

ON THE TANGENT BUNDLE OF A NORDEN MANIFOLD ENDOWED WITH A WEYL CONNECTION

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Abstract. The aim of this paper is to study the tangent bundle of Norden manifolds endowed with the complete lift of a Weyl connection. Firstly, the condition for an almost Norden manifold of being a Norden manifold endowed with the Weyl connection is investigated. Finally, the condition for the tangent bundle of an almost Norden manifold of being a Norden manifold endowed with the Weyl connection is investigated.

1. Introduction

The tangent bundle of a differentiable manifold has an important role for providing many new problems in the study of modern differential geometry. Some studies about metric and connection have been built on the tangent bundle of a differentiable manifold by Yano and Davies [23]. Some extension of metric and connection have been studied on tangent bundle by many authors [1, 4, 5, 8, 14].

H. Weyl introduced unified field theory to study a generalization of Riemannian geometry in 1918 [22]. Weyl's theory provided mathematicians to study a new topic. There are many studies devoted to Weyl structure described as a conformal class of metrics preserved by Weyl connection. Folland showed that a Weyl structure is equivalent to a connection on a certain bundle and derived a characterization of Weyl structures in terms of their induced linear connection [6]. Hall has made a study of Weyl connection and their associated holonomy groups [9]. Bejan and Gül have constructed a Weyl structure on tangent bundle of Weyl manifold by using the Sasaki metric [2]. Tarafdar and Kundu have studied different types of geometric properties on almost Hermitian manifolds admitting Einstein-Weyl connection [21].

Since 1960, one of the fields in differential geometry that has developed rapidly is the theory of Norden manifolds. Norden investigated the B-metric with respect to almost complex structure on four-dimensional pseudo-Riemannian neutral space [13]. In some studies, different names (anti-Hermitian, pure, Norden) are used for B-metric

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[4, 7, 11, 12, 14, 19]. If (M_{2n}, φ) is an almost complex manifold with Norden metric g , we say that (M_{2n}, φ, g) is an almost Norden manifold. If φ is integrable, we say that (M_{2n}, φ, g) is a Norden manifold. In this study, we investigate the Weyl connection with respect to the complete lift of Riemannian connection on the tangent bundle of an almost Norden manifold. Under a condition, we show that the quadruple $(M_{2n}, \varphi, g, \bar{\nabla})$ with Weyl connection $\bar{\nabla}$ is a Norden manifold. Finally, we show that the quadruple $(TM_{2n}, {}^C\varphi, {}^Cg, {}^C\bar{\nabla})$ with the complete lift of Weyl connection $\bar{\nabla}$ is a Norden manifold. To avoid repetition, manifolds, tensor fields, linear connections are assumed to be differentiable and of class C^∞ . We denote by $\mathfrak{S}_s^r(M_{2n})$ the set of all differentiable tensor fields of type (r, s) on M_{2n} .

2. Preliminaries

2.1. Norden Manifold. Let (M_{2n}, φ, g) be a $2n$ -dimensional almost complex manifold with Norden metric, i.e., φ is an almost complex structure and g is a Norden metric on M_{2n} such that:

$$\varphi^2 X + X = 0, \quad g(\varphi X, \varphi Y) = -g(X, Y) \quad (2.1)$$

for any vector fields $X, Y \in \mathfrak{S}_0^1(M_{2n})$. The triple (M_{2n}, φ, g) is called an almost Norden manifold in many studies [10, 11, 15, 17]. Let ∇ be the Riemannian connection on M_{2n} . The tensor field F of type $(0, 3)$ on M_{2n} is defined by

$$F(X, Y, Z) = g((\nabla_X \varphi)Y, Z). \quad (2.2)$$

It has the following symmetries

$$F(X, Y, Z) = F(X, Z, Y) = F(X, \varphi Y, \varphi Z). \quad (2.3)$$

Then the quadruple $(M_{2n}, \varphi, g, \nabla)$ is called complex manifold with Norden metric, in another words it is called Norden manifold, if

$$F(X, Y, \varphi Z) + F(Y, Z, \varphi X) + F(Z, X, \varphi Y) = 0 \quad (2.4)$$

or equivalently $N = 0$, where

$$N(X, Y) = [\varphi X, \varphi Y] - [X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] \quad (2.5)$$

for any vector fields $X, Y, Z \in \mathfrak{S}_0^1(M_{2n})$ [7].

2.2. Weyl Connection. A linear connection $\bar{\nabla}$ on M_{2n} is called a Weyl connection if it given as

$$\bar{\nabla}_X Y = \nabla_X Y - \frac{1}{2}(\omega(X)Y + \omega(Y)X - g(X, Y)P) \quad (2.6)$$

and satisfies

$$(\bar{\nabla}_X g)(Y, Z) = \omega(X)g(Y, Z) \quad (2.7)$$

for any vector fields $X, Y, Z \in \mathfrak{S}_0^1(M_{2n})$, where ∇ is a Riemannian connection and P is a vector field defined by $g(P, X) = \omega(X)$. The Weyl connection is a symmetric recurrent metric connection. The torsion tensor of Weyl connection is equal to zero, i.e.

$$\bar{T}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0. \quad (2.8)$$

The Weyl connection has been studied by many authors [3, 9, 21].

2.3. Tangent Bundle. Let M_{2n} be a differentiable manifold and $T_P(M_{2n})$ the tangent space at any point $P \in M_{2n}$, i.e., $T_P(M_{2n})$ the tangent space is set of all tangent vectors of M_{2n} at the point P . Then the set $TM_{2n} = \bigcup_{P \in M_{2n}} T_P(M_{2n})$ is called the total space of the tangent bundle of M_{2n} . The local coordinate system (x^i, y^i) induced from the base coordinate (x^i) is defined on tangent bundle $T(M_{2n})$. For any point $\tilde{P} \in TM_{2n}$, the compatibility $\tilde{P} \rightarrow P$ determines the bundle projection $\pi : TM_{2n} \rightarrow M_{2n}$, i.e., $\pi(\tilde{P}) = P$. The set $\pi^{-1}(P)$ is called the fibre over $P \in M_{2n}$ and M_{2n} is called the base space.

The composition of two maps $\pi : TM_{2n} \rightarrow M_{2n}$ and $f : M_{2n} \rightarrow \mathbb{R}$ defined by ${}^V f = f \circ \pi$ is called the vertical lift of $f \in \mathfrak{S}_0^0(M_{2n})$. For any point $\tilde{P}$ according to induced coordinates (x^i, y^i) , the equation

$${}^V f(\tilde{P}) = {}^V f(x, y) = f \circ \pi(\tilde{P}) = f(P) = f(x) \quad (2.9)$$

is obtained. The equation (2.9) show that the value of ${}^V f(\tilde{P})$ is constant along each fibre $T_P(M_{2n})$ and equal to the value $f(P)$ of f .

For $f \in \mathfrak{S}_0^0(M_{2n})$, the complete lift of f is defined by

$${}^C f = \iota(df) = y^i \partial_i f \quad (2.10)$$

with respect to the induced coordinates (x^i, y^i) on TM_{2n} .

Let $f, t \in \mathfrak{S}_0^0(M_{2n})$, $X \in \mathfrak{S}_0^1(M_{2n})$, $\omega \in \mathfrak{S}_1^0(M_{2n})$, $\varphi \in \mathfrak{S}_1^1(M_{2n})$, $g \in \mathfrak{S}_2^0(M_{2n})$, $T \in \mathfrak{S}_2^1(M_{2n})$ be a function, a vector field, a 1-form, type- (1, 1), type-(0, 2), type-(1, 2) tensor field, respectively. We denote, respectively, by ${}^V f$, ${}^V X$, ${}^V \omega$, ${}^V \varphi$, ${}^V g$, ${}^V T$ their vertical lifts and by ${}^C f$, ${}^C X$, ${}^C \omega$, ${}^C \varphi$, ${}^C g$, ${}^C T$ their complete lifts. This lifts have the properties [24]:

$$\begin{aligned} [{}^C X, {}^C Y] &= {}^C [X, Y]; & {}^C \varphi({}^C X) &= {}^C (\varphi(X)) \\ {}^V \omega({}^C X) &= {}^V (\omega(X)); & {}^C \omega({}^C X) &= {}^C (\omega(X)) \\ {}^C g({}^V X, {}^C Y) &= {}^C g({}^C X, {}^V Y) = {}^V (g(X, Y)); & {}^C g({}^C X, {}^C Y) &= {}^C (g(X, Y)) \\ {}^C \nabla_{{}^C X} {}^C Y &= {}^C (\nabla_X Y); & {}^C \nabla_{{}^C X} {}^V Y &= {}^V (\nabla_X Y) \\ {}^C T({}^C X, {}^C Y) &= {}^C (T(X, Y)); & {}^C f \cdot {}^V t + {}^V f \cdot {}^C t &= {}^C (ft). \end{aligned} \quad (2.11)$$

3. Main Results

3.1. Complete Lift of Weyl Connection on a Tangent Bundle of Norden Manifold. Let the triple (M_{2n}, φ, g) be an almost Norden manifold and the triple $(TM_{2n}, {}^C \varphi, {}^C g)$ be its tangent bundle where the complete lift of φ and g are ${}^C \varphi$ and ${}^C g$, respectively, satisfying

$$({}^C \varphi)^2 {}^C X + {}^C X = 0, \quad {}^C g({}^C (\varphi X), {}^C (\varphi Y)) = - {}^C g({}^C X, {}^C Y). \quad (3.12)$$

Let ${}^C\nabla$ be complete lift of the Riemannian connection on TM_{2n} . Then the quadruple $(TM_{2n}, {}^C\varphi, {}^Cg, {}^C\nabla)$ is called Norden manifold endowed with complete lift of the Riemannian connection if

$${}^CN({}^CX, {}^CY) = 0 \quad (3.13)$$

for all $X, Y \in \mathfrak{S}_0^1(M_{2n})$, where we denoted by ${}^CN$ the Nijenhuis tensor field of $(TM_{2n}, {}^C\varphi, {}^Cg, {}^C\nabla)$.

Theorem 3.1. *Let $\bar{\nabla}$ be a Weyl connection with respect to the Riemannian connection ∇ on almost Norden Manifold (M_{2n}, φ, g) . Then ${}^C\bar{\nabla}$ is also a Weyl connection with respect to the complete lift of the Riemannian connection on $(TM_{2n}, {}^C\varphi, {}^Cg)$.*

Proof. Applying the complete lift to both sides of the equation (2.6), we obtain

$$\begin{aligned} {}^C(\bar{\nabla}_X Y) &= {}^C(\nabla_X Y) - \frac{1}{2} {}^C(\omega(X) Y) - \frac{1}{2} {}^C(\omega(Y) X) + \frac{1}{2} {}^C(g(X, Y) P) \\ {}^C(\bar{\nabla}_X Y) &= {}^C(\nabla_X Y) - \frac{1}{2} {}^C(\omega(X)) {}^VY - \frac{1}{2} {}^V(\omega(X)) {}^CY \\ &\quad - \frac{1}{2} {}^C(\omega(Y)) {}^VX - \frac{1}{2} {}^V(\omega(Y)) {}^CX \\ &\quad + \frac{1}{2} {}^C(g(X, Y)) {}^VP + \frac{1}{2} {}^V(g(X, Y)) {}^CP \\ {}^C\bar{\nabla}_{cX} {}^CY &= {}^C\nabla_{cX} {}^CY - \frac{1}{2} {}^C\omega({}^CX) {}^VY - \frac{1}{2} {}^V\omega({}^CX) {}^CY \\ &\quad - \frac{1}{2} {}^C\omega({}^CY) {}^VX - \frac{1}{2} {}^V\omega({}^CY) {}^CX \\ &\quad + \frac{1}{2} {}^Cg({}^CX, {}^CY) {}^VP + \frac{1}{2} {}^Cg({}^VX, {}^CY) {}^CP. \end{aligned} \quad (3.14)$$

From (2.11) it follows that

$$\begin{aligned} {}^C\bar{\nabla}_{cY} {}^CX &= {}^C\nabla_{cY} {}^CX - \frac{1}{2} {}^C\omega({}^CY) {}^VX - \frac{1}{2} {}^V\omega({}^CY) {}^CX \\ &\quad - \frac{1}{2} {}^C\omega({}^CX) {}^VY - \frac{1}{2} {}^V\omega({}^CX) {}^CY \\ &\quad + \frac{1}{2} {}^Cg({}^CY, {}^CX) {}^VP + \frac{1}{2} {}^Cg({}^VY, {}^CX) {}^CP. \end{aligned} \quad (3.15)$$

Then using the equations (3.14) and (3.15) we get the complete lift of the torsion tensor of the Weyl connection

$$\begin{aligned} {}^C\bar{T}({}^CX, {}^CY) &= {}^C\bar{\nabla}_{cX} {}^CY - {}^C\bar{\nabla}_{cY} {}^CX - [{}^CX, {}^CY] \\ &= {}^C\nabla_{cX} {}^CY - {}^C\nabla_{cY} {}^CX - [{}^CX, {}^CY] \\ &= {}^CT({}^CX, {}^CY) \\ &= {}^C(T(X, Y)) \\ &= 0. \end{aligned} \quad (3.16)$$

Then we obtain

$$\begin{aligned}
& {}^Cg({}^C\overline{\nabla}_{{}^C X} {}^C Y, {}^C Z) + {}^Cg({}^C Y, {}^C\overline{\nabla}_{{}^C X} {}^C Z) \\
&= {}^Cg({}^C\nabla_{{}^C X} {}^C Y - \frac{1}{2} {}^C\omega({}^C X) {}^V Y - \frac{1}{2} {}^V\omega({}^C X) {}^C Y \\
&\quad - \frac{1}{2} {}^C\omega({}^C Y) {}^V X - \frac{1}{2} {}^V\omega({}^C Y) {}^C X \\
&\quad + \frac{1}{2} {}^Cg({}^C X, {}^C Y) {}^V P + \frac{1}{2} {}^Cg({}^V X, {}^C Y) {}^C P, {}^C Z) \\
&\quad + {}^Cg({}^C Y, {}^C\nabla_{{}^C X} {}^C Z - \frac{1}{2} {}^C\omega({}^C X) {}^V Z - \frac{1}{2} {}^V\omega({}^C X) {}^C Z \\
&\quad - \frac{1}{2} {}^C\omega({}^C Z) {}^V X - \frac{1}{2} {}^V\omega({}^C Z) {}^C X \\
&\quad + \frac{1}{2} {}^Cg({}^C X, {}^C Z) {}^V P + \frac{1}{2} {}^Cg({}^V X, {}^C Z) {}^C P) \\
&= {}^Cg({}^C\nabla_{{}^C X} {}^C Y, {}^C Z) + {}^Cg({}^C Y, {}^C\nabla_{{}^C X} {}^C Z) \\
&\quad - \frac{1}{2} {}^C(\omega(X)) {}^Cg({}^V Y, {}^C Z) - \frac{1}{2} {}^V(\omega(X)) {}^Cg({}^C Y, {}^C Z) \\
&\quad - \frac{1}{2} {}^C(\omega(Y)) {}^Cg({}^V X, {}^C Z) - \frac{1}{2} {}^V(\omega(Y)) {}^Cg({}^C X, {}^C Z) \\
&\quad - \frac{1}{2} {}^C(\omega(X)) {}^Cg({}^C Y, {}^V Z) - \frac{1}{2} {}^V(\omega(X)) {}^Cg({}^C Y, {}^C Z) \\
&\quad - \frac{1}{2} {}^C(\omega(Z)) {}^Cg({}^C Y, {}^V X) - \frac{1}{2} {}^V(\omega(Z)) {}^Cg({}^C Y, {}^C X) \\
&\quad + \frac{1}{2} {}^Cg({}^C X, {}^C Y) {}^Cg({}^V P, {}^C Z) + \frac{1}{2} {}^Cg({}^V X, {}^C Y) {}^Cg({}^C P, {}^C Z) \\
&\quad + \frac{1}{2} {}^Cg({}^C X, {}^C Z) {}^Cg({}^C Y, {}^V P) + \frac{1}{2} {}^Cg({}^V X, {}^C Z) {}^Cg({}^C Y, {}^C P) \\
&= {}^Cg({}^C\nabla_{{}^C X} {}^C Y, {}^C Z) + {}^Cg({}^C Y, {}^C\nabla_{{}^C X} {}^C Z) \\
&\quad - \frac{1}{2} {}^C(\omega(X)) {}^V(g(Y, Z)) - \frac{1}{2} {}^V(\omega(X)) {}^C(g(Y, Z)) \\
&\quad - \frac{1}{2} {}^C(\omega(Y)) {}^V(g(X, Z)) - \frac{1}{2} {}^V(\omega(Y)) {}^C(g(X, Z)) \\
&\quad - \frac{1}{2} {}^C(\omega(X)) {}^V(g(Y, Z)) - \frac{1}{2} {}^V(\omega(X)) {}^C(g(Y, Z)) \\
&\quad - \frac{1}{2} {}^C(\omega(Z)) {}^V(g(Y, X)) - \frac{1}{2} {}^V(\omega(Z)) {}^C(g(Y, X)) \\
&\quad + \frac{1}{2} {}^C(g({}^V X, {}^V Y)) {}^V(\omega(Z)) + \frac{1}{2} {}^V(g({}^V X, {}^V Y)) {}^C(\omega(Z)) \\
&\quad + \frac{1}{2} {}^C(g({}^V X, {}^V Z)) {}^V(\omega(Y)) + \frac{1}{2} {}^V(g({}^V X, {}^V Z)) {}^C(\omega(Y)) \\
&= {}^Cg({}^C\nabla_{{}^C X} {}^C Y, {}^C Z) + {}^Cg({}^C Y, {}^C\nabla_{{}^C X} {}^C Z) \\
&\quad - {}^C(\omega(X)) {}^V(g(Y, Z)) - {}^V(\omega(X)) {}^C(g(Y, Z)) \\
&= {}^Cg({}^C\nabla_{{}^C X} {}^C Y, {}^C Z) + {}^Cg({}^C Y, {}^C\nabla_{{}^C X} {}^C Z) - {}^C(\omega(X)) {}^V(g(Y, Z))
\end{aligned}$$

$$= ({}^C X) {}^C g ({}^C Y, {}^C Z) - {}^C (\omega(X) g(Y, Z)). \quad (3.17)$$

And we write following equation

$$\begin{aligned} ({}^C X) {}^C g ({}^C Y, {}^C Z) &= ({}^C \bar{\nabla}_{{}^C X} {}^C g) ({}^C Y, {}^C Z) \\ &+ {}^C g ({}^C \bar{\nabla}_{{}^C X} {}^C Y, {}^C Z) \\ &+ {}^C g ({}^C Y, {}^C \bar{\nabla}_{{}^C X} {}^C Z). \end{aligned} \quad (3.18)$$

By taking into account the relations (3.18) and (2.1) we obtain from (3.17) the condition (3.19)

$$({}^C \bar{\nabla}_{{}^C X} {}^C g) ({}^C Y, {}^C Z) = {}^C (\omega(X) g(Y, Z)). \quad (3.19)$$

From (3.15) and (3.19) it follows that ${}^C \bar{\nabla}$ is a Weyl connection with respect to ${}^C \nabla$ on $(TM_{2n}, {}^C \varphi, {}^C g, {}^C \nabla)$. $\square$

Now we show that the equation (2.3) is valid on the almost Norden manifold with the Weyl connection $\bar{\nabla}$.

Theorem 3.2. *Let g be a Norden metric on the almost Norden manifold with the Weyl connection $\bar{\nabla}$. Then*

$$g((\bar{\nabla}_X \varphi) Y, Z) = g(Y, (\bar{\nabla}_X \varphi) Z).$$

Proof. We will show that the equation (2.3) defined for Riemannian connection ∇ is provided for Weyl connection $\bar{\nabla}$ on almost Norden Manifold (M_{2n}, φ, g) . By using (2.1) and

$$Xg(Y, Z) = (\bar{\nabla}_X g)(Y, Z) + g(\bar{\nabla}_X Y, Z) + g(Y, \bar{\nabla}_X Z) \quad (3.20)$$

from $Xg(\varphi Y, Z) = Xg(Y, \varphi Z)$, we have

$$\begin{aligned} (\bar{\nabla}_X g)(\varphi Y, Z) + g(\bar{\nabla}_X \varphi Y, Z) &= (\bar{\nabla}_X g)(Y, \varphi Z) + g(\bar{\nabla}_X Y, \varphi Z) \\ &+ g(\varphi Y, \bar{\nabla}_X Z) + g(Y, \bar{\nabla}_X \varphi Z) \end{aligned}$$

$$\begin{aligned} \omega(X)g(\varphi Y, Z) + g(\bar{\nabla}_X \varphi Y, Z) &= \omega(X)g(Y, \varphi Z) + g(\bar{\nabla}_X Y, \varphi Z) \\ &+ g(\varphi Y, \bar{\nabla}_X Z) + g(Y, \bar{\nabla}_X \varphi Z) \end{aligned}$$

$$\begin{aligned} \omega(X)g(\varphi Y, Z) + g(\bar{\nabla}_X \varphi Y, Z) &= \omega(X)g(\varphi Y, Z) + g(\bar{\nabla}_X Y, \varphi Z) \\ &+ g(\varphi Y, \bar{\nabla}_X Z) + g(Y, \bar{\nabla}_X \varphi Z) \end{aligned}$$

$$g(\bar{\nabla}_X \varphi Y, Z) + g(\varphi Y, \bar{\nabla}_X Z) = g(\bar{\nabla}_X Y, \varphi Z) + g(Y, \bar{\nabla}_X \varphi Z)$$

$$g(\varphi Y, \bar{\nabla}_X Z) - g(Y, \bar{\nabla}_X \varphi Z) = g(\bar{\nabla}_X Y, \varphi Z) - g(\bar{\nabla}_X \varphi Y, Z)$$

$$g(Y, \varphi(\bar{\nabla}_X Z)) - g(Y, \bar{\nabla}_X \varphi Z) = g(\varphi(\bar{\nabla}_X Y), Z) - g(\bar{\nabla}_X \varphi Y, Z)$$

$$g(Y, \varphi(\bar{\nabla}_X Z) - \bar{\nabla}_X \varphi Z) = g(\varphi(\bar{\nabla}_X Y) - \bar{\nabla}_X \varphi Y, Z)$$

$$g(Y, (\bar{\nabla}_X \varphi) Z) = g((\bar{\nabla}_X \varphi) Y, Z) \quad (3.21)$$

The equation (3.21) shows that $\bar{F}(X, Y, Z) = \bar{F}(X, Z, Y)$ where $\bar{F}(X, Y, Z) = g((\bar{\nabla}_X \varphi) Y, Z)$. Similarly, the equation $\bar{F}(X, Y, Z) = \bar{F}(X, \varphi Y, \varphi Z)$ is shown. $\square$

Theorem 3.3. *Let $(M_{2n}, \varphi, g, \bar{\nabla})$ be an almost Norden manifold with Weyl connection. The Nijenhuis tensor $\bar{N}$ with respect to the Weyl connection $\bar{\nabla}$ coincides with the Nijenhuis tensor N with respect to the Riemannian connection ∇ .*

Proof. The Nijenhuis tensor $N(X, Y)$ on an almost Norden manifold with respect to the Riemannian connection is given in equation (2.5). Another form of Nijenhuis tensor with respect to Riemannian connection is written by

$$\begin{aligned} N(X, Y) &= \nabla_{\varphi X} \varphi Y - \nabla_{\varphi Y} \varphi X - \nabla_X Y + \nabla_Y X \\ &\quad - \varphi \nabla_{\varphi X} Y + \varphi \nabla_Y \varphi X - \varphi \nabla_X \varphi Y + \varphi \nabla_{\varphi Y} X. \end{aligned} \quad (3.22)$$

In the same way, we define that the Nijenhuis tensor $\bar{N}(X, Y)$ on an almost Norden manifold with respect to the Weyl connection $\bar{\nabla}$ is written by

$$\begin{aligned} \bar{N}(X, Y) &= \bar{\nabla}_{\varphi X} \varphi Y - \bar{\nabla}_{\varphi Y} \varphi X - \bar{\nabla}_X Y + \bar{\nabla}_Y X \\ &\quad - \varphi \bar{\nabla}_{\varphi X} Y + \varphi \bar{\nabla}_Y \varphi X - \varphi \bar{\nabla}_X \varphi Y + \varphi \bar{\nabla}_{\varphi Y} X. \end{aligned} \quad (3.23)$$

By using the equation (2.6), we have

$$\begin{aligned} \bar{N}(X, Y) &= \nabla_{\varphi X} \varphi Y - \nabla_{\varphi Y} \varphi X - \nabla_X Y + \nabla_Y X \\ &\quad - \varphi \nabla_{\varphi X} Y + \varphi \nabla_Y \varphi X - \varphi \nabla_X \varphi Y + \varphi \nabla_{\varphi Y} X \\ &\quad - \frac{1}{2} \omega(\varphi X) \varphi Y - \frac{1}{2} \omega(\varphi Y) \varphi X + \frac{1}{2} g(\varphi X, \varphi Y) P \\ &\quad + \frac{1}{2} \omega(\varphi Y) \varphi X - \frac{1}{2} \omega(\varphi X) \varphi Y + \frac{1}{2} g(\varphi Y, \varphi X) P \\ &\quad + \frac{1}{2} \omega(X) Y + \frac{1}{2} \omega(Y) X - \frac{1}{2} g(X, Y) P \\ &\quad - \frac{1}{2} \omega(Y) X - \frac{1}{2} \omega(X) Y + \frac{1}{2} g(Y, X) P \\ &\quad + \frac{1}{2} \varphi \omega(\varphi X) Y + \frac{1}{2} \varphi \omega(Y) \varphi X - \frac{1}{2} \varphi g(\varphi X, Y) P \\ &\quad - \frac{1}{2} \varphi \omega(Y) \varphi X - \frac{1}{2} \varphi \omega(\varphi X) Y + \frac{1}{2} g(Y, \varphi X) P \\ &\quad + \frac{1}{2} \varphi \omega(X) \varphi Y + \frac{1}{2} \varphi \omega(\varphi Y) X - \frac{1}{2} \varphi g(X, \varphi Y) P \\ &\quad - \frac{1}{2} \omega(Y) X - \frac{1}{2} \omega(X) Y + \frac{1}{2} g(Y, X) P \\ &= N(X, Y). \end{aligned} \quad (3.24)$$

We see that the Nijenhuis tensor $\bar{N}$ coincides with the Nijenhuis tensor N . $\square$

We know that $(M_{2n}, \varphi, g, \nabla)$ is a Norden manifold if the Nijenhuis tensor N vanishes. The Nijenhuis tensor N being zero indicates that φ is integrable. In other words, the almost Norden manifold $(M_{2n}, \varphi, g, \nabla)$ is a Norden manifold if φ is integrable. Hence, we have

Theorem 3.4. *An almost Norden manifold endowed with the Weyl connection $\bar{\nabla}$ is a Norden manifold if and only if it is a Norden manifold with respect to the Riemannian connection ∇ .*

The Nijenhuis tensor ${}^CN$ with respect to the Riemannian connection ${}^C\nabla$ on $(TM_{2n}, {}^C\varphi, {}^Cg, {}^C\nabla)$ is given by

$$\begin{aligned} {}^CN({}^CX, {}^CY) &= {}^C\nabla_{{}^C(\varphi X)} {}^C(\varphi Y) - {}^C\nabla_{{}^C(\varphi Y)} {}^C(\varphi X) - {}^C\nabla_{{}^CX} {}^CY \\ &\quad + {}^C\nabla_{{}^CY} {}^CX - {}^C\varphi {}^C\nabla_{{}^C(\varphi X)} {}^CY + {}^C\varphi {}^C\nabla_{{}^CY} {}^C(\varphi X) \\ &\quad - {}^C\varphi {}^C\nabla_{{}^CX} {}^C(\varphi Y) + {}^C\varphi {}^C\nabla_{{}^C(\varphi Y)} {}^CX. \end{aligned} \quad (3.25)$$

Next, we are going to give the relationship between Nijenhuis tensor ${}^C\bar{N}$ with respect to the Weyl connection ${}^C\bar{\nabla}$ and Nijenhuis tensor ${}^CN$ with respect to the Riemannian connection ${}^C\nabla$.

Theorem 3.5. *Let $(TM_{2n}, {}^C\varphi, {}^Cg, {}^C\bar{\nabla})$ be the tangent bundle of an almost Norden manifold with the Weyl connection. The Nijenhuis tensor ${}^C\bar{N}$ with respect to the Weyl connection ${}^C\bar{\nabla}$ coincides with the Nijenhuis tensor ${}^CN$ with respect to the Riemannian connection ${}^C\nabla$.*

Proof. The Nijenhuis tensor ${}^C\bar{N}$ on tangent bundle of an almost Norden manifold with respect to the Weyl connection ${}^C\bar{\nabla}$ is given by

$$\begin{aligned} {}^C\bar{N}({}^CX, {}^CY) &= {}^C\bar{\nabla}_{{}^C(\varphi X)} {}^C(\varphi Y) - {}^C\bar{\nabla}_{{}^C(\varphi Y)} {}^C(\varphi X) - {}^C\bar{\nabla}_{{}^CX} {}^CY \\ &\quad + {}^C\bar{\nabla}_{{}^CY} {}^CX - {}^C\varphi {}^C\bar{\nabla}_{{}^C(\varphi X)} {}^CY + {}^C\varphi {}^C\bar{\nabla}_{{}^CY} {}^C(\varphi X) \\ &\quad - {}^C\varphi {}^C\bar{\nabla}_{{}^CX} {}^C(\varphi Y) + {}^C\varphi {}^C\bar{\nabla}_{{}^C(\varphi Y)} {}^CX. \end{aligned} \quad (3.26)$$

By using (2.6) and (2.11), we obtain

$${}^C\bar{N}({}^CX, {}^CY) = {}^CN({}^CX, {}^CY). \quad (3.27)$$

□

The almost Norden manifold $(TM_{2n}, {}^C\varphi, {}^Cg, {}^C\nabla)$ is a Norden manifold if ${}^C\varphi$ is integrable. Hence, we have

Corollary 3.6. The tangent bundle of an almost Norden manifold $(M_{2n}, \varphi, g, \bar{\nabla})$, where $\bar{\nabla}$ is a Weyl connection, is a Norden manifold with respect to ${}^C\bar{\nabla}$ if and only if it is a Norden manifold with respect to the Riemannian connection ${}^C\nabla$.

G defined by $G(X, Y) = g(\varphi X, Y)$ for any $X, Y \in \mathfrak{S}_0^1(M_{2n})$ is called the twin Norden metric on a Norden manifold $(M_{2n}, \varphi, g, \nabla)$ [16, 20]. The complete lift of twin Norden metric G is denoted by

$${}^CG({}^CX, {}^CY) = {}^Cg({}^C(\varphi X), {}^CY). \quad (3.28)$$

Proposition 3.7. Let G be a twin Norden metric on a Norden manifold $(M_{2n}, \varphi, g, \nabla)$. Then

$$({}^C\bar{\nabla}_{{}^CZ} {}^CG)({}^CX, {}^CY) = {}^C(\omega(Z) g(\varphi X, Y))$$

for any $X, Y, Z \in \mathfrak{S}_0^1(M_{2n})$.

Proof. For twin Norden metric G , we have

$$\begin{aligned}
 ({}^C \bar{\nabla}_{{}^C Z} {}^C G)({}^C X, {}^C Y) &= ({}^C Z) {}^C G({}^C X, {}^C Y) \\
 &\quad - {}^C G({}^C \bar{\nabla}_{{}^C Z} {}^C X, {}^C Y) \\
 &\quad - {}^C G({}^C X, {}^C \bar{\nabla}_{{}^C Z} {}^C Y) \\
 &= {}^C (\omega(Z) G(X, Y)) \\
 &= {}^C (\omega(Z) g(\varphi X, Y)). \tag{3.29}
 \end{aligned}$$

After similar computations to those from the proof of Theorem 3.1, the relation in the statement is proved. $\square$

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