

RESULTS ON UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING TWO VALUES

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Abstract. In this paper, we investigate the possible relation between two meromorphic functions $f^n f^{(k)}$ and $g^n g^{(k)}$ that share two values and obtain some results which improve and extend a result of Wang and Gao [8] and supplement some other results earlier given by Yang and Hua [11] and Fang and Qiu [3].

1. Introduction and main results

Throughout this paper, a meromorphic function always means meromorphic in the whole complex plane, unless specifically stated otherwise. Let k be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. Set $E(a, f) = \{z : f(z) - a = 0\}$, where a zero point with multiplicity k is counted k times in the set. If these zeros points are only counted once, then we denote the set by $\bar{E}(a, f)$. Let f and g be two nonconstant meromorphic functions. If $E(a, f) = E(a, g)$, then we say that f and g share the value a CM; if $\bar{E}(a, f) = \bar{E}(a, g)$, then we say that f and g share the value a IM. We denote by $E_k(a, f)$ the set of all a -points of f with multiplicities not exceeding k , where an a -point is counted according to its multiplicity. Also we denote by $\bar{E}_k(a, f)$ the set of distinct a -points of f with multiplicities not greater than k . It is assumed that the reader is familiar with the notations of Nevanlinna theory such as $T(r, f)$, $m(r, f)$, $N(r, f)$, $\bar{N}(r, f)$, $S(r, f)$ and so on, that can be found, for instance in [4, 10].

Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \cup_{a \in S} \{z : f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $E_f(S) = \cup_{a \in S} \{z : f(z) - a = 0\}$ is denoted by $\bar{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the otherhand, if $\bar{E}_f(S) = \bar{E}_g(S)$, we say that f and g share the set S IM. For a set S of distinct elements of $\mathbb{C}$ we define $E_k(S, f) = \cup_{a \in S} E_k(a, f)$.

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Due to Nevanlinna [6], it is well known that if f and g share four distinct values CM, then f is a Möbius transformation of g .

In 1997, corresponding to one famous question of Hayman [5], Yang and Hua [11] showed that similar conclusions hold for certain types of differential polynomials when they share only one value. They proved the following result.

Theorem A. Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and let $n \geq 11$ be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = 1$ or $f(z) \equiv tg(z)$ for a constant such that $t^{n+1} = 1$.

In 2002, Fang and Qiu [3] investigated meromorphic functions sharing fixed points which is an improvement of Theorem A.

Theorem B. Let $f(z)$ and $g(z)$ be two nonconstant meromorphic (entire) functions and let $n \geq 11$ ($n \geq 6$) be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share z CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1 c_2)^{n+1} c^2 = -1$ or $f(z) \equiv tg(z)$ for a constant such that $t^{n+1} = 1$.

Corresponding to transcendental meromorphic functions, Wang and Gao [8] further extended the above results as follows.

Theorem C. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions and let $a(z) (\neq 0)$ be a common small function with respect to them and let $n \geq 11$ be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share $a(z)$ CM, then either $f^n(z)f'(z)g^n(z)g'(z) \equiv a^2(z)$ or $f(z) \equiv tg(z)$ for a constant such that $t^{n+1} = 1$.

Now one may ask the following questions which are the motivation of this paper.

- (i) Can the lower bound of n be further reduced in Theorem C?
- (ii) What happens if f' is replaced by $f^{(k)}$ in Theorem C?

In this paper, we investigate the solution of the above questions. We now state the following theorems which are the main results of the paper.

Theorem 1.1. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions and let $n \geq k + 8$ be a positive integer. If $f^n(z)f^{(k)}(z)$ and $g^n(z)g^{(k)}(z)$ share $(1, 2)$, $f(z)$ and $g(z)$ share (∞, ∞) , then one of the following three cases holds:

- (1) $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$,
- (2) $f^n f^{(k)} = g^n g^{(k)}$, if $\frac{f}{g}$ is not a constant;
- (3) $f = c_3 e^{dz}$ and $g = c_4 e^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k (c_3 c_4)^{n+1} d^{2k} = 1$.

Theorem 1.2. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions and let $n \geq k + 9$ be a positive integer. If $f^n(z)f^{(k)}(z)$ and $g^n(z)g^{(k)}(z)$ share $(1, 2)$, $f(z)$ and $g(z)$ share $(\infty, 0)$, then the conclusion of Theorem 1.1 holds.

Theorem 1.3. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions and let $n \geq k + 8$ be a positive integer. If $f(z)$ and $g(z)$ share (∞, ∞) and $E_3(1, f^n(z)f^{(k)}(z)) =$

$E_3(1, g^n(z)g^{(k)}(z))$, then the conclusion of Theorem 1.1 holds.

Theorem 1.4. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions and let $n \geq k+9$ be a positive integer. If $f(z)$ and $g(z)$ share $(\infty, 0)$ and $E_3(1, f^n(z)f^{(k)}(z)) = E_3(1, g^n(z)g^{(k)}(z))$, then the conclusion of Theorem 1.1 holds.

Remark 1.1. If $k = 1$ in Theorem 1.1-Theorem 1.4, then Theorem 1.1-Theorem 1.4 improves Theorem A-Theorem C.

Remark 1.2. If we set $S_1 = \{1, w, \dots, w^{n-1}\}$, $S_2 = \{\infty\}$, we can get the following equivalent forms.

Theorem 1.5. Let S_1, S_2 be given as above. Suppose $f(z)$ and $g(z)$ be two transcendental meromorphic functions such that

$$E_{f(z)}(S_1, 2) = E_{g(z)}(S_1, 2), E_{f(z)}(S_2, \infty) = E_{g(z)}(S_2, \infty).$$

If $n \geq k+8$, then the conclusion of Theorem 1.1 holds.

Theorem 1.6. Let S_1, S_2 be given as above. Suppose $f(z)$ and $g(z)$ be two transcendental meromorphic functions such that

$$E_{f(z)}(S_1, 2) = E_{g(z)}(S_1, 2), E_{f(z)}(S_2, 0) = E_{g(z)}(S_2, 0).$$

If $n \geq k+9$, then the conclusion of Theorem 1.1 holds.

Theorem 1.7. Let S_1, S_2 be given as above. Suppose $f(z)$ and $g(z)$ be two transcendental meromorphic functions such that

$$E_3(S_1, f(z)) = E_3(S_1, g(z)), E_{f(z)}(S_2, \infty) = E_{g(z)}(S_2, \infty).$$

If $n \geq k+8$, then the conclusion of Theorem 1.1 holds.

Theorem 1.8. Let S_1, S_2 be given as above. Suppose $f(z)$ and $g(z)$ be two transcendental meromorphic functions such that

$$E_3(S_1, f(z)) = E_3(S_1, g(z)), E_{f(z)}(S_2, 0) = E_{g(z)}(S_2, 0).$$

If $n \geq k+9$, then the conclusion of Theorem 1.1 holds.

2. Preliminary Lemmas

In this section, we state some lemmas which will be required in the sequel. We will denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right).$$

Lemma 2.1 ([9]). Let f be a non-constant meromorphic function and let $a_1, a_2, \dots, a_n$ be finite complex numbers, $a_n \neq 0$. Then

$$T(r, a_n f^n + \dots + a_2 f^2 + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.2 ([12]). Suppose that $f(z)$ is a nonconstant meromorphic function in the complex plane and k is a positive integers. Then

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\bar{N}(r, f) + S(r, f). \quad (2.1)$$

Lemma 2.3 ([1]). Let F, G be two non-constant meromorphic functions. If F, G share $(1, 2)$ and (∞, k) , where $0 \leq k \leq \infty$. If $H \neq 0$, then

$$T(r, F) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \overline{N}(r, F) + \overline{N}(r, G) + \overline{N}_*(r, \infty; F, G) + S(r, F) + S(r, G),$$

where $\overline{N}_*(r, \infty; F, G)$ denotes the reduced counting function of those a -points of F whose multiplicities differ from the multiplicities of the corresponding a -points of G .

Lemma 2.4 ([1]). Let F, G be two non-constant meromorphic functions. If F, G share (∞, k) and $E_3(1, F) = E_3(1, G)$, where $0 \leq k \leq \infty$. If $H \neq 0$, then

$$T(r, F) + T(r, G) \leq 2N_2\left(r, \frac{1}{F}\right) + 2N_2\left(r, \frac{1}{G}\right) + 2\overline{N}(r, F) + 2\overline{N}(r, G) + 2\overline{N}_*(r, \infty; F, G) + S(r, F) + S(r, G),$$

where $\overline{N}_*(r, \infty; F, G)$ denotes the reduced counting function of those a -points of F whose multiplicities differ from the multiplicities of the corresponding a -points of G .

3. Proof of Theorems

Proof of Theorem 1.1. Let $F = f^n f^{(k)}$ and $G = g^n g^{(k)}$. Then it is easy to verify F and G share $(1, 2)$ and (∞, ∞) . Let H be defined as above. Suppose that $H \neq 0$. It follows from Lemma 2.3 that

$$T(r, F) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \overline{N}(r, F) + \overline{N}(r, G) + \overline{N}_*(r, \infty; F, G) + S(r, F) + S(r, G). \quad (3.1)$$

It's obvious that

$$\begin{aligned} N_2\left(r, \frac{1}{F}\right) &\leq N_2\left(r, \frac{1}{f^n f^{(k)}}\right) \\ &\leq 2\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f) \\ &= \frac{2}{n} \left(n\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)}}\right) \right) + \left(1 - \frac{2}{n}\right) N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f) \\ &\leq \frac{2}{n} N\left(r, \frac{1}{f^n f^{(k)}}\right) + \\ &\quad \left(1 - \frac{2}{n}\right) \frac{1}{n+1} \left[nN\left(r, \frac{1}{f}\right) + nk\overline{N}(r, f) + N\left(r, \frac{1}{f^{(k)}}\right) \right] + S(r, f) \\ &\leq \frac{2}{n} N\left(r, \frac{1}{f^n f^{(k)}}\right) + \left(1 - \frac{2}{n}\right) \frac{1}{n+1} N\left(r, \frac{1}{f^n f^{(k)}}\right) + \\ &\quad \left(1 - \frac{2}{n}\right) \frac{nk}{n+1} \overline{N}(r, f) + S(r, f) \\ &\leq \frac{3}{n+1} N\left(r, \frac{1}{f^n f^{(k)}}\right) + \frac{(n-2)k}{n+1} \overline{N}(r, f) + S(r, f), \end{aligned} \quad (3.2)$$

$$N_2\left(r, \frac{1}{G}\right) \leq \frac{3}{n+1}N\left(r, \frac{1}{g^n g^{(k)}}\right) + \frac{(n-2)k}{n+1}\overline{N}(r, g) + S(r, g), \quad (3.3)$$

$$\overline{N}_*(r, \infty; F, G) = 0. \quad (3.4)$$

Using (3.2), we get

$$\begin{aligned} N_2\left(r, \frac{1}{F}\right) + \overline{N}(r, F) &\leq \frac{3}{n+1}N\left(r, \frac{1}{f^n f^{(k)}}\right) + \frac{(n-2)k}{n+1}\overline{N}(r, f) + \overline{N}(r, f) + S(r, f) \\ &= \frac{3}{n+1}N\left(r, \frac{1}{f^n f^{(k)}}\right) + \frac{(n-2)k + n + 1}{n+1}\overline{N}(r, f) + S(r, f) \\ &\leq \frac{3}{n+1}N\left(r, \frac{1}{F}\right) + \frac{(n-2)k + n + 1}{(n+1)(n+k+1)}N(r, F) + S(r, f) \\ &\leq \frac{k+4}{n+k+1}T(r, F) + S(r, f), \end{aligned} \quad (3.5)$$

we have $S(r, f) = S(r, F)$. Combining with (3.5), we obtain

$$N_2\left(r, \frac{1}{F}\right) + \overline{N}(r, F) \leq \frac{k+4}{n+k+1}T(r, F) + S(r, F). \quad (3.6)$$

Similarly, we get

$$N_2\left(r, \frac{1}{G}\right) + \overline{N}(r, G) \leq \frac{k+4}{n+k+1}T(r, G) + S(r, G). \quad (3.7)$$

Combining (3.1), (3.6) and (3.7), we deduce

$$T(r, F) \leq \frac{k+4}{n+k+1}T(r, F) + \frac{k+4}{n+k+1}T(r, G) + S(r, F) + S(r, G). \quad (3.8)$$

In a similar way we can obtain

$$T(r, G) \leq \frac{k+4}{n+k+1}T(r, G) + \frac{k+4}{n+k+1}T(r, F) + S(r, F) + S(r, G). \quad (3.9)$$

Adding (3.8) and (3.9), yields

$$\frac{n-k-7}{n+k+1}\{T(r, F) + T(r, G)\} \leq S(r, F) + S(r, G). \quad (3.10)$$

Since both F and G are transcendental meromorphic functions and $n \geq k+8$, then we deduce a contradiction from (3.10). Therefore $H \equiv 0$. By integration, we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \quad (3.11)$$

where $A \neq 0$ and B are constants. From (3.11) we have

$$G = \frac{(B-A)F + (A-B-1)}{BF - B - 1}. \quad (3.12)$$

We discuss the following three cases.

Case I. Suppose that $B \neq 0, -1$. From (3.12), we have

$$\overline{N}\left(r, \frac{1}{F - \frac{B+1}{B}}\right) = \overline{N}(r, G). \quad (3.13)$$

From the second fundamental theorem, we have

$$\begin{aligned} T(r, F) &\leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \frac{B+1}{B}}\right) + S(r, F) \\ &= \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}(r, G) + S(r, F) + S(r, G) \\ &\leq \frac{3n + 2kn + 2k + 3}{(n+1)(n+k+1)}T(r, F) + \frac{1}{n+k+1}T(r, G) + S(r, F) + S(r, G). \end{aligned}$$

Similarly,

$$T(r, G) \leq \frac{3n + 2kn + 2k + 3}{(n+1)(n+k+1)}T(r, G) + \frac{1}{n+k+1}T(r, F) + S(r, F) + S(r, G).$$

Thus we obtain

$$\frac{n-k-3}{n+k+1}\{T(r, F) + T(r, G)\} \leq S(r, F) + S(r, G),$$

which contradicts with $n \geq k + 8$.

Case II. Suppose that $B = 0$. From (3.12), we have

$$G = AF - (A - 1). \quad (3.14)$$

If $A \neq 1$, from (3.14), we obtain

$$\overline{N}\left(r, \frac{1}{F - \frac{A-1}{A}}\right) = \overline{N}\left(r, \frac{1}{G}\right). \quad (3.15)$$

From the second fundamental theorem, we have

$$\begin{aligned} T(r, F) &\leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \frac{A-1}{A}}\right) + S(r, F) \\ &= \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + S(r, F) + S(r, G) \\ &\leq \frac{3n + 2kn + 2k + 3}{(n+1)(n+k+1)}T(r, F) + \frac{2kn + 2n + 2k + 2}{(n+1)(n+k+1)}T(r, G) + S(r, F) + S(r, G). \end{aligned}$$

Similarly,

$$T(r, G) \leq \frac{3n + 2kn + 2k + 3}{(n+1)(n+k+1)}T(r, G) + \frac{2kn + 2n + 2k + 2}{(n+1)(n+k+1)}T(r, F) + S(r, F) + S(r, G).$$

Thus we obtain

$$\frac{n-3k-4}{n+k+1}\{T(r, F) + T(r, G)\} \leq S(r, F) + S(r, G),$$

which contradicts with $n \geq k + 8$. Thus $A = 1$. From (3.14) we have $F = G$, that is

$$f^n f^{(k)} = g^n g^{(k)} \quad (3.16)$$

Let $h = \frac{f}{g}$. If h is constant, by putting $f = gh$ in (3.16) we get $h^{n+1} = 1$. Thus $f = tg$ for a constant t such that $t^{n+1} = 1$. If h is not constant then we must have $f^n f^{(k)} = g^n g^{(k)}$.

Case III. Suppose that $B = -1$. From (3.12), we have

$$G = \frac{(A+1)F - A}{F}. \quad (3.17)$$

If $A \neq -1$, we obtain from (3.17) that

$$\overline{N}\left(r, \frac{1}{F - \frac{A-1}{A}}\right) = \overline{N}\left(r, \frac{1}{G}\right).$$

By the same reasoning discussed in Case II, we obtain a contradiction. Hence $A = -1$. From (3.17), we get $FG = 1$, that is

$$f^n f^{(k)} g^n g^{(k)} = 1. \quad (3.18)$$

Let z_0 be a zero of f with multiplicity p , then z_0 be a zero of $f^n f^{(k)}$ with multiplicity $(n+1)p - k$. Clearly z_0 is a pole of g with multiplicity q . Note that z_0 will be a pole of $g^n g^{(k)}$ with multiplicity $(n+1)q + k$. Thus, by virtue of (3.18), we get

$$(n+1)p - k = (n+1)q + k$$

implies that

$$(n+1)(p-q) = 2k. \quad (3.19)$$

Since $n \geq 2k$, we get a contradiction from (3.19).

Hence as a result f has no zero. Similarly we can prove that g has no zero. Therefore, we conclude that f and g are meromorphic functions. By using the same argument as in Lemma 3.14 [7] and Lemma 2.6 [2], we get $f(z) = c_3 e^{dz}$ and $g(z) = c_4 e^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k (c_3 c_4)^{n+1} d^{2k} = 1$.

Proof of Theorem 1.2. Let $F = f^n f^{(k)}$ and $G = g^n g^{(k)}$. Then it is easy to verify F and G share $(1, 2)$ and $(\infty, 0)$. Let H be defined as above. Suppose that $H \not\equiv 0$. It follows from Lemma 2.3 that

$$\begin{aligned} T(r, F) &\leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \overline{N}(r, F) + \overline{N}(r, G) + \overline{N}_*(r, \infty; F, G) + S(r, F) \\ &\quad + S(r, G). \end{aligned} \quad (3.20)$$

It's obvious that

$$\overline{N}_*(r, \infty; F, G) \leq \overline{N}(r, f). \quad (3.21)$$

Combining (3.6), (3.7), (3.20) and (3.21), we deduce

$$T(r, F) \leq \frac{k+5}{n+k+1} T(r, F) + \frac{k+4}{n+k+1} T(r, G) + S(r, F) + S(r, G). \quad (3.22)$$

In a similar way we can obtain

$$T(r, G) \leq \frac{k+5}{n+k+1} T(r, G) + \frac{k+4}{n+k+1} T(r, F) + S(r, F) + S(r, G). \quad (3.23)$$

Adding (3.22) and (3.23), yields

$$\frac{n-k-8}{n+k+1} \{T(r, F) + T(r, G)\} \leq S(r, F) + S(r, G),$$

which contradicts with $n \geq k+9$. Therefore $H \equiv 0$. Similar to the arguments in Theorem 1.1, we see that Theorem 1.2 holds.

Proof of Theorem 1.3. Let $F = f^n f^{(k)}$ and $G = g^n g^{(k)}$. Then it is easy to verify F and G share $E_3(1, F) = E_3(1, G)$ and (∞, ∞) . So $\overline{N}_*(r, \infty; F, G) = 0$. Let H be defined as above. Suppose that $H \not\equiv 0$. It follows from Lemma 2.4 that

$$\begin{aligned} T(r, F) + T(r, G) &\leq 2N_2\left(r, \frac{1}{F}\right) + 2N_2\left(r, \frac{1}{G}\right) + 2\overline{N}(r, F) + 2\overline{N}(r, G) + 2\overline{N}_*(r, \infty; F, G) \\ &\quad + S(r, F) + S(r, G). \end{aligned} \quad (3.24)$$

Combining (3.6), (3.7) and (3.24), we deduce

$$\begin{aligned} T(r, F) + T(r, G) &\leq 2\left(\frac{k+4}{n+k+1}\right)\{T(r, F) + T(r, G)\} + S(r, F) + S(r, G) \\ \frac{n-k-7}{n+k+1}\{T(r, F) + T(r, G)\} &\leq S(r, F) + S(r, G), \end{aligned}$$

which contradicts with $n \geq k+8$. Therefore $H \equiv 0$. Similar to the arguments in Theorem 1.1, we see that Theorem 1.3 holds.

Proof of Theorem 1.4. Let $F = f^n f^{(k)}$ and $G = g^n g^{(k)}$. Then it is easy to verify F and G share $E_3(1, F) = E_3(1, G)$ and $(\infty, 0)$. So $\overline{N}_*(r, \infty; F, G) \leq \overline{N}(r, \infty; F) = \overline{N}(r, \infty; G)$. Let H be defined as above. Suppose that $H \not\equiv 0$. It follows from Lemma 2.4 that

$$\begin{aligned} T(r, F) + T(r, G) &\leq 2N_2\left(r, \frac{1}{F}\right) + 2N_2\left(r, \frac{1}{G}\right) + 3\overline{N}(r, F) + 3\overline{N}(r, G) + S(r, F) \\ &\quad + S(r, G). \end{aligned} \quad (3.25)$$

Combining (3.6), (3.7) and (3.25), we deduce

$$\begin{aligned} T(r, F) + T(r, G) &\leq \frac{2k+9}{n+k+1}\{T(r, F) + T(r, G)\} + S(r, F) + S(r, G) \\ \frac{n-k-8}{n+k+1}\{T(r, F) + T(r, G)\} &\leq S(r, F) + S(r, G), \end{aligned}$$

which contradicts with $n \geq k+9$. Therefore $H \equiv 0$. Similar to the arguments in Theorem 1.1, we see that Theorem 1.4 holds.

4. Open Problems

1. What happens if in Theorem 1.1-Theorem 1.8 the two transcendental meromorphic functions f and g are replaced by two transcendental entire functions?
2. Can the lower bound of n be further reduced in Theorem 1.1-Theorem 1.8 to get the same conclusion?
3. What happens if $f^n f^{(k)}$ is replaced by $f^n (f-1)^m f^{(k)}$ in Theorem 1.1-Theorem 1.8?

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