

θ_ω -CONNECTED SPACE AND θ_ω -CONTINUITY IN THE PRODUCT SPACE

LYLE LEON L. BUTANAS AND MHELMAR A. LABENDIA[†]

Date of Receiving : 21.03.2020

Date of Revision : 12.05.2020

Date of Acceptance : 18.05.2020

Abstract. In this paper, the concepts of θ_ω -connected space and θ_ω -continuous function from an arbitrary topological space into the product space will be introduced and characterized.

1. Introduction and Preliminaries

Substituting several concepts in topology with concepts acquiring either of weaker or stronger properties is often studied. The first attempt was done by Levine [23] when he introduced the concepts of semi-open sets, semi-closed sets, and semi-continuity of a function. Since then, several mathematicians became interested in presenting new topological concepts which can substitute the concept of the classical open set.

In 1968, Velicko [26] introduced the concepts of θ -continuity between topological spaces and subsequently defined the concepts of θ -closure and θ -interior of a subset of topological space. The concept of θ -open sets and its related topological concepts had been deeply studied and investigated by numerous authors, see [2, 6, 7, 14, 15, 19, 20, 21, 24, 25].

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The θ -closure and θ -interior of A are, respectively, denoted and defined by $Cl_\theta(A) = \{x \in X : Cl(U) \cap A \neq \emptyset \text{ for every open set } U \text{ containing } x\}$ and $Int_\theta(A) = \{x \in X : Cl(U) \subseteq A \text{ for some open set } U \text{ containing } x\}$, where $Cl(U)$ is the closure of U in X . A subset A of X is θ -closed if $Cl_\theta(A) = A$ and θ -open if $Int_\theta(A) = A$. Equivalently, A is θ -open if and only if $X \setminus A$ is θ -closed.

In 1982, Hdeib [17] introduced the concepts of ω -open and ω -closed sets and ω -closed mappings on a topological space. He showed that ω -closed mappings are strictly weaker than closed mappings and also showed that the Lindelöf property is preserved by counter

2010 *Mathematics Subject Classification.* 54A10, 54A05.

Key words and phrases. θ_ω -open, θ_ω -closed, θ_ω -connected, θ_ω -continuous.

Communicated by. Saeid Jafari

[†]Corresponding author.

images of ω -closed mappings with Lindelöf counter image of points. The concepts of ω -open sets and its corresponding topological concepts had been studied in several papers, see [4, 5, 8, 9, 10, 11, 12, 13]. In 2010, Ekici et al. [16] introduced the concepts of ω_θ -open and ω_θ -closed sets on a topological space. They showed that the family of all ω_θ -open sets in a topological space X forms a topology on X . They also introduced the notions of ω_θ -interior and ω_θ -closure of a subset of a topological space.

A point x of a topological space X is called a condensation point of $A \subseteq X$ if for each open set G containing x , $G \cap A$ is uncountable. A subset B of X is ω -closed if it contains all of its condensation points. The complement of B is ω -open. Equivalently, a subset U of X is ω -open (resp., ω_θ -open) if and only if for each $x \in U$, there exists an open set O containing x such that $O \setminus U$ (resp., $O \setminus Int_\theta(U)$) is countable. A subset B of X is ω_θ -closed if its complement $X \setminus B$ is ω_θ -open. The ω -closure (resp., ω_θ -closure) and ω -interior (resp., ω_θ -interior) of $A \subseteq X$ are, respectively, denoted and defined by $Cl_\omega(A) = \cap\{F : F \text{ is an } \omega\text{-closed set containing } A\}$ (resp., $Cl_{\omega_\theta}(A) = \cap\{F : F \text{ is an } \omega_\theta\text{-closed set containing } A\}$) and $Int_\omega(A) = \cup\{G : G \text{ is an } \omega\text{-open set contained in } A\}$ (resp., $Int_{\omega_\theta}(A) = \cup\{G : G \text{ is an } \omega_\theta\text{-open set contained in } A\}$).

In 2017, Al Ghour et al. [1] defined the notions of θ_ω -closure operator as a new topological operator and showed that θ_ω -closure of a subset of a topological space is strictly between its usual closure and its θ -closure. They also used the θ_ω -closure operator to introduce θ_ω -open sets and proved that the class of all θ_ω -open sets lies strictly between the class of open sets and the class of θ -open sets.

A point $x \in X$ is in θ_ω -closure of A ($x \in Cl_{\theta_\omega}(A)$) if $Cl_\omega(A) \cap A \neq \emptyset$ for any open set U containing x . A subset A of X is θ_ω -closed if $Cl_{\theta_\omega}(A) = A$. The complement of a θ_ω -closed set is called a θ_ω -open set. It is worth noting that $A \subseteq Cl_{\theta_\omega}(A)$ (resp., $A \subseteq Cl_\omega(A)$, $A \subseteq Cl_{\omega_\theta}(A)$) and $Int_{\theta_\omega}(A) \subseteq A$ (resp., $Int_\omega(A) \subseteq A$, $Int_{\omega_\theta}(A) \subseteq A$). Let $\mathcal{T}_{\theta_\omega}$ (resp., $\mathcal{T}_\omega$, $\mathcal{T}_{\omega_\theta}$) be the family of all θ_ω -open (resp., ω -open, ω_θ -open) subsets of a topological space X . Since $\mathcal{T}_{\theta_\omega}$ (resp., $\mathcal{T}_\omega$, $\mathcal{T}_{\omega_\theta}$) is a topology on X , for any set $A \subseteq X$, $Int_{\theta_\omega}(A)$ (resp., $Int_\omega(A)$, $Int_{\omega_\theta}(A)$) is θ_ω -open (resp., ω -open, ω_θ -open) and the largest θ_ω -open (resp., ω -open, ω_θ -open) set contained in A . Moreover, for any set $A \subseteq X$, $Cl_{\theta_\omega}(A)$ (resp., $Cl_\omega(A)$, $Cl_{\omega_\theta}(A)$) is θ_ω -closed (resp., ω -closed, ω_θ -closed) and the smallest θ_ω -closed (resp., ω -closed, ω_θ -closed) set containing A .

A topological space X is said to be θ_ω -connected (resp., θ -connected, ω -connected, ω_θ -connected) if X cannot be written as the union of two nonempty disjoint θ_ω -open (resp., θ -open, ω -open, ω_θ -open) sets. Otherwise, X is θ_ω -disconnected (resp., θ -disconnected, ω -disconnected, ω_θ -disconnected). A subset B of X is θ_ω -connected (resp., θ -connected, ω -connected, ω_θ -connected) if it is θ_ω -connected (resp., θ -connected, ω -connected, ω_θ -connected) as a subspace of X .

A function f from a topological space X to another topological space Y is said to be

- (i) θ_ω -open (resp., θ -open, ω -open, ω_θ -open) if $f(G)$ is θ_ω -open (resp., θ -open, ω -open, ω_θ -open) in Y for every open set G in X ;
- (ii) θ_ω -closed (resp., θ -closed, ω -closed, ω_θ -closed) if $f(G)$ is θ_ω -closed (resp., θ -closed, ω -closed, ω_θ -closed) in Y for every closed set G in X ;
- (iii) θ_ω -continuous if $f^{-1}(V)$ is θ_ω -open in X for every open subset V of Y .

Let $\mathcal{A}$ be an indexing set and $\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a family of topological spaces. For each $\alpha \in \mathcal{A}$, let $\mathcal{T}_\alpha$ be the topology on Y_α . The Tychonoff topology on $\Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ is

the topology generated by a subbase consisting of all sets $p_\alpha^{-1}(U_\alpha)$, where the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in \mathcal{A}\} \rightarrow Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$, U_α ranges over all members of $\mathcal{T}_\alpha$, and α ranges over all elements of $\mathcal{A}$. Corresponding to $U_\alpha \subseteq Y_\alpha$, denote $p_\alpha^{-1}(U_\alpha)$ by $\langle U_\alpha \rangle$. Similarly, for finitely many indices $\alpha_1, \alpha_2, \dots, \alpha_n$, and sets $U_{\alpha_1} \subseteq Y_{\alpha_1}$, $U_{\alpha_2} \subseteq Y_{\alpha_2}, \dots, U_{\alpha_n} \subseteq Y_{\alpha_n}$, the subset

$$\langle U_{\alpha_1} \rangle \cap \langle U_{\alpha_2} \rangle \cap \dots \cap \langle U_{\alpha_n} \rangle = p_{\alpha_1}^{-1}(U_{\alpha_1}) \cap p_{\alpha_2}^{-1}(U_{\alpha_2}) \cap \dots \cap p_{\alpha_n}^{-1}(U_{\alpha_n})$$

is denoted by $\langle U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n} \rangle$. We note that for each open set U_α subset of Y_α , $\langle U_\alpha \rangle = p_\alpha^{-1}(U_\alpha) = U_\alpha \times \prod_{\beta \neq \alpha} Y_\beta$. Hence, a basis for the Tychonoff topology consists of sets of the form $\langle B_{\alpha_1}, B_{\alpha_2}, \dots, B_{\alpha_k} \rangle$, where B_{α_i} is open in Y_{α_i} for every $i \in K = \{1, 2, \dots, k\}$.

Now, the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in \mathcal{A}\} \rightarrow Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$ for each $\alpha \in \mathcal{A}$. It is known that every projection map is a continuous open surjection. Also, it is well known that a function f from an arbitrary space X into the Cartesian product Y of the family of spaces $\{Y_\alpha : \alpha \in \mathcal{A}\}$ with the Tychonoff topology is continuous if and only if each coordinate function $p_\alpha \circ f$ is continuous, where p_α is the α -th coordinate projection map.

In this paper, we revisit the concept of θ_ω -open sets and characterize its related topological concepts such as θ_ω -connected space and θ_ω -continuous functions from an arbitrary topological space into the product space.

2. θ_ω -Open and θ_ω -Closed Functions

In this section, we investigate the connection of θ_ω -open (resp., θ_ω -closed) function to the other well-known functions such as the classical open, θ -open, ω -open, and ω_θ -open (resp., closed, θ -closed, ω -closed, ω_θ -closed) functions. We also give some characterizations of θ_ω -open and θ_ω -closed functions. Throughout, if no confusion arises, let X and Y be topological spaces.

The following three results can be proved by modifying the construction given in [1], [3], and [16], respectively.

Theorem 2.1. *Let $f : X \rightarrow Y$ be a function.*

- (i) *If f is θ -open (resp., θ -closed), then f is θ_ω -open (resp., θ_ω -closed), but not conversely.*
- (ii) *If f is θ_ω -open (resp., θ_ω -closed), then f is open (resp., closed), but not conversely.*

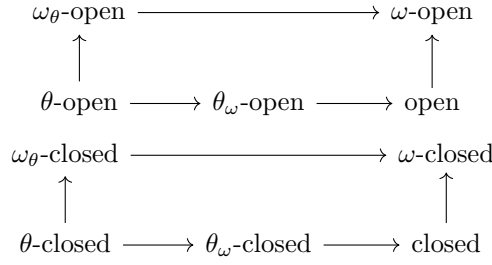
Theorem 2.2. *Let $f : X \rightarrow Y$ be a function.*

- (i) *If f is open (resp., closed), then f is ω -open (resp., ω -closed), but not conversely.*

Theorem 2.3. *Let $f : X \rightarrow Y$ be a function.*

- (i) *If f is θ -open (resp., θ -closed), then f is ω_θ -open (resp., ω_θ -closed), but not conversely.*
- (ii) *If f is ω_θ -open (resp., ω_θ -closed), then f is ω -open (resp., ω -closed), but not conversely.*

Remark 2.4. The following diagrams hold for a function $f : X \rightarrow Y$.



Remark 2.5. Open (resp., closed) sets and ω_θ -open (resp., ω_θ -closed) sets are two independent notions.

Consider the topological space $Y = \{a, b, c, d\}$ with topology $\mathcal{T}_Y = \{\emptyset, Y, \{a\}, \{d\}, \{a, d\}, \{a, b\}, \{a, b, d\}\}$. Since Y is finite, every subset of Y is both ω_θ -open and ω_θ -closed. Note that $\{a, b, c\}$ is ω_θ -open but not open and $\{d\}$ is ω_θ -closed but not closed in Y .

Next, consider the topological space $\mathbb{R}$ with topology $\mathcal{T} = \{\emptyset, \mathbb{R}, \mathbb{R} \setminus (1, 2)\}$. Then consider the open set $\mathbb{R} \setminus (1, 2)$. Since $\text{Int}_\theta(\mathbb{R} \setminus (1, 2)) = \emptyset$, for every $x \in \mathbb{R} \setminus (1, 2)$, we cannot find an open set O containing x such that $O \setminus \text{Int}_\theta(\mathbb{R} \setminus (1, 2)) = O$ is countable. Thus, $\mathbb{R} \setminus (1, 2)$ is not ω_θ -open on $\mathbb{R}$. Similarly, $(1, 2)$ is closed but not ω_θ -closed.

Remark 2.6. Open (resp., closed) functions and ω_θ -open (resp., ω_θ -closed) functions are two independent notions.

Corollary 2.7. [1, Corollary 2.4] Every open ω -closed set in a topological space is θ_ω -open.

Remark 2.8. θ_ω -open (resp., θ_ω -closed) sets and ω_θ -open (resp., ω_θ -closed) sets are two independent notions.

By Remark 2.5, ω_θ -open (resp., ω_θ -closed) set is not necessarily θ_ω -open (resp., θ_ω -closed). To show that θ_ω -open is not necessarily ω_θ -open, consider $\mathbb{R}$ as the real number line with topology $\mathcal{T} = \{\emptyset, \mathbb{R}, \mathbb{R} \setminus \mathbb{N}, \{0\} \cup \mathbb{N}, \{0\}\}$. Let $A = \mathbb{R} \setminus \mathbb{N}$. Note that $Cl(\mathbb{R} \setminus \mathbb{N}) = \mathbb{R} \not\subseteq A$, $Cl(\{0\} \cup \mathbb{N}) = \mathbb{R} \not\subseteq A$, and $Cl(\{0\}) = \mathbb{R} \not\subseteq A$. Thus, $\text{Int}_\theta(A) = \emptyset$. Let $x \in \mathbb{R} \setminus (\mathbb{N} \cup \{0\})$. Then, $\mathbb{R}$ and $\mathbb{R} \setminus \mathbb{N}$ are the only open sets containing x . However, $\mathbb{R} \setminus \text{Int}_\theta(A) = \mathbb{R}$ and $(\mathbb{R} \setminus \mathbb{N}) \setminus \text{Int}_\theta(A) = \mathbb{R} \setminus \mathbb{N}$ are both uncountable. Therefore, A is not ω_θ -open.

Next, we will verify that A is θ_ω -open. Let $y \in \mathbb{N}$. Observe that $\{0\} \cup \mathbb{N}$ is an open set containing y , and $(\{0\} \cup \mathbb{N}) \setminus \mathbb{N} = \{0\}$ is countable. Hence, $\mathbb{N}$ is ω -open. Thus, its complement $\mathbb{R} \setminus \mathbb{N} = A$ is ω -closed. By Corollary 2.7, A is θ_ω -open. Using the same topology given above, we can verify that $\mathbb{N}$ is θ_ω -closed but not ω_θ -closed.

Remark 2.9. θ_ω -open (resp., θ_ω -closed) functions and ω_θ -open (resp., ω_θ -closed) functions are two independent notions.

We shall now give some characterizations of θ_ω -open and θ_ω -closed functions.

Theorem 2.10. Let X and Y be topological spaces and $f : X \rightarrow Y$ be a function. Then the following statements are equivalent:

- (i) f is θ_ω -open on X .
- (ii) $f(\text{Int}(A)) \subseteq \text{Int}_{\theta_\omega}(f(A))$ for every $A \subseteq X$.

- (iii) $f(B)$ is θ_ω -open for every basic open set B in X .
- (iv) For each $p \in X$ and every open set O in X containing p , there exists an open set W in Y containing $f(p)$ such that $Cl_\omega(W) \subseteq f(O)$.

Proof. (i) $\Rightarrow$ (ii): Suppose that f is θ_ω -open on X . Then $f(G)$ is θ_ω -open in Y for all open set G in X . Let $A \subseteq X$. Then $f(Int(A)) \subseteq f(A)$. Since $Int(A)$ is open, $f(Int(A))$ is a θ_ω -open set contained in $f(A)$. Thus, $f(Int(A)) \subseteq Int_{\theta_\omega}(f(A))$.

(ii) $\Rightarrow$ (iii): Let B be a basic open set in X . Then $f(B) = f(Int(B)) \subseteq Int_{\theta_\omega}(f(B)) \subseteq f(B)$. Hence, $f(B) = Int_{\theta_\omega}(f(B))$ so that $f(B)$ is θ_ω -open in Y .

(iii) $\Rightarrow$ (iv): Let $p \in X$ and let O be an open set in X such that $p \in O$. Since O is open, there exist a basic open set B containing p such that $B \subseteq O$. This implies that $f(p) \in f(B) \subseteq f(O)$. By assumption, there exists an open set W in Y containing $f(p)$ such that $W \subseteq Cl_\omega(W) \subseteq f(B) \subseteq f(O)$.

(iv) $\Rightarrow$ (i): Let O be open in X and $y \in f(O)$. Then there exists $x \in O$ such that $f(x) = y$. By assumption, there exists an open set W in Y containing y such that $Cl_\omega(W) \subseteq f(O)$, which implies that $Cl_\omega(W) \cap f(O) \neq \emptyset$. Hence, $f(O)$ is θ_ω -open in Y . Thus, f is θ_ω -open on X . $\square$

Theorem 2.11. Let X and Y be topological spaces and $f : X \rightarrow Y$ be a function. Then the following statements are equivalent:

- (i) f is θ_ω -closed on X .
- (ii) $Cl_{\theta_\omega}(f(A)) \subseteq f(Cl(A))$ for every $A \subseteq X$.

Proof. (i) $\Rightarrow$ (ii): Suppose that f is θ_ω -closed on X . Then $f(A) \subseteq f(Cl(A))$. Since $Cl(A)$ is closed in X , $f(Cl(A))$ is θ_ω -closed in Y . Thus, $Cl_{\theta_\omega}(f(A)) \subseteq f(Cl(A))$ since $Cl_{\theta_\omega}(f(A))$ is the smallest θ_ω -closed set containing $f(A)$.

(ii) $\Rightarrow$ (i): Let B be closed in X . Then $f(B) \subseteq Cl_{\theta_\omega}(f(B)) \subseteq f(Cl(B)) = f(B)$. Hence, $f(B) = Cl_{\theta_\omega}(f(B))$ so that $f(B)$ is θ_ω -closed in Y . Thus, f is θ_ω -closed on X . $\square$

3. θ_ω -Continuous Function in the Product Space

In this section, a characterization of a θ_ω -continuous function from an arbitrary topological space into the product space will be presented.

We shall give first a characterization of a θ_ω -continuous function from a topological space to another topological space.

Theorem 3.1. Let $f : X \rightarrow Y$ be a function. Then the following statements are equivalent.

- (i) f is θ_ω -continuous on X .
- (ii) $f^{-1}(F)$ is θ_ω -closed in X for each closed subset F of Y .
- (iii) $f^{-1}(B)$ is θ_ω -open in X for each (subbasic) basic open set B in Y .
- (iv) For every $p \in X$ and every open set V of Y containing $f(p)$, then exists a θ_ω -open set U containing p such that $f(U) \subseteq V$.
- (v) $f(Cl_{\theta_\omega}(A)) \subseteq Cl(f(A))$ for each $A \subseteq X$.
- (vi) $Cl_{\theta_\omega}(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$ for each $B \subseteq Y$.

Proof. (i) $\Leftrightarrow$ (ii): Let F be closed in Y . Then $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is θ_ω -open in X . Thus, $f^{-1}(F)$ is θ_ω -closed in X . The converse is proved similarly.

(i) $\Rightarrow$ (iii): Trivial since (subbasic) basic open sets are open sets.

(iii) $\Rightarrow$ (i): Suppose that $f^{-1}(B)$ is θ_ω -open in X for each $B \in \mathcal{B}$ where $\mathcal{B}$ is a basis for the topology in Y . Let G be an open set in Y . Then $G = \cup \{B : B \in \mathcal{B}^*\}$, where $\mathcal{B}^* \subseteq \mathcal{B}$. It follows that $f^{-1}(G) = \cup \{f^{-1}(B) : B \in \mathcal{B}^*\}$. Since $\mathcal{T}_{\theta_\omega}$ is a topology, $f^{-1}(G)$ is θ_ω -open in X .

(i) $\Rightarrow$ (iv): Let $p \in X$ and V be an open set in Y containing $f(p)$. Since f is θ_ω -continuous, $U := f^{-1}(V)$ is θ_ω -open in X containing p . Therefore, $f(U) = f(f^{-1}(V)) \subseteq V$.

(iv) $\Rightarrow$ (v): Let $A \subseteq X$ and $p \in Cl_{\theta_\omega}(A)$. Let G be an open subset of Y containing $f(p)$. Then there exists a θ_ω -open subset O of X containing p such that $f(O) \subseteq G$. Since $p \in Cl_{\theta_\omega}(A)$, $O \cap A \neq \emptyset$. It follows that $\emptyset \neq f(O \cap A) \subseteq f(O) \cap f(A) \subseteq G \cap f(A)$. This implies that $f(p) \in Cl(f(A))$. Hence, $f(Cl_{\theta_\omega}(A)) \subseteq Cl(f(A))$.

(v) $\Rightarrow$ (vi): Let $B \subseteq Y$ and let $A = f^{-1}(B) \subseteq X$. Then $f(A) = f(f^{-1}(B)) \subseteq B$. It follows that $Cl(f(A)) \subseteq Cl(B)$. Using the assumption, $Cl_{\theta_\omega}(f^{-1}(B)) \subseteq f^{-1}(f(Cl_{\theta_\omega}(A))) \subseteq f^{-1}(Cl(f(A))) \subseteq f^{-1}(Cl(B))$.

(vi) $\Rightarrow$ (ii): Let F be a closed subset of Y . Then $Cl(F) = F$. By assumption,

$$Cl_{\theta_\omega}(f^{-1}(F)) \subseteq f^{-1}(Cl(F)) = f^{-1}(F) \subseteq Cl_{\theta_\omega}(f^{-1}(F)).$$

It follows that $Cl_{\theta_\omega}(f^{-1}(F)) = f^{-1}(F)$ and so $f^{-1}(F)$ is θ_ω -closed in X . Thus, f is θ_ω -continuous on X . $\square$

We shall use the following lemma later. The proof is standard, hence omitted.

Lemma 3.2. *Let X be a topological space and $A \subseteq X$.*

- (i) $x \in Cl_\omega(A)$ if and only if $B \cap A$ is uncountable for every basic open set B in X with $x \in B$.
- (ii) $x \in Cl_{\theta_\omega}(A)$ if and only if $Cl_\omega(B) \cap A \neq \emptyset$ for every basic open set B in X with $x \in B$.
- (iii) A is θ_ω -open if and only if for each $x \in A$, there exists a basic open set B containing x such that $Cl_\omega(B) \subseteq A$.

In the following results, if $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ is a product space and $A_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$, we denote $A_{\alpha_1} \times \cdots \times A_{\alpha_n} \times \Pi\{Y_\alpha : \alpha \notin K\}$ by $\langle A_{\alpha_1}, \dots, A_{\alpha_n} \rangle$, where $K = \{\alpha_1, \dots, \alpha_n\}$. If $Y = \Pi\{Y_i : i = 1, 2, \dots, n\}$ is a finite product, denote $A_1 \times \cdots \times A_n$ by $\langle A_1, \dots, A_n \rangle$.

Theorem 3.3. *Let $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ and $A_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$. Then in Y with the Tychonoff topology, $\Pi\{Cl_\omega(A_\alpha) : \alpha \in \mathcal{A}\} \subseteq Cl_\omega(\Pi\{A_\alpha : \alpha \in \mathcal{A}\})$.*

Proof. Let $x = \langle a_\alpha \rangle \in \Pi\{Cl_\omega(A_\alpha) : \alpha \in \mathcal{A}\}$. Then, $a_\alpha \in Cl_\omega(A_\alpha)$ for each $\alpha \in \mathcal{A}$. This means that for every open set U_α in Y_α with $a_\alpha \in U_\alpha$, $U_\alpha \cap A_\alpha$ is uncountable. Let $V = \langle V_{\alpha_1}, \dots, V_{\alpha_n} \rangle$ be a basic open set in Y with $x \in V$. Let $K = \{\alpha_1, \dots, \alpha_n\}$. Then

$$\begin{aligned} & \langle V_{\alpha_1}, \dots, V_{\alpha_n} \rangle \cap \Pi\{A_\alpha : \alpha \in \mathcal{A}\} \\ &= (V_{\alpha_1} \times \cdots \times V_{\alpha_n} \times \Pi\{Y_\alpha : \alpha \notin K\}) \cap (A_{\alpha_1} \times \cdots \times A_{\alpha_n} \times \Pi\{A_\alpha : \alpha \notin K\}) \\ &= (V_{\alpha_1} \cap A_{\alpha_1}) \times \cdots \times (V_{\alpha_n} \cap A_{\alpha_n}) \times \Pi\{A_\alpha : \alpha \notin K\} \end{aligned}$$

is uncountable. Hence, $x \in Cl_\omega(\Pi\{A_\alpha : \alpha \in \mathcal{A}\})$. $\square$

Theorem 3.4. *Let $Y = \Pi\{Y_i : i = 1, 2, \dots, n\}$ be a product space and $A_i \subseteq Y_i$ for each $i \in \{1, 2, \dots, n\}$. Then $Int_{\theta_\omega}(\Pi\{A_i : i = 1, 2, \dots, n\}) \subseteq \Pi\{Int_{\theta_\omega}(A_i) : i = 1, 2, \dots, n\}$.*

Proof. Let $x = \langle a_i \rangle \in \text{Int}_{\theta_\omega}(\Pi\{A_i : i = 1, 2, \dots, n\})$. Then there exists an open set $U = U_1 \times \dots \times U_n$ containing x such that $Cl_\omega(U) \subseteq \Pi\{A_i : i = 1, 2, \dots, n\}$. Thus, $a_i \in U_i$ for all i . By Theorem 3.3, $\langle Cl_\omega(U_1), \dots, Cl_\omega(U_n) \rangle \subseteq Cl_\omega(\langle U_1, \dots, U_n \rangle) \subseteq \Pi\{A_i : i = 1, 2, \dots, n\}$. Hence, $Cl_\omega(U_i) \subseteq A_i$ so that for each $i \in \{1, 2, \dots, n\}$, $a_i \in \text{Int}_{\theta_\omega}(A_i)$. Accordingly, $x \in \Pi\{\text{Int}_{\theta_\omega}(A_i) : i = 1, 2, \dots, n\}$. $\square$

Theorem 3.5. Let X be a topological space and $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ a product space. A function $f : X \rightarrow Y$ is θ_ω -continuous on X if and only if each coordinate function $p_\alpha \circ f$ is θ_ω -continuous on X .

Proof. Suppose that f is θ_ω -continuous on X . Let $\alpha \in \mathcal{A}$, and U_α be open in Y_α . Since p_α is continuous, $p_\alpha^{-1}(U_\alpha)$ is open in Y so that $f^{-1}(p_\alpha^{-1}(U_\alpha)) = (p_\alpha \circ f)^{-1}(U_\alpha)$ is θ_ω -open in X . Thus, $p_\alpha \circ f$ is θ_ω -continuous for every $\alpha \in \mathcal{A}$.

Conversely, suppose that each coordinate function $p_\alpha \circ f$ is θ_ω -continuous. Let G_α be open in Y_α . Since p_α is continuous, $p_\alpha^{-1}(G_\alpha) = \langle G_\alpha \rangle$ is open in Y . Then $(p_\alpha \circ f)^{-1}(G_\alpha) = f^{-1}(p_\alpha^{-1}(G_\alpha)) = f^{-1}(\langle G_\alpha \rangle)$ is θ_ω -open in X . Therefore, f is θ_ω -continuous on X . $\square$

The following result is a consequence of Theorem 3.5.

Corollary 3.6. Let X be a topological space, $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ a product space, and $f_\alpha : X \rightarrow Y_\alpha$ a function for each $\alpha \in \mathcal{A}$. Let $f : X \rightarrow Y$ be the function defined by $f(x) = \langle f_\alpha(x) \rangle$. Then, f is θ_ω -continuous on X if and only if each f_α is θ_ω -continuous for each $\alpha \in \mathcal{A}$.

Theorem 3.7. Let $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a product space and $\emptyset \neq O_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$. Let $K = \{\alpha_1, \dots, \alpha_n\} \subseteq \mathcal{A}$. If $O = \langle O_{\alpha_1}, \dots, O_{\alpha_n} \rangle$ is θ_ω -open in Y , then each O_{α_i} is θ_ω -open in Y_{α_i} .

Proof. Let $x = \langle a_\alpha \rangle \in O$. Since O is θ_ω -open in Y , there exists a basic open set $U = \langle U_{\alpha_1}, \dots, U_{\alpha_n} \rangle$ containing x such that $Cl_\omega(U) \subseteq O$. Thus for each $i \in \{1, 2, \dots, n\}$, $a_{\alpha_i} \in U_{\alpha_i}$ such that U_{α_i} is open in Y_{α_i} . It follows that $\langle Cl_\omega(U_{\alpha_1}), \dots, Cl_\omega(U_{\alpha_n}) \rangle \subseteq Cl_\omega(\langle U_{\alpha_1}, \dots, U_{\alpha_n} \rangle) \subseteq O$ implying that $Cl_\omega(U_{\alpha_i}) \subseteq O_{\alpha_i}$ for each $i \in \{1, 2, \dots, n\}$. Accordingly, each O_{α_i} is θ_ω -open in Y_{α_i} . $\square$

Theorem 3.8. Let $Y = \Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ and $A_\alpha \subseteq Y_\alpha$ for each $\alpha \in \mathcal{A}$. Then in Y with the Tychonoff topology,

$$\Pi\{Cl_{\theta_\omega}(A_\alpha) : \alpha \in \mathcal{A}\} \subseteq Cl_{\theta_\omega}(\Pi\{A_\alpha : \alpha \in \mathcal{A}\}).$$

Proof. Let $x = \langle a_\alpha \rangle \in \Pi\{Cl_{\theta_\omega}(A_\alpha) : \alpha \in \mathcal{A}\}$. Then $a_\alpha \in Cl_{\theta_\omega}(A_\alpha)$ for all $\alpha \in \mathcal{A}$. This means that for every open set U_α in Y_α with $a_\alpha \in U_\alpha$, $Cl_\omega(U_\alpha) \cap A_\alpha \neq \emptyset$. For each $\alpha \in \mathcal{A}$, choose $b_\alpha \in Cl_\omega(U_\alpha) \cap A_\alpha$. Let $V = \langle V_{\alpha_1}, \dots, V_{\alpha_n} \rangle$ be a basic open set in Y containing x . Then $a_{\alpha_i} \in V_{\alpha_i}$ for all $i \in J = \{1, 2, \dots, n\}$. For each $i \in J$, choose $c_{\alpha_i} \in Cl_\omega(V_{\alpha_i}) \cap A_{\alpha_i}$. Define $y = \langle d_\alpha \rangle$ by $d_\alpha = b_\alpha$ for each $\alpha \in \mathcal{A} \setminus K$ and $d_{\alpha_i} = c_{\alpha_i}$ for all $i \in J$, where $K = \{\alpha_1, \dots, \alpha_n\}$. Then $y \in \Pi\{A_\alpha : \alpha \notin K\} \times Cl_\omega(V_{\alpha_1}) \cap A_{\alpha_1} \times \dots \times Cl_\omega(V_{\alpha_n}) \cap A_{\alpha_n}$. Hence,

$$\begin{aligned} & \Pi\{A_\alpha : \alpha \notin K\} \times Cl_\omega(V_{\alpha_1}) \cap A_{\alpha_1} \times \dots \times Cl_\omega(V_{\alpha_n}) \cap A_{\alpha_n} \\ &= \langle Cl_\omega(V_{\alpha_1}), \dots, Cl_\omega(V_{\alpha_n}) \rangle \cap \Pi\{A_\alpha : \alpha \in \mathcal{A}\} \neq \emptyset. \end{aligned}$$

By Theorem 3.3, $Cl_\omega(\langle V_{\alpha_1}, \dots, V_{\alpha_n} \rangle) \cap \Pi\{A_\alpha : \alpha \in \mathcal{A}\} \neq \emptyset$. Hence $x \in Cl_{\theta_\omega}(\Pi\{A_\alpha : \alpha \in \mathcal{A}\})$. $\square$

Theorem 3.9. *Let $X = \Pi\{X_i : i = 1, 2, \dots, n\}$ and $Y = \Pi\{Y_i : i = 1, 2, \dots, n\}$ be product spaces, and for each $i \in \{1, 2, \dots, n\}$, let $f_i : X_i \rightarrow Y_i$ be a function. If $f : X \rightarrow Y$ defined by $f(\langle x_i \rangle) = \langle f_i(x_i) \rangle$ is θ_ω -continuous on X , then each f_i is θ_ω -continuous on X_i .*

Proof. Let O_i be an open set in Y_i . For each $i \in \{1, 2, \dots, n\}$, let $a_i \in f_i^{-1}(O_i) = G_i$. Since each O_i is open in Y_i , $O = \langle O_1, \dots, O_n \rangle$ is open in Y . Since f is θ_ω -continuous on X ,

$$f^{-1}(O) = f^{-1}(\langle O_1, \dots, O_n \rangle) = \langle f^{-1}(O_1), \dots, f^{-1}(O_n) \rangle = \langle G_1, \dots, G_n \rangle$$

is θ_ω -open in X and $x = \langle a_i \rangle \in f^{-1}(O) = \langle G_1, \dots, G_n \rangle$. It follows that there exists an open set $U = \langle U_1, \dots, U_n \rangle$ containing x such that $Cl_\omega(U) \subseteq f^{-1}(O)$. By Theorem 3.3, $\langle Cl_\omega(U_1), \dots, Cl_\omega(U_n) \rangle \subseteq \langle G_1, \dots, G_n \rangle$. This means that $Cl_\omega(U_i) \subseteq G_i = f_i^{-1}(O_i)$ for each $i \in \{1, 2, \dots, n\}$. Hence, each $f_i^{-1}(O_i)$ is θ_ω -open in X_i . Thus, each f_i is θ_ω -continuous on X_i . $\square$

4. θ_ω -Connectedness

This section provides a characterization of θ_ω -connectedness and its relationship to connected, θ -connected, ω -connected, and ω_θ -connected topological spaces. Denote by $\mathcal{D}$, the topological space $\{0, 1\}$ with the discrete topology.

The proof of the following lemma is straightforward, hence omitted.

Lemma 4.1. *Let X be any topological space and $\chi_A : X \rightarrow \mathcal{D}$ the characteristic function of a subset A of X . Then χ_A is θ_ω -continuous if and only if A is both θ_ω -open and θ_ω -closed.*

Theorem 4.2. *Let X be a topological space. Then the following statements are equivalent:*

- (i) X is θ_ω -connected.
- (ii) The only subsets of X that are both θ_ω -open and θ_ω -closed are $\emptyset$ and X .
- (iii) No θ_ω -continuous function from X to $\mathcal{D}$ is surjective.

Proof. (i) $\Rightarrow$ (ii): Let $G \subseteq X$ which is both θ_ω -open and θ_ω -closed. Then $X \setminus G$ is also both θ_ω -open and θ_ω -closed. Moreover, $X = G \cup (X \setminus G)$. Since X is θ_ω -connected, either $G = \emptyset$ or $G = X$.

(ii) $\Rightarrow$ (iii): Suppose that $f : X \rightarrow \mathcal{D}$ is a θ_ω -continuous surjection. Then $f^{-1}(\{0\}) \neq \emptyset, X$. Since $\{0\}$ is open and closed in $\mathcal{D}$, by Theorem 3.1, $f^{-1}(\{0\})$ is both θ_ω -open and θ_ω -closed in X . This is a contradiction.

(iii) $\Rightarrow$ (i): If $X = A \cup B$, where A and B are disjoint nonempty θ_ω -open sets, then A and B are also θ_ω -closed sets. Consider the characteristic function $\chi_A : X \rightarrow \mathcal{D}$ of $A \subseteq X$. By Lemma 4.1, χ_A is θ_ω -continuous. This is a contradiction. Thus, X is θ_ω -connected. $\square$

Remark 4.3. It is worth to notice that θ -connectedness is equivalent to connectedness [18].

Remark 4.4. [22, p.839] Let X be a topological space. If X is ω -connected, then X is ω_θ -connected, but not conversely.

Remark 4.5. Let X be a topological space. If X is ω_θ -connected, then X is θ -connected, but not conversely.

Since every θ -open is ω_θ -open, ω_θ -connectedness implies θ -connectedness. To show that a θ -connected space is not necessarily an ω_θ -connected space, consider $X = \{a, b, c, d\}$ with topology $\mathcal{T} = \{\emptyset, X, \{a\}, \{d\}, \{a, d\}, \{a, b\}, \{a, b, d\}\}$. Then X is θ -connected. Since $\{a, c, d\}$ and $\{b\}$ are disjoint ω_θ -open sets with $X = \{a, c, d\} \cup \{b\}$, it follows that X is ω_θ -disconnected.

Theorem 4.6. Let X be a topological space. Then X is θ_ω -connected if and only if X is θ -connected.

Proof. Since every θ -open is θ_ω -open, θ_ω -connectedness implies θ -connectedness.

Conversely, suppose that X is θ -connected, which is also connected by Remark 4.3. Hence, X is not the union of two nonempty disjoint open sets in X . Since every θ_ω -open is open, X cannot be the union of two nonempty disjoint θ_ω -open sets. Hence, X is θ_ω -connected. $\square$

In view of Remark 4.3 and Theorem 4.6, the following result follows.

Corollary 4.7. Let X be a topological space. Then X is θ_ω -connected if and only if X is connected.

Remark 4.8. The following diagram holds for any topological space.

$$\begin{array}{ccc} \omega_\theta\text{-connected} & \longleftarrow & \omega\text{-connected} \\ \downarrow & & \\ \theta\text{-connected} & \longleftrightarrow \theta_\omega\text{-connected} \longleftrightarrow & \text{connected} \end{array}$$

Acknowledgment. The authors would like to thank the referee for his helpful comments for the improvement of this paper.

References

- [1] S. Al Ghour and B. Irshedat, The topology of θ_ω -open sets, FILOMAT, 31 (2017), 5369-5377.
- [2] T.A. Al-Hawary, On supper continuity of topological spaces, MATEMATIKA, 21 (2005), 43-49.
- [3] K. Al-Zoubi and K. Al-Nashef, The topology of ω -open subsets, Al-Manarah Journal, 9 (2003), 169-179.
- [4] H. Aljarrah, M. Md Noorani, and T. Noiri, Contra $\omega\beta$ -continuity, Bol. Soc. Parana. Mat., 32 (2014), 9-22.
- [5] H. Aljarrah, M. Md Noorani, and T. Noiri, On generalized $\omega\beta$ -closed sets, Missouri J. Math. Sci., 26 (2014), 70-87.
- [6] M. Caldas, S. Jafari, and M.M. Kovar, Some properties of θ -open sets, Divulg. Mat., 12 (2004), 161-169.
- [7] M. Caldas, S. Jafari, and R.M. Latif, Sobriety via θ -open sets, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.), 56 (2010), 163-167.
- [8] C. Carpintero, N. Rajesh, E. Rosas, and S. Saranyasri, Properties of faintly ω -continuous functions, Bol. Mat., 20 (2013), 135-143.
- [9] C. Carpintero, N. Rajesh, E. Rosas, and S. Saranyasri, Some properties of upperlower ω -continuous multifunctions, Sci. Stud. Res. Ser. Math. Inform., 23 (2013), 35-55.

- [10] C. Carpintero, N. Rajesh, E. Rosas, and S. Saranyasri, On upper and lower almost contra- ω -continuous multifunctions, *Ital. J. Pure Appl. Math.*, 32 (2014), 445-460.
- [11] C. Carpintero, N. Rajesh, E. Rosas, and S. Saranyasri, Somewhat ω -continuous functions, *Sarajevo J. Math.*, 11 (2015), 131-137.
- [12] C. Carpintero, N. Rajesh, E. Rosas, and S. Saranyasri, Upper and lower ω -continuous multifunctions, *Afr. Mat.*, 26 (2015), 399-405.
- [13] H. Darwesh, Between preopen and open sets in topological spaces, *Thai J. Math.*, 11 (2013), 143-155.
- [14] R.F. Dickman Jr. and J.R. Porter, θ -closed subsets of Hausdorff spaces, *Pacific J. Math.*, 59 (1975), 407-415.
- [15] R.F. Dickman Jr. and J.R. Porter, θ -perfect and θ -absolutely closed functions, *Illinois J. Math.*, 21 (1977), 42-60.
- [16] E. Ekici, S. Jafari, and R. M. Latif, On a finer topological space than τ_θ and some maps, *Ital. J. Pure Appl. Math.*, 27 (2010), 293-304.
- [17] H.Z. Hdeib, ω -closed mappings, *Rev. Colombiana Mat.*, 1-2 (1982), 65-78.
- [18] S. Jafari, Some properties of quasi- θ -continuous functions, *Far East J. Math. Sci.*, 6 (1998), 689-696.
- [19] D.S. Jankovic, θ -regular spaces, *Internat. J. Math. & Math. Sci.*, 8 (1986), 615-619.
- [20] J.E. Joseph, θ -closure and θ -subclosed graphs, *Math. Chronicle*, 8 (1979), 99-117.
- [21] M.M. Kovar, On θ -regular spaces, *Internat. J. Math. & Math. Sci.*, 17 (1994), 687-692.
- [22] M. Labendia and J.A. Sasam, On ω -connectedness and ω -continuity in the product space, *European J. of Pure & Appl. Math.*, 11 (2018), 834-843.
- [23] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Month.*, 70 (1963), 36-41.
- [24] P.E. Long and L.L. Herrington, The τ_θ -topology and faintly continuous functions, *Kyungpook Math. J.*, 22 (1982), 7-14.
- [25] T. Noiri and S. Jafari, Properties of (θ, s) -continuous functions, *Topology Appl.*, 123 (2002), 167-179.
- [26] N. Velicko, H-closed topological spaces, *Trans. AMS.*, 78 (1968), 103-118.

(Lyle Leon L. Butanas and Mhelmar A. Labendia) DEPARTMENT OF MATHEMATICS AND STATISTICS, MINDANAO STATE UNIVERSITY- ILIGAN INSTITUTE OF TECHNOLOGY, 9200 ILIGAN CITY, PHILIPPINES.

E-mail address, Lyle Leon L. Butanas: lyleleon.butanas@g.msuiit.edu.ph

E-mail address, Mhelmar A. Labendia: mhelmar.labendia@g.msuiit.edu.ph