \text{ and} \\ \mathcal{M}(E) &= \{\Phi|E \mid \Phi \in \mathcal{M}(E_{\mathbb{C}})\}.\end{aligned}\tag{3.2}$$

Lemma 3.4. *Let E be a commutative strictly real m -convex Q -algebra. Then $\mathcal{M}(E) \neq \emptyset$ and E is an algebra with functional spectrum.*

Proof. By a result of Żelazko ([20, Theorem, p. 226]), the complexification $E_{\mathbb{C}}$ of E is a Q -algebra. Then $E_{\mathbb{C}}$ is a commutative complex m -convex algebra, which is also a Q -algebra. By the Wiener property and equality (3.2), $Sp(x) = Sp_{E_{\mathbb{C}}}(x) = \{\varphi(x) \mid \varphi \in \mathcal{M}(E)\}$, if $x \in E$. $\square$

Proposition 3.5. *Let E be an almost commutative m -convex Q -algebra. If E is a complex algebra or a strictly real algebra, then $\mathcal{M}(E) \neq \emptyset$ and*

$$Sp(x) = \widehat{x}(\mathcal{M}(E)) = \{\varphi(x) \mid \varphi \in \mathcal{M}(E)\}$$

for every $x \in E$.

Proof. The algebra $B = E/\text{Rad}(E)$ is a commutative m -convex Q -algebra by hypothesis, and it is a complex algebra or a strictly real algebra depending on which type of algebra is E . From the Wiener property, in the complex case, and from Lemma 3.4, when E is strictly real, we have that $\mathcal{M}(B) \neq \emptyset$ and

$$Sp_E(x) = Sp_B(s(x)) = \{\psi(s(x)) \mid \psi \in \mathcal{M}(B)\}$$

for every $x \in E$, where $s : E \rightarrow E/\text{Rad}(E)$ is the canonical surjection. Since $\psi \circ s \in \mathcal{M}(E)$ for every $\psi \in \mathcal{M}(B)$ and $\varphi(x) \in Sp_E(x)$ for every $\varphi \in \mathcal{M}(E)$, we get the result. $\square$

The next lemma is inspired by results of Michael ([15, Proposition 7.3, p. 29]) and Hadjigeorgiou and Oudadess ([11, Proposition 13, p. 5]), that refer to real complete Hausdorff m -convex algebras.

Lemma 3.6. *Suppose $(E, (|\cdot|_{\lambda}))$ is a real m -convex algebra which is also a Q -algebra. Then*

$$\text{Rad}(E) = \bigcap_{\lambda} \pi_{\lambda}^{-1}(\text{Rad}(E_{\lambda})),$$

where $E_{\lambda} = E/\ker |\cdot|_{\lambda}$ and $\pi_{\lambda} : E \rightarrow E_{\lambda}$ is the canonical surjection.

Proof. Inclusion left to right is clear. For the other direction, let $\overline{E}_{\lambda}$ be the completion of the normed algebra E_{λ} . Since E is a Q -algebra, E is advertibly complete. Then $x \in G(E)$ if and only if $\pi_{\lambda}(x) = x_{\lambda} \in G(\overline{E}_{\lambda})$ for all λ ([10, Proposition 6.10 (4), p. 74]). In particular, $e - x \in G(E)$ for every $x \in \bigcap_{\lambda} \pi_{\lambda}^{-1}(\text{Rad}(E_{\lambda}))$. Since $\bigcap_{\lambda} \pi_{\lambda}^{-1}(\text{Rad}(E_{\lambda}))$ is a bilateral ideal I , $e - xy, e - yx \in G(E)$ for $x \in I$ and $y \in E$, i.e., $x \in I$ implies $x \in \text{Rad}(E)$. $\square$

For our next result we shall use the following one.

Theorem 3.7. ([4, Theorem 7.4.17, p. 333]) *Every strictly real semi-simple p -normed algebra is commutative.*

Remark 3.8. We believe that an extra condition is necessary in this theorem that guarantees the closedness of primitive ideals, for example the Q -property. Anyway, in our case there is no problem because we shall deal with Q -algebras.

Theorem 3.9. Suppose $(E, (|\cdot|_\lambda))$ is a strictly real m -convex Q -algebra. Then

- (i) E and $E_{\mathbb{C}}$ are almost commutative;
- (ii) $\mathcal{M}(E) \neq \emptyset$ and

$$Sp(x) = \widehat{x}(\mathcal{M}(E)) = \{\varphi(x) \mid \varphi \in \mathcal{M}(E)\}$$

for every $x \in E$, where $\widehat{x}$ is the Gelfand transform of x ;

- (iii) $\mathcal{M}(E_{\mathbb{C}}) \neq \emptyset$ and

$$Sp(x + iy) = \widehat{x + iy}(\mathcal{M}(E_{\mathbb{C}})) = \{\widetilde{\varphi}(x + iy) \mid \varphi \in \mathcal{M}(E)\}$$

for every $x + iy \in E_{\mathbb{C}}$;

- (iv) $\mathcal{M}(E)$ is compact in the weak* topology $\sigma(E^*, E)$ on E^* . $Sp(x)$ is compact in $\mathbb{R}$ and the spectral radius function ρ is finite valued; in fact, it is a submultiplicative seminorm on E ;

- (v) $(\mathcal{M}(E), \sigma(E^*, E))$ is isomorphic to $(\mathcal{M}(E_{\mathbb{C}}), \sigma(E_{\mathbb{C}}^*, E_{\mathbb{C}}))$ and the space $(\mathcal{M}(E_{\mathbb{C}}), \sigma(E_{\mathbb{C}}^*, E_{\mathbb{C}}))$ is compact.

Proof. (i) By Remark 3.2, the normed algebra $E_\lambda = E / \ker |\cdot|_\lambda$ is a strictly real Q -algebra, $Rad(E_\lambda)$ is closed in E_λ and then the semi-simple algebra $E_\lambda / Rad(E_\lambda)$ is a normed Q -algebra. It follows from Theorem 3.7 that $E_\lambda / Rad(E_\lambda)$ is commutative. Then $x_\lambda y_\lambda - y_\lambda x_\lambda \in Rad(E_\lambda)$ for each λ and $x, y \in E$. By Lemma 3.6, $xy - yx \in Rad(E)$ for every $x, y \in E$. Thus, $E / Rad(E)$ is commutative.

The complexification $(E / Rad(E))_{\mathbb{C}}$ of $E / Rad(E)$ is then commutative. On the other hand, $E_{\mathbb{C}} / Rad(E_{\mathbb{C}})$ and $(E / Rad(E))_{\mathbb{C}}$ are algebraically isomorphic via the map $s'(x + iy) \rightarrow s(x) + is(y)$, where s and s' are the corresponding canonical surjections from E and $E_{\mathbb{C}}$, respectively. Therefore, $E_{\mathbb{C}} / Rad(E_{\mathbb{C}})$ is commutative.

Since E and $E_{\mathbb{C}}$ are m -convex Q -algebras, the items (ii) and (iii) follow from Proposition 3.5 and equality (3.2).

(iv) $\mathcal{M}(E)$ is an equicontinuous subset of the topological dual E^* of E , because E is a Q -algebra ([14, Proposition 7.1, p. 75]). Thus, $\mathcal{M}(E)$ is relatively compact in the weak* topology $\sigma(E^*, E)$, but $\mathcal{M}(E)$ is $\sigma(E^*, E)$ -closed, because E is unital. Hence, $\mathcal{M}(E)$ is $\sigma(E^*, E)$ -compact.

For $x \in E$, the Gelfand transform $\widehat{x} : (E, \sigma(E^*, E)) \rightarrow \mathbb{R}$ is continuous. Therefore, $Sp(x)$ is compact and

$$\rho(x) = \max_{\varphi \in \mathcal{M}(E)} |\varphi(x)| < \infty \quad (3.3)$$

for every $x \in X$. Clearly ρ is a submultiplicative seminorm.

(v) The map $\varphi \rightarrow \widetilde{\varphi}$ is bijective and $\sigma(E^*, E) - \sigma(E_{\mathbb{C}}^*, E_{\mathbb{C}})$ continuous. It is an isomorphism, since $(\mathcal{M}(E), \sigma(E^*, E))$ is compact. $\square$

Remark 3.10. In [3, p. 12], Arizmendi and Velov named the equality (3.1) in the Wiener property as (m). They observed that this equality holds for all elements of any p -normed algebra. In [9, Corollary, p. 190], it is exhibited another class of complex algebras for which this is true. Now we can add to all these the classes of strictly real m -convex (Hausdorff) algebras, which are also Q -algebras, and their complexifications.

4. The set of quasi-nilpotent elements

For any real or complex unital m -convex algebra E , $N(E)$ will denote the set of all *quasi-nilpotent* elements x of E , i.e., such $x \in E$, for which $Sp_E(x) = \{0\}$ (equivalently, $\rho(x) = 0$).

Proposition 4.1. If E is an m -convex Q -algebra that is also strictly real or complex and commutative, then

- (i) $Sp(xy) \subset Sp(x) \cdot Sp(y)$ and $Sp(x+y) \subset Sp(x) + Sp(y)$ for $x, y \in E$;
- (ii) $N(E) = \{x \in E \mid \varphi(x) = 0 \ \forall \ \varphi \in \mathcal{M}(E)\}$;
- (iii) $N(E)$ is an ideal of E ;
- (iv) $N(E_{\mathbb{C}}) = \{x + iy \in E_{\mathbb{C}} \mid \varphi(x) = \varphi(y) = 0 \ \forall \ \varphi \in \mathcal{M}(E)\}$, in case of strict reality of E .

Proof. The results (i) and (ii) follow from item (ii) of Theorem 3.9, in the strictly real case. In the complex one, they are consequences of the Arens-Michael decomposition and the Wiener property. From item (ii) we obtain (iii). Item (iv) follows from (iii) of Theorem 3.9. $\square$

Remark 4.2. It is well known that $Rad(E) \subset N(E)$ for any complex m -convex algebra. Indeed, $x \in Rad(E)$ implies $e_E - \frac{1}{\lambda}x \in G(E)$ for all $\lambda \neq 0$. Hence, $Sp_E(x) = \{0\}$. Therefore, $N(E) = Rad(E)$ if $N(E) \subset Rad(E)$. Such is the case when $N(E)$ is an ideal, because $\rho(xy) = 0$ if $\rho(x) = 0$ and then $e - xy \in G(E)$ for all $y \in E$, i.e., $x \in N(E)$ implies $x \in Rad(E)$.

Proposition 4.3. If E is a strictly real m -convex Q -algebra, then

- (i) $N(E) = Rad(E)$;
- (ii) $N(E_{\mathbb{C}}/Rad(E_{\mathbb{C}})) = Rad(E_{\mathbb{C}}/Rad(E_{\mathbb{C}}))$;
- (iii) $N(E_{\mathbb{C}}) = Rad(E_{\mathbb{C}}) = Rad(E) + iRad(E)$;
- (iv) $N((E/Rad(E))_{\mathbb{C}}) = Rad((E/Rad(E))_{\mathbb{C}})$.

Proof. (i) By Proposition 4.1, $N(E)$ is an ideal of E . Thus $N(E) = Rad(E)$, by Remark 4.2.

(ii) Let $s' : E_{\mathbb{C}} \rightarrow B_C = E_{\mathbb{C}}/Rad(E_{\mathbb{C}})$ be the canonical surjection. The algebra B_C is a complex commutative (Theorem 3.9) m -convex Q -algebra. By Proposition 4.1 and Remark 4.2, $N(B_C)$ is an ideal of B_C and $N(B_C) = Rad(B_C)$.

(iii) To get the first equality, it is only necessary to prove the left to right inclusion: $x + iy \in N(E_{\mathbb{C}})$ implies $s'(x + iy) \in N(B_C) = \{s'(0_{E_{\mathbb{C}}})\}$. Then $N(E_{\mathbb{C}}) \subset Rad(E_{\mathbb{C}})$. From Proposition 4.1 it is clear that $N(E_{\mathbb{C}}) = N(E) + iN(E)$ and, by item (i), $Rad(E_{\mathbb{C}}) = Rad(E) + iRad(E)$.

(iv) Follows from Remark 3.2 and item (iii). $\square$

Corollary 4.4. Suppose $(E, (|\cdot|_{\lambda}))$ is a strictly real m -convex Q -algebra. Let $s : E \rightarrow B = E/Rad(E)$ and $s' : E_{\mathbb{C}} \rightarrow B_C = E_{\mathbb{C}}/Rad(E_{\mathbb{C}})$ be the canonical surjections. Then

- (i) $\mathcal{M}(B) = \{\psi_{\varphi} \mid \varphi \in \mathcal{M}(E)\}$, where $\psi_{\varphi} : B \rightarrow \mathbb{R}$ is defined by $\psi_{\varphi}(s(x)) = \varphi(x)$;
- (ii) $(\mathcal{M}(E), \sigma(E^*, E))$ and $(\mathcal{M}(B), \sigma(B^*, B))$ are isomorphic and the space $(\mathcal{M}(B), \sigma(B^*, B))$ is compact;
- (iii) $\mathcal{M}(B_C) = \{\Psi_{\varphi} \mid \varphi \in \mathcal{M}(E)\}$, where $\Psi_{\varphi} : B_C \rightarrow \mathbb{R}$ is defined by $\Psi_{\varphi}(s'(x + iy)) = \varphi(x) + i\varphi(y)$;

(iv) $(\mathcal{M}(E), \sigma(E^*, E))$ and $(\mathcal{M}(B_C), \sigma(B_C^*, B_C))$ are isomorphic and the space $(\mathcal{M}(B_C), \sigma(B_C^*, B_C))$ is compact.

Proof. (i) By the above proposition, $\psi_\varphi(s(x)) = \varphi(x)$ is a well defined function, for $\varphi \in \mathcal{M}(E)$ and $x \in E$. It belongs to $\mathcal{M}(B)$, because $|\varphi(x)| \leq M|x|_\lambda$ for all $x \in E$ and some λ , whence $|\psi_\varphi(s(x))| \leq M|s(x)|_{B,\lambda}$, where $|\cdot|_{B,\lambda}$ is the seminorm on the quotient B , corresponding to $|\cdot|_\lambda$. On the other hand, if $\psi \in \mathcal{M}(B)$ and $\varphi = \psi \circ s$, then $\varphi \in \mathcal{M}(E)$ and $\psi = \psi_\varphi$.

(ii) The map $\varphi \rightarrow \psi_\varphi$ is $\sigma(E^*, E) - \sigma(B^*, B)$ continuous and it is an isomorphism because $(\mathcal{M}(E), \sigma(E^*, E))$ is compact (Theorem 3.9). Then $\mathcal{M}(B)$ is $\sigma(B^*, B)$ -compact.

Items (iii) and (iv) can be proved using similar arguments and Theorem 3.9. $\square$

Remark 4.5. $\Psi \in \mathcal{M}(E_C / \text{Rad}(E_C))$ implies $(\Psi \circ s')|E \in \mathcal{M}(E)$.

Corollary 4.6. If $(E, (|\cdot|_\lambda))$ is a strictly real m -convex Q -algebra, then

- (i) $Sp_{E/\text{Rad}(E)}(s(x)) = \{\varphi(x) \mid \varphi \in \mathcal{M}(E)\}$ for every $x \in X$;
- (ii) $Sp_{E_C/\text{Rad}(E_C)}(s'(x + iy)) = \{\varphi(x) + i\varphi(y) \mid \varphi \in \mathcal{M}(E)\}$.

Proof. The results follow from item (i) of Theorem 3.9, Proposition 3.5 and Corollary 4.4. $\square$

5. Principal component

Recall that the *principal component* G_1 of a real m -convex algebra E is the connected component in $G(E)$, containing the unit element e . Since the inversion is continuous in E , G_1 is a closed normal subgroup of $G(E)$ ([4, Proposition 5.2.8, p. 230]).

Since the map $x \rightarrow zx$ is a homeomorphism from $G(E)$ onto $G(E)$ for every $z \in G(E)$, we have that any component of $G(E)$ is of the form zG_1 , with $z \in G(E)$.

Denote the *resolvent* $\mathbb{C} \setminus Sp(x)$, of an element x , by $Res(x)$.

For $a, b \in E$, $[a, b] = \{(1 - \alpha)a + \alpha b; 0 \leq \alpha \leq 1\}$ is the *closed segment* joining a to b . Put

$$M = \{a \in G_1 : [a, e] \subset G_1\},$$

and

$$M^k = \{a_1 \cdots a_k : a_1, \dots, a_k \in M\}.$$

We first collect together some facts stated in the Banach case ([16]). They do not rely on the existence of a norm, but only on algebraic calculations and connectedness.

Proposition 5.1. For any real m -convex algebra E we have:

- (i) $[a, e] \subset G_1$ if and only if $\mathbb{R}_- \subset Res(a)$;
- (ii) if E is also a Q -algebra, then M^k is the set of all $a \in E$ such that a can be joined to e by a polygonal arc of k closed segments in G_1 .

Theorem 5.2. If E is a real m -convex Q -algebra, then

- (i) $G_1 = \bigcup_k M^k$;
- (ii) the principal component of $s(E) = E / \text{Rad}(E)$ is exactly $s(G_1)$, where $s : E \rightarrow E / \text{Rad}(E)$ is the canonical surjection.

A subset A of an algebra is called a *semialgebra* if A is closed under the operations of addition, multiplication and multiplication by non-negative real numbers.

Proposition 5.3. If E is a strictly real m -convex Q -algebra, then

- (i) $G_1 = M = \{x \in E : Sp(x) \subset \mathbb{R}_+ \setminus \{0\}\} \cup \{0_E\}$ and G_1 is closed under the addition and multiplication by scalars $t \in \mathbb{R}_+ \setminus \{0\}$;
- (ii) $K = \{x \in E : Sp(x) \subset \mathbb{R}_+\}$ is a semialgebra of E .

Proof. (i) By definition of strict reality, the spectrum of every element is real. By (i) of Proposition 5.1, $M = \{x \in E : Sp(x) \subset \mathbb{R}_+ \setminus \{0\}\}$. It remains then to show that $G_1 = M$. For this, we claim that $M^2 = M$. Let $x = a_1 a_2$, with a_1 and a_2 in M . Using (i) of Proposition 4.1, $Sp(x) \subset (\mathbb{R}_+ \setminus \{0\}) \cdot (\mathbb{R}_+ \setminus \{0\}) = \mathbb{R}_+ \setminus \{0\}$. So $M^2 \subset M$; but also $M \subset M^2$, since $e \in M$. By induction, one gets $M^k = M$, for every k . To finish this part, use (i) of Theorem 5.2.

To see that G_1 is closed under the addition, use Proposition 4.1. If $x \in G_1$ and $t \in \mathbb{R}_+ \setminus \{0\}$, $Sp(tx) = tSp(x) \subset \mathbb{R}_+ \setminus \{0\}$. Then $tx \in G_1$.

(ii) It is clear that $tK \subset K$ for $t \in \mathbb{R}_+$. By item (i) of Proposition 4.1, K is closed under the operations of addition and multiplication. $\square$

Remark 5.4. Notice that

$$0_E \notin G_1, (\mathbb{R}_+ \setminus \{0\})G_1 = G_1, G_1 + G_1 = G_1, G_1 \cdot G_1 = G_1.$$

Remark 5.5. The semialgebra K contains G_1 and all quasi-nilpotent elements.

6. The natural ordering

We now turn our attention to the natural ordering defined by Miller in [16]. Using the considerations in the previous sections, the same order appears to be meaningful.

Definition 6.1. Let E be a real m -convex Q -algebra and $x, y \in E$. $x \prec y$ means $y - x \in G_1$ and $x \preceq y$ that $x \prec y$ or $x = y$. This relation is called the natural ordering in E .

From items (ii) of Theorem 3.9 and (i) of Proposition 5.3, we get the next equivalent and very useful formulation for the relation $x \prec y$. This formulation was not indicated by Miller in the Banach case ([16]).

Proposition 6.2. If E is a strictly real m -convex Q -algebra, then

$$x \prec y \Leftrightarrow \varphi(x) < \varphi(y)$$

for every $\varphi \in \mathcal{M}(E)$.

Remark 6.3. $(E, \preceq)$ is an ordered vector space, with positive cone $G_1 \cup \{0_E\}$, since

$$(x \preceq y, a \in E) \implies x + a \preceq y + a$$

and

$$(x \preceq y, \alpha \in \mathbb{R}_+) \implies \alpha x \preceq \alpha y.$$

Moreover, the order is compatible with the algebra structure, that is

$$(x \preceq y \text{ and } a \succ 0_E) \implies (ax \preceq ay \text{ and } xa \preceq ya).$$

Thus, $(E, \preceq)$ is an ordered algebra.

Observe also that

$$a \succ 0_E \Leftrightarrow (a \in G(E), a^{-1} \succ 0_E),$$

$$0_E \prec x \prec y \Leftrightarrow 0_E \prec y^{-1} \prec x^{-1}$$

and

$$(\alpha < \min Sp(x) \text{ and } \alpha < \min Sp(y)) \Leftrightarrow \alpha e \prec x, y. \quad (6.4)$$

$$(\alpha > \max Sp(x) \text{ and } \alpha > \max Sp(y)) \Leftrightarrow \alpha e \succ x, y. \quad (6.5)$$

In particular, the order directs E .

The following statement means that G_1 generates the algebra E in some sense.

Proposition 6.4. Let E be a strictly real m -convex Q -algebra. Then $E = G_1 - G_1$.

Proof. Let $x \in E$. Take $\alpha > \max Sp(x)$ and $\alpha > 0$. Then $\alpha e \succ x, 0_E$ and $x = \alpha e - (\alpha e - x)$. $\square$

The order defined is good enough, to have the following interpolation properties.

Proposition 6.5. Suppose $(E, (|\cdot|_\lambda))$ is a strictly real m -convex Q -algebra. Then

TR (1,2): $(a, x, y \in E, a \prec x, a \prec y) \implies (\exists u \in E : a \prec u \prec x, y)$

and

TR (2,1): $(a, x, y \in E, x \prec a, y \prec a) \implies (\exists v \in E : x, y \prec v \prec a)$.

The elements x and y do not have to be distinct from each other.

Proof. It is sufficient to consider the case TR (1,2), with $a = 0_E$. Let $0 < \alpha < \min Sp(x), \min Sp(y)$. By (6.4), $0_E \prec \alpha e \prec x, y$. $\square$

Remark 6.6. Thus E is order dense with respect to $\preceq$. By induction, we get that properties TR (1,n) and TR (n,1) hold for all $n \geq 2$. Besides this, join $x \vee y$ or meet $x \wedge y$ exists only when $x \preceq y$ or $y \preceq x$.

In the following statement, the order is more than archimedean; Miller says that it is strongly archimedean.

Proposition 6.7. Let E be a strictly real m -convex Q -algebra. Then

- (i) if $a \succ 0_E$, then, for every $x \in E$, there exists $n \in \mathbb{N}$ such that $x \prec na$. Equivalently, if $a \succ 0_E$, then, for every $x \in E$, there exists $\epsilon \in \mathbb{R}_+ \setminus \{0\}$ such that $\epsilon x \prec a$;
- (ii) if $a \succ e$, then, for every $x \in E$, there exists $m \in \mathbb{N}$ such that $x \prec a^m$.

Proof. (i) Take φ_0 and φ_1 in $\mathcal{M}(E)$ such that $\varphi_0(a) = \min Sp(a)$ and $\varphi_1(x) = \max Sp(x)$. Then $\varphi_0(a) > 0$ and $\varphi_1(x) < n\varphi_0(a)$ for n large enough. From Proposition 6.2, we get the result.

(ii) The same argumentation. $\square$

Proposition 6.8. Let E be a strictly real m -convex Q -algebra. The natural order is total if and only if E is isomorphic to the real field $\mathbb{R}$.

Proof. Any isomorphism between E and $\mathbb{R}$ preserves invertibility, and hence, the order relation. Thus, the natural order in E is total.

Suppose the partial order is total and $x \in E$. If $Sp(x)$ were not a singleton, then, for $\gamma \in Sp(x)$, $x \prec \gamma e$ implies $\max Sp(x) < \gamma$ (by (6.5)); a contradiction. In a similar way it follows that $x \succ \gamma e$ or $x = \gamma e$ can not hold. Therefore, $Sp(x) = \{\gamma_x\}$ and $x = \gamma_x e$. The map $x \rightarrow \gamma_x$ defines an isomorphism with the real field $\mathbb{R}$. $\square$

7. The order topology

Assume that $(E, (|\cdot|_\lambda))$ is a strictly real m -convex algebra which is also a Q -algebra.

For $a, b \in E$, with $a \prec b$, define the open order-interval $(a, b) = \{x \in E : a \prec x \prec b\}$.

We have:

- (1) for $x \in E$, $x \in (\alpha e, \beta e)$ if and only if $\alpha < \min Sp(x)$ and $\beta > \max Sp(x)$;
- (2) let $x \in (a_1, b_1) \cap (a_2, b_2)$. There exist u and v such that $a_1, a_2 \prec u \prec x$ and $x \prec v \prec b_1, b_2$ (Proposition 6.5). Then $(u, v) \subset (a_1, b_1) \cap (a_2, b_2)$.

The topology τ_o , which is formed by the set of all open order-intervals as its base, is called the *order topology*, induced by the natural order $\preceq$.

From Proposition 6.2, we obtain the next result.

Lemma 7.1. *Suppose E is a strictly real m -convex Q -algebra and let $\epsilon \in \mathbb{R}_+ \setminus \{0\}$. Then*

$$x \in (-\epsilon e, \epsilon e) \Leftrightarrow |\varphi(x)| < \epsilon \quad (7.6)$$

for every $\varphi \in \mathcal{M}(E)$, i.e.,

$$-\epsilon e \prec x \prec \epsilon e \Leftrightarrow \rho(x) < \epsilon, \quad (7.7)$$

where $\rho(x)$ is the spectral radius of x . Actually, the unit ball for ρ is exactly $(-e, e)$.

Proposition 7.2. Suppose E is a strictly real m -convex Q -algebra. Then

- (i) $(E, \preceq, \tau_o)$ is an ordered l.c.a. with jointly continuous multiplication. The set $\mathcal{I}$ of all the intervals $(-\epsilon e, \epsilon e)$, with $\epsilon \in \mathbb{R}_+ \setminus \{0\}$, forms a base at 0_E for τ_o ;
- (ii) (E, τ_o) is an m -convex algebra. In fact, the topology τ_o coincides with the topology τ_ρ generated by the submultiplicative seminorm ρ ;
- (iii) $(G(E), \tau_o|_{G(E)})$ is a topological group with respect to the multiplication.

Proof. (i) Using equivalence (7.7), it is easy to see that the interval $(-\epsilon e, \epsilon e)$ is an absorbing, balanced and convex set for any $\epsilon \in \mathbb{R}_+ \setminus \{0\}$. Since $\mathcal{I}$ is a filter base and $\alpha(-\epsilon e, \epsilon e) = (-\epsilon \alpha e, \epsilon \alpha e)$ for $\alpha, \epsilon \in \mathbb{R}_+ \setminus \{0\}$, there is a locally convex topology in E for which the family $\mathcal{I}$ is a base at 0_E .

This topology coincides with τ_o , since, given $x \in (a, b)$, there exist, by Proposition 6.7, $\epsilon_1, \epsilon_2 \in \mathbb{R}_+ \setminus \{0\}$ such that $\epsilon_1 e \prec x - a$, $\epsilon_2 e \prec b - x$, and then $x + (-\epsilon e, \epsilon e) \subset (a, b)$, if $0 < \epsilon < \min(\epsilon_1, \epsilon_2)$.

The multiplication in the algebra (E, τ_o) is jointly continuous, because, given $(-\epsilon e, \epsilon e)$ and taking $\delta = \sqrt{\epsilon}$, we have, by equivalence (7.6), that $x, y \in (-\delta e, \delta e)$ implies $|\varphi(xy)| < \delta^2$ for every $\varphi \in \mathcal{M}(E)$, i.e., $xy \in (-\epsilon e, \epsilon e)$.

(ii) That the order topology τ_o coincides with the topology τ_ρ , follows from the equivalence (7.7).

(iii) Since (E, τ_o) is an m -convex algebra, inversion is continuous. $\square$

Remark 7.3. If $x \in N(E)$, i.e., $x \in E$ is quasi-nilpotent, then $Sp(x + y) = Sp(y)$ (Theorem 3.9 (ii)), for every y . Whence $a \prec y \prec b$ if and only if $a \prec y + x \prec b$. So, x and $x + y$ can not be separated in E . Thus, separatedness is related to semi-simplicity.

Proposition 7.4. Let E be a strictly real m -convex Q -algebra. Then (E, τ_o) is separated if and only if it is semi-simple.

Proof. By item (i) of Proposition 4.3, $N(E) = Rad(E)$. Suppose (E, τ_o) is separated, then it is semi-simple, otherwise $x + y$ and y can not be separated if $y \in Rad(E) \setminus \{0_E\}$.

Reciprocally, assume that the elements $x_1 \neq x_2$ can not be separated. For each $\epsilon \in \mathbb{R}_+ \setminus \{0\}$, there exists $z_\epsilon \in (x_1 - \epsilon e, x_1 + \epsilon e) \cap (x_2 - \epsilon e, x_2 + \epsilon e)$. By equivalence (7.6), $|\varphi(x_i - z_\epsilon)| < \epsilon$ for $i = 1, 2$ if $\varphi \in \mathcal{M}(E)$. Therefore, $|\varphi(x_1 - x_2)| = 0$ for all $\varphi \in \mathcal{M}(E)$ and so, $x_1 - x_2 \in \text{Rad}(E) \setminus \{0_E\}$. Then (E, τ_o) is not semi-simple. $\square$

Remark 7.5. If (E, τ_o) is separated, then $\text{Rad}(E) = \{0_E\}$. Since $E/\text{Rad}(E)$ is commutative (Theorem 3.9 (i)), E is commutative. Then (E, τ_o) can not be separated if it is not commutative.

Though the algebra (E, τ_o) may not be Hausdorff, the algebraic operations behave well as we have seen in Proposition 7.2.

Remark 7.6. Since the sequence $((1 - \frac{n-1}{n})e)_n$ tends to 0_E , G_1 is open but not closed. The cone $G_1 \cup \{0_E\}$ is not open, because the sequence $(-\frac{1}{n}e)_n$ in its complement tends to 0_E . If the natural order is not total, then $G_1 \cup \{0_E\}$ is not closed. Indeed, $Sp(x)$ is not a singleton for some $x \in E$, otherwise E would be isomorphic to $\mathbb{R}$ (see the final part of the proof of Proposition 6.8) and the natural order would be total. Taking $\alpha = \min Sp(x)$, we have that $x - (\alpha - \frac{1}{n})e \in G_1$ for each n and $(x - (\alpha - \frac{1}{n})e)_n$ tends to $x - \alpha e \in E \setminus (G_1 \cup \{0_E\})$.

The next lemma follows immediately from Theorem 3.9 (ii) and Proposition 6.2.

Lemma 7.7. Suppose $(E, (|\cdot|_\lambda)_\lambda)$ is a strictly real m -convex Q -algebra.

(i) If $x \prec y$, then

$$\forall \alpha \in Sp(x) \exists \beta \in Sp(y) : \alpha < \beta$$

and

$$\forall \gamma \in Sp(y) \exists \delta \in Sp(x) : \delta < \gamma;$$

(ii) if $0_E \prec x \prec y$, then $\rho(x) < \rho(y)$, i.e., G_1 is τ_o -normal.

We recall here a very simple result that we need in the proofs of the next result and Proposition 9.1.

Lemma 7.8. Let $p, q : V \rightarrow \mathbb{R}_+$ be two positively homogeneous functions on a vector space V and $\alpha, \beta > 0$. Suppose $p(x) \leq \alpha$ if $q(x) \leq \beta$. Then $p(x) \leq \frac{\alpha}{\beta} q(x)$ for all $x \in V$.

Proof. We have that $q\left(\frac{\beta}{q(x)+\epsilon}x\right) < \beta$ for every $x \in X$ and $\epsilon > 0$. By our assumption, this implies $p\left(\frac{\beta}{q(x)+\epsilon}x\right) \leq \alpha$, i.e., $p(x) \leq \frac{\alpha}{\beta}(q(x) + \epsilon)$ for all $\epsilon > 0$, hence the result follows. $\square$

Proposition 7.9. Suppose $(E, (|\cdot|_\lambda)_\lambda)$ is a strictly real m -convex Q -algebra.

(i) There exist λ and $c \in \mathbb{R}_+ \setminus \{0\}$ such that

$$\rho(x) \leq c|x|_\lambda, \quad (7.8)$$

for all $x \in E$, i.e., $\tau_o \subset \tau$;

(ii) if G_1 is τ -normal, i.e.,

$$0_E \prec x \prec y \implies |x|_\lambda \leq |y|_\lambda, \text{ for all } \lambda,$$

then $\tau_o = \tau$ and the spectral radius ρ is a norm.

Proof. (i) As $\mathcal{M}(E)$ is an equicontinuous subset of the topological dual E^* of E , because E is a Q -algebra ([14, Proposition 7.1, p. 75]), there exist λ and $c \in \mathbb{R}_+ \setminus \{0\}$ such that $\rho(x) \leq c|x|_\lambda$ for all $x \in E$.

(ii) By hypothesis, $x \in (-\frac{1}{4}e, \frac{1}{4}e)$, i.e., $\rho(x) < \frac{1}{4}$, implies $|x + \frac{1}{4}e|_\lambda < \frac{1}{2}$, for all λ . Recalling that we have assumed that $|e|_\lambda = 1$ for every λ , we get $|x|_\lambda < \frac{3}{4}$ if $\rho(x) < \frac{1}{4}$. This implies, through positive homogeneity of $|x|_\lambda$ and $\rho(x)$ and using Lemma 7.8, that $|x|_\lambda \leq 3\rho(x)$ for all $x \in E$ and λ . Then ρ is a norm (remember that unless otherwise stated, we assumed that all topological algebras are Hausdorff) and $\tau_o = \tau$ because ρ generates the topology τ_o . $\square$

A submultiplicative seminorm p in a topological algebra is called *spectral seminorm* if

$$\rho(x) \leq p(x)$$

for every element x in the algebra. In that case ([17, Theorem 2.2.5, p. 212]),

$$\rho(x) = \lim p(x^n)^{\frac{1}{n}}.$$

Remark 7.10. Due to the strength of Proposition 7.9, the following curious fact is worthwhile to mention.

$0_E \prec x \prec y$ implies $|x^n|_\lambda < |y^n|_\lambda$, for some λ and all n large enough.

Indeed, by Proposition (7.9), there exist λ and $c \in \mathbb{R}_+ \setminus \{0\}$ such that $c|x|_\lambda$ is a spectral seminorm. Thus, $\rho(x) = \lim |x^n|_\lambda^{\frac{1}{n}}$ and $\rho(y) = \lim |y^n|_\lambda^{\frac{1}{n}}$. Since $0_E \prec x \prec y$ imply $\rho(x) < \rho(y)$, one has $|x^n|_\lambda < |y^n|_\lambda$, for all n large enough.

Proposition 7.11. Suppose (E, τ) is a strictly real m -convex Q -algebra. If G_1 is τ -normal, then:

- (i) (E, τ_o) is a commutative normed algebra;
- (ii) it is isometrically isomorphic to a subalgebra of the algebra $C_{\mathbb{R}}(X)$ of real continuous functions on the carrier space of $X = E$ or $E_{\mathbb{C}}$.

Proof. (i) The spectral radius ρ is a norm. Therefore, τ_o is a Hausdorff topology and (E, τ_o) is a normed strictly real semi-simple algebra (Proposition 7.4). By Theorem 3.7, E is commutative.

- (ii) The Gelfand transform $x \rightarrow \hat{x}$, gives the desired isometric isomorphism. $\square$

8. The square root operation

It will appear that, modulo completion and the Q -property, this operation has good properties in E . For Banach spaces we have ([5, Proposition 13, p. 44]): if $\rho(e - x) < 1$, then there exists a unique square root y of x such that $\rho(e - y) < 1$ and y belongs to the Banach algebra generated by e and x . Therefore, y is in the bicommutant of x .

Suppose $(E, (|\cdot|_\lambda))$ is a unital real complete m -convex algebra and $\rho(e - x) < 1$. It follows from a result of Sterbova ([10, Corollary 5.5, p. 67]), that there is a unique square root y of x such that $yx = xy$ and $\rho(e - y) < 1$. In fact, if $\overline{E}_\lambda$ is the completion of $E_\lambda = E / \ker |\cdot|_\lambda$, then y corresponds to the element $(y_\lambda)_\lambda$ in the Arens-Michael decomposition $E = \varprojlim \overline{E}_\lambda$, where y_λ is, for each λ , the unique square root of the image

x_λ of x in the Banach algebra $\overline{E}_\lambda$ such that $\rho_{\overline{E}_\lambda}(e_\lambda - y_\lambda) < 1$. Then y_λ is in the bicommutant of x_λ in $\overline{E}_\lambda$ and y is in the bicommutant of x in E .

Proposition 8.1. Let E be a strictly real complete m -convex Q -algebra. If $x \in E$ and $x \succ 0_E$, then it admits a unique positive square root, denoted by $\sqrt{x}$. It belongs to the bicommutant of x and $\rho(e - \sqrt{x}) < 1$, when $\rho(e - x) < 1$.

Proof. Let $w \succ 0_E$ in E and $\alpha \in \mathbb{R}_+ \setminus \{0\}$. From equality (3.3), we get:

$$\rho(e - \alpha w) < 1 \Leftrightarrow \alpha < \frac{2}{\max Sp(w)}. \quad (8.9)$$

Observe the following two facts, that are easy consequences of equivalences (7.7) and (8.9): $\rho(e - \alpha w) < 1$ and $0 < \beta < \alpha$ imply $\rho(e - \beta w) < 1$; and for each $w \succ 0_E$ there is $\alpha_w \in \mathbb{R}_+ \setminus \{0\}$ such that $\rho(e - \alpha_w w) < 1$.

For $x \succ 0_E$ take $\alpha_x \in \mathbb{R}_+ \setminus \{0\}$ for which $\rho(e - \alpha_x x) < 1$. From Sterbova's result, $\alpha_x x$ admits a unique square root y with $\rho(e - y) < 1$ and y is in the bicommutant of x . Thus, $y \succ 0_E$. Putting $z = \frac{1}{\sqrt{\alpha_x}} y$, we get $z^2 = x$, $z \succ 0_E$ and z belongs to the bicommutant of x .

To prove the uniqueness of z , suppose $z_1 \succ 0_E$ is also a positive square root of x . If $0 < \beta < \alpha_{z_1}^2, \alpha_x$, we have that $\sqrt{\beta} z_1$ and $\sqrt{\frac{\beta}{\alpha_x}} y$ are two square roots of βx in the unit ρ -ball about e . Thus $z_1 = z$.

Finally, by equivalence (7.6),

$$\rho(e - x) < 1 \Rightarrow |1 - \varphi(\sqrt{x})| \leq |1 - \varphi(x)| < 1 \quad \forall \varphi \in \mathcal{M}(E),$$

i.e., $\rho(e - \sqrt{x}) < 1$. □

Using the above proposition, Theorem 3.9 and Proposition 7.2 we shall prove the next result given in [16, Theorem 23, p. 550] for strictly real Banach algebras.

Proposition 8.2. Let E be a strictly real complete m -convex Q -algebra. Define $\sqrt{0_E} = 0_E$. Then

(i) $\sqrt{\cdot} : (G_1 \cup \{0_E\}, \tau_o) \rightarrow (G_1 \cup \{0_E\}, \tau_o)$ is uniformly continuous and strictly monotone;

(ii) $\sqrt{xy} = \sqrt{x}\sqrt{y}$ if and only if x and y commute.

Proof. (i) The strictly monotonicity follows from $\varphi(\sqrt{x}) = \sqrt{\varphi(x)}$ for any $x \succ 0_E$ and $\varphi \in \mathcal{M}(E)$.

Since $|\varphi(\sqrt{y}) - \varphi(\sqrt{x})| \leq \sqrt{|\varphi(y) - \varphi(x)|}$ for all $x, y \succeq 0_E$ and $\varphi \in \mathcal{M}(E)$, then $\rho(\sqrt{y} - \sqrt{x}) \leq \sqrt{\rho(y - x)}$ and the function $\sqrt{\cdot}$ is τ_o -uniformly continuous.

(ii) If x and y commute, then $\sqrt{x}\sqrt{x}\sqrt{y}\sqrt{y} = \sqrt{x}\sqrt{y}\sqrt{x}\sqrt{y}$, because $\sqrt{x}$ and $\sqrt{y}$ are in the bicommutants of x and y , respectively. Thus, $\sqrt{xy} = \sqrt{x}\sqrt{y}$. Conversely, this equality implies, by the invertibility of $\sqrt{x}$ and $\sqrt{y}$, that $\sqrt{x}\sqrt{y} = \sqrt{y}\sqrt{x}$, and then $xy = yx$. □

9. The dual cone

Suppose E is a strictly real m -convex algebra which is also a Q -algebra. Write $E^\#$ and E^* (respectively, $E_\mathbb{C}^\#$ and $E_\mathbb{C}^*$) to denote the algebraic and topological duals of (E, τ) (respectively, its complexification $(E_\mathbb{C}, \tau_\mathbb{C})$).

The dual cone for the natural ordering $\succeq$ on E is

$$K_{\succeq} = \{h \in E^{\#} | h(x) \geq 0 \text{ if } x \succeq 0_E\} = \{h \in E^{\#} | h(x) \geq 0 \text{ if } x \succ 0_E\}.$$

We can extend the order $\succeq$ to a preorder on $E_{\mathbb{C}}$ defining $x + iy \succeq z + iw$ if $x \succeq z$. For this preorder on $E_{\mathbb{C}}$, the dual wedge is

$$K_{\succeq}^{\mathbb{C}} = \{f \in E_{\mathbb{C}}^{\#} | f(x) \geq 0 \text{ if } x \succeq 0_E\} = \{f \in E_{\mathbb{C}}^{\#} | f(x) \geq 0 \text{ if } x \succ 0_E\}.$$

Put

$$\begin{aligned} \mathcal{P} &= \{h \in E^{\#} : h(x) > 0 \text{ if } x \succ 0_E\}, \\ \mathcal{P}_{\mathbb{C}} &= \{f \in E_{\mathbb{C}}^{\#} : f(x) > 0 \text{ if } x \succ 0_E\}. \end{aligned}$$

None of the two sets contains the zero functional and both of them are closed under the sum and multiplication by scalars $t \in \mathbb{R}_+ \setminus \{0\}$. These sets are referred by Miller ([16, p. 551]) as the dual cone of G_1 .

Since $\varphi \in \mathcal{M}(E)$ (respectively, $\Phi \in \mathcal{M}(E_{\mathbb{C}})$) and $x \in E$ imply $\varphi(x) \in Sp(x)$ (respectively, $\Phi(x) \in Sp(x)$), we have that $\varphi(x) > 0$ (respectively, $\Phi(x) > 0$) for $x \succ 0_E$. From this and the definitions we have

$$\begin{aligned} \mathcal{M}(E) &\subset \mathcal{P} \subset K_{\succeq}, \\ \mathcal{M}(E_{\mathbb{C}}) &\subset \mathcal{P}_{\mathbb{C}} \subset K_{\succeq}^{\mathbb{C}}. \end{aligned} \tag{9.10}$$

Since $\mathcal{M}(E)$ is not empty as well as $\mathcal{M}(E_{\mathbb{C}})$, $\mathcal{P}$ and $\mathcal{P}_{\mathbb{C}}$ are not empty sets.

For any $h \in E^{\#}$ (respectively, E^*), the functional $\tilde{h}(x + iy) = h(x) + ih(y)$, defined in $E_{\mathbb{C}}$, is a complex functional and belongs to $E_{\mathbb{C}}^{\#}$ (respectively, $E_{\mathbb{C}}^*$). On the other hand, if $f \in E_{\mathbb{C}}^{\#}$ (respectively, $E_{\mathbb{C}}^*$) is such that $f(x) \geq 0$ for every $x \succ 0_E$, then $f|E$ is a real functional and belongs to $E^{\#}$ (respectively, E^*); indeed, each $x \in E$ can be written as $x = x_1 - x_2$ with $x_1, x_2 \in G_1$ (Proposition 6.4). Then

$$\begin{aligned} K_{\succeq}^{\mathbb{C}} &= \{\tilde{h} | h \in K_{\succeq}\}, \\ \mathcal{P}_{\mathbb{C}} &= \{\tilde{h} | h \in \mathcal{P}\}. \end{aligned} \tag{9.11}$$

Proposition 9.1. Let $(E, (|\cdot|_{\lambda}))$ be a strictly real m -convex algebra which is also a Q -algebra. Then

(i) $\mathcal{K}_{\succeq} \subset (E, \tau_0)^* \subset (E, \tau)^*$. In fact, there exist λ and $c \in \mathbb{R}_+ \setminus \{0\}$ such that

$$|h(x)| \leq h(e) \rho(x) \leq c \cdot h(e) |x|_{\lambda} \tag{9.12}$$

for all $x \in E$ and $h \in \mathcal{K}_{\succeq}$;

(ii) $\mathcal{K}_{\succeq}$ is compact in the weak* topology $\sigma(E^*, E)$;

(iii) $\mathcal{K}_{\succeq}^{\mathbb{C}} \subset (E_{\mathbb{C}}, \tau_{\mathbb{C}})^*$;

(iv) $\mathcal{K}_{\succeq}^{\mathbb{C}}$ is compact in the weak* topology $\sigma(E_{\mathbb{C}}^*, E_{\mathbb{C}})$.

Proof. (i) Suppose $x \in E$ and $\rho(x) < 1$. Then $e - x, e + x \succ 0_E$ because $Sp(e \pm x) = Sp(e) + Sp(\pm x) \subset (0, 2) \subset \mathbb{R} \setminus \{0\}$. For $h \in \mathcal{K}_{\succeq}$, we have $h(e) > 0$ and $\rho(x) < 1$ implies $0 \leq h(e \pm x)$. Therefore, $\pm h(x) \leq h(e)$, i.e., $|h(x)| \leq h(e)$ if $\rho(x) < 1$. Since $|h(x)|$ and $\rho(x)$ are positively homogeneous functions we obtain by Lemma 7.8 that

$|h(x)| \leq h(e) \rho(x)$ for all $x \in E$. By Proposition 7.9, there exist λ and $c \in \mathbb{R}_+ \setminus \{0\}$ such that $\rho(x) \leq c|x|_\lambda$ for all $x \in E$. Thus,

$$h \in K_{\succeq} \Rightarrow h \text{ is } \tau_0\text{-continuous} \Rightarrow h \text{ is } \tau\text{-continuous.} \quad (9.13)$$

(ii) By inequalities (9.12), $K_{\succeq}$ is relatively compact in the weak* topology $\sigma(E^*, E)$. But it is clear that $K_{\succeq}$ is also weak* closed. Therefore, $K_{\succeq}$ is weak*-compact.

(iii) Take $f \in K_{\succeq}^{\mathbb{C}}$. Since $f = \tilde{h}$ for some $h \in K_{\succeq}$ and $h \in (E, \tau)^*$, one has $f \in (E_{\mathbb{C}}, \tau_{\mathbb{C}})^*$.

(iv) The map $h \rightarrow \tilde{h}$ is a $\sigma(E^*, E) - \sigma(E_{\mathbb{C}}^*, E_{\mathbb{C}})$ homeomorphism from $K_{\succeq}$ onto $K_{\succeq}^{\mathbb{C}}$. $\square$

Proposition 9.2. Let (E, τ) be a strictly real m -convex algebra which is also a Q -algebra. Then

$$G_1 = \{x \in E \mid h(x) > 0 \ \forall h \in \mathcal{P}\}$$

and

$$\overline{G}_1^{\tau} = \{x \in E \mid h(x) \geq 0 \ \forall h \in \mathcal{P}\}.$$

Proof. The first equality follows from the first inclusion in (9.10) and the next one: $G_1 = \{x \in E \mid x \succ 0_E\} = \{x \in E \mid \varphi(x) > 0 \ \forall \varphi \in \mathcal{M}(E)\}$. On the other hand, $\overline{G}_1^{\tau} \subset \{x \in E \mid h(x) \geq 0 \ \forall h \in \mathcal{P}\}$, by the continuity of each $h \in \mathcal{P}$. We have the reverse inclusion, since $x + \frac{1}{n}e \in G_1$ if $h(x) \geq 0$ for every $h \in \mathcal{P}$. $\square$

From implications (9.13), we get the next result.

Proposition 9.3. Let E be a strictly real m -convex Q -algebra. Then

- (i) $\mathcal{P} \cup \{0_{E^{\#}}\} = K_{\succeq}$;
- (ii) $h(x) = 0$ if $h \in \mathcal{P}$ and $x \in \text{Rad}(E)$;
- (iii) $h(xy) = h(yx)$ for $h \in \mathcal{P}$ and $x, y \in E$;
- (iv) the topological dual cone of $(E, \preceq, \tau_0)$ is $\mathcal{P} \cup \{0_{E^{\#}}\}$ and $(E, \tau_0)^* = \mathcal{P} - \mathcal{P}$;
- (v) the order dual for $(E, \succeq)$ coincides with the topological dual $(E, \tau_0)^*$.

Proof. (i) The inclusion left to right is clear. To check the other inclusion, take $h \in K_{\succeq}$. If $h(a) > 0$, whenever $a \succ 0_E$, then $h \in \mathcal{P}$. Suppose $h(a) = 0$ for some $a \succ 0_E$. By Proposition 6.7, given $x \succ 0_E$, there exists $n \geq 1$ such that $x \prec na$. Therefore, $h(x) = 0$ and then $h|_{G_1} = 0$. Since $E = G_1 - G_1$, $h = 0_{E^{\#}}$.

(ii) Let $x \in \text{Rad}(E)$ and $h \in \mathcal{P}$. By Proposition 4.3, $\varphi(x) = 0$ if $\varphi \in \mathcal{M}(E)$. Then $\frac{1}{n}e \pm x \succ 0_E$ and $\frac{1}{n}h(e) \pm h(x) > 0$ for all $n \geq 1$. Therefore, $h(x) = 0$.

(iii) Let $x, y \in E$. The quotient algebra $E/\text{Rad}(E)$ is commutative (Theorem 3.9). Then $xy - yx \in \text{Rad}(E)$. By item (ii), $h(xy) = h(yx)$ for $h \in \mathcal{P}$.

(iv) The space $(E, \preceq, \tau_0)$ is a topological ordered linear space. Its topological dual cone is $(E, \tau_0)^* \cap K_{\succeq}$. By Proposition 9.1 and item (i), $(E, \tau_0)^* \cap K_{\succeq} = K_{\succeq} = \mathcal{P} \cup \{0_{E^{\#}}\}$. As the seminorm ρ defines the topology τ_0 and the cone is τ_0 -normal, then $\rho(x) \leq \rho(y)$ if $x \preceq y$. From ([13, Theorem 23.6, p. 228]), we obtain

$$(E, \tau_0)^* = (\mathcal{P} \cup \{0_{E^{\#}}\}) - (\mathcal{P} \cup \{0_{E^{\#}}\}) = \mathcal{P} - \mathcal{P}.$$

(v) By definition, the order dual for $(E, \succeq)$ is $K_{\succeq} - K_{\succeq}$. By (i), this difference is equal to $\mathcal{P} - \mathcal{P}$. $\square$

Remark 9.4. We have already proved that $\text{Rad}(E_{\mathbb{C}}) = \text{Rad}(E) + i\text{Rad}(E)$, $\mathcal{P}_{\mathbb{C}} = \{\tilde{h} | h \in \mathcal{P}\}$ and $K_{\geq}^{\mathbb{C}} = \{\tilde{h} | h \in K_{\geq}\}$. By all this, we can replace $E, E^*, \mathcal{P}$ and $K_{\geq}$ by $E_{\mathbb{C}}, E_{\mathbb{C}}^*, \mathcal{P}_{\mathbb{C}}$ and $K_{\geq}^{\mathbb{C}}$, respectively, in items (i) to (iii) of the above proposition.

Corollary 9.5. Let $\mathcal{Q} = \{h \in \mathcal{P} | h(e) = 1\}$. Then $\mathcal{Q}$ is a convex compact set in the weak* topology $\sigma(E^*, E)$ and $\frac{h}{h(e)} \in \mathcal{Q}$ for every $h \in \mathcal{P}$.

Proof. It is clear that $\mathcal{Q}$ is a convex set and $\frac{h}{h(e)} \in \mathcal{Q}$ for every $h \in \mathcal{P}$. By inequalities (9.12), $\mathcal{Q}$ is an equicontinuous subset of E^* . Thus, $\mathcal{Q}$ is relatively compact in the weak* topology $\sigma(E^*, E)$. The weak*-limit h in E^* of any weak*-convergent net in $\mathcal{Q}$ satisfies $h(x) \geq 0$ for every $x \succ 0_E$. Hence, $h \in K_{\geq} = \mathcal{P} \cup \{0_{E^{\#}}\}$, but since $h(e) = 1$, $h \in \mathcal{P}$. Therefore, $\mathcal{Q}$ is weak*-compact. $\square$

For its implications in the theory of representations by boundary integrals and based on a result of Hervé ([2, Theorem I.4.3, p. 32]) about metrizability of compact convex subsets of Hausdorff locally convex spaces, Miller proved ([16, Theorem 35, p. 555]) the next theorem for strictly real Banach algebras. Now we state it in the framework that we have used here, weakening the boundedness hypothesis of (x_n) , assuming only that it is τ_0 -bounded.

Theorem 9.6. Let $(E, (|\cdot|_{\lambda}))$ be a strictly real m -convex Q -algebra. The weak*-compact convex set $\mathcal{Q}$ is metrizable if G_1 has a τ_0 -bounded countable separating set (x_n) for $\mathcal{Q}$. In particular, $\mathcal{Q}$ is metrizable if $(G_1, \tau_0|_{G_1})$ is separable.

Miller's proof remains valid if boundedness of (x_n) is assumed to be with respect to the topology τ_0 . It also works in our case, using in some point the inequality (9.12) in order to obtain $|h(x)| \leq \rho(x)$ for all $x \in E$ and $h \in \mathcal{Q}$. If the space $(G_1, \tau_0|_{G_1})$ is separable, then a τ_0 -bounded countable separating set for $\mathcal{Q}$ in G_1 is the sequence $\left(\frac{x_n}{\rho(x_n)+1}\right)$, where (x_n) is any countable τ_0 -dense set in G_1 .

Dedication

The publication of this article has been overshadowed by the death of Professor Mohamed Oudadess. In sorrow, the co-authors dedicate this work to his memory.

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