

BLOCK SEQUENCES OF RETRO BANACH FRAMES

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Abstract. In this note, we introduce block sequences for a retro Banach frame and exhibit the existence with examples and counter examples. Also, we give a necessary and sufficient condition for a block sequence of a retro Banach frame to be a retro Banach frame. We give a condition (a necessary and sufficient) under which a block sequence of an exact retro Banach frame is an exact retro Banach frame. Finally, we discuss exact retro Banach frames and prove a result related to a geometric property of the underlying space.

1. Introduction and Preliminaries

Throughout H (with inner product $\langle \cdot, \cdot \rangle$) will denote a separable Hilbert space, X a Banach space, X^* denote the conjugate space of X and $\text{Codim}_{(X^*)}[f_n]$ is the dimension of quotient spaces of $(X^*/[f_n])$. While working with families of exponentials $\{e^{i\lambda_n t}\}_{n \in \mathbb{Z}}$, Duffin and Schaeffer [6], tried to determine when they are complete or form a Riesz basis for $L^2[a, b]$ etc, and in the process defined frames. They gave the following definition of frame:

A sequence $\{x_n\}$ in a Hilbert space H (with inner product $\langle \cdot, \cdot \rangle$) is said to be a frame (Hilbert) for H , if there exist positive constants A and B such that

$$A\|x\|^2 \leq \sum_n |\langle x, x_n \rangle|^2 \leq B\|x\|^2, \quad \text{for all } x \in H \quad (1.1)$$

The positive constants A and B , respectively, are called *lower* and *upper frame bound* for the frame $\{x_n\}$ and collectively known as *frame bounds* for the frame $\{x_n\}$. But they are not unique. In fact, any pair of positive constants A and B for which the inequality (1.1), called *frame inequality* for the frame $\{x_n\}$, hold good are called frame bounds for the frame $\{x_n\}$. The supremum of all lower frame bounds and the infimum of all upper

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frame bounds (of the frame $\{x_n\}$) are called *optimal bounds* or *best bounds* for the frame $\{x_n\}$. A frame $\{x_n\}$ is said to be *tight* if we can choose $A = B$ as frame bounds. A frame $\{x_n\}$ is said to be *Parseval frame* if we can choose $A = B = 1$ as frame bounds. A frame $\{x_n\}$ for H is called an *exact frame* if removal of one x_i renders the collection $\{x_n\}_{n \neq i}$ no longer a frame for H . If $\{x_n\}$ satisfies the upper frame inequality in (1.1), then $\{x_n\}$ is called a *Bessel sequence* for H . Given a frame $\{x_n\}$ for a Hilbert space H , the operator $T : \ell^2 \rightarrow H$ given by

$$T(\{c_n\}) = \sum_{n=1}^{\infty} c_n x_n, \quad \{c_n\} \in \ell^2$$

is called the *pre-frame operator* and the adjoint operator $T^* : H \rightarrow \ell^2$ is given by

$$T^*(x) = \{\langle x, x_n \rangle\}_{n=1}^{\infty}, \quad x \in H,$$

is called the analysis operator. The operator $S = TT^* : H \rightarrow H$ given by

$$S(x) = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n, \quad x \in H,$$

is called the *frame operator* associated with the frame $\{x_n\}$.

Coifman and Weiss [4], introduced the notion of atomic decomposition for certain Function spaces. Later, the notion of atomic decomposition to certain Banach spaces was extended by Feichtinger and Gröchenig [7]. They gave the following definition of atomic decomposition:

Let X be a Banach space and let X_d be an associated Banach space of scalar-valued sequences indexed by $\mathbb{N}$. Let $\{x_n\}$ be a sequence in X and let $\{f_n\}$ be a sequence in X^* . Then, the pair $(\{f_n\}, \{x_n\})$ is called an *atomic decomposition* of X with respect to X_d , if

- (1) $\{f_n(x)\} \in X_d$, for all $x \in X$,
- (2) there exist constants A, B with $0 < A \leq B < \infty$ such that

$$A\|x\|_X \leq \|\{f_n(x)\}\|_{X_d} \leq B\|x\|_X, \quad x \in X,$$

- (3) $x = \sum f_n(x)x_n$, for all $x \in X$.

The positive constants A and B are called *atomic bounds* for the atomic decomposition $(\{f_n\}, \{x_n\})$.

The notion of Banach frame which is more general than the notion of atomic decomposition for Banach spaces was given by Gröchenig [10]. He gave the following definition of Banach frame for Banach spaces:

Let X be a non-trivial Banach space and let X_d be an associated Banach space of scalar-valued sequences indexed by $\mathbb{N}$. Let $\{f_n\}$ be a sequence in X^* and let $S : X_d \rightarrow X$ be an operator. Then the pair $(\{f_n\}, S)$ is called a *Banach frame* for X with respect to X_d , if

- (1) $\{f_n(x)\} \in X_d$, $x \in X$,
- (2) there exist constants A, B with $0 < A \leq B < \infty$ such that

$$A\|x\|_X \leq \|\{f_n(x)\}\|_{X_d} \leq B\|x\|_X, \quad x \in X, \tag{1.2}$$

- (3) S is a bounded linear operator such that

$$S(\{f_n(x)\}) = x, \quad x \in X.$$

As in case of frame for a Hilbert space the positive constants A and B , respectively, are called *lower* and *upper* frame bound for the Banach frame $(\{f_n\}, S)$ and collectively known as *frame bounds* for $(\{f_n\}, S)$. The operator $S : X_d \rightarrow X$ is called the *reconstruction operator* (or the *pre-frame operator*). The inequality (1.2) is called the *frame inequality* for the Banach frame $(\{f_n\}, S)$. Banach frames were further studied in [8, 11, 12].

Aldroubi, Sun and Tang [1], introduced p -frames in Banach spaces as a generalization of the concept of frames. They gave the following definition of p -frames:

Let $1 < p < \infty$. A countable family $\{f_i\} \subset X^*$ is a p -frame for X if there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_X \leq \left(\sum_i |f_i(x)|^p \right)^{\frac{1}{p}} \leq B\|x\|_X, \quad \text{for all } x \in X.$$

This concept of p -frames was further studied by Christensen and Stoeva [2].

Retro Banach frames for conjugate Banach Spaces were introduced and studied in 2004 by Jain, Kaushik and Vashisht [14]. Retro Banach frames for Banach spaces were defined as follows:

Let X be a Banach space and X^* be its conjugate space. Let $(X^*)_d$ be a Banach space of scalar-valued sequences associated with X^* , indexed by $\mathbb{N}$. Let $\{x_n\}$ be a sequence in X and let $T : (X^*)_d \rightarrow X^*$ be given. The pair $(\{x_n\}, T)$ is called a *retro Banach frame* for X^* with respect to $(X^*)_d$ if

- (a) $\{f(x_n)\} \in (X^*)_d$, for all $f \in X^*$,
- (b) there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|_{X^*} \leq \|\{f(x_n)\}\|_{(X^*)_d} \leq B\|f\|_{X^*}, \quad f \in X^*, \quad (1.3)$$

- (c) T is a bounded linear operator such that
 $T(\{f(x_n)\}) = f$, for all $f \in X^*$.

The positive constants A and B respectively, are called *lower* and *upper frame bound* for the retro Banach frame $(\{x_n\}, T)$ and collectively known as *retro frame bounds* (or simply bounds) for the retro Banach frame $(\{x_n\}, T)$. The operator $T : (X^*)_d \rightarrow X^*$ is called *the reconstruction operator* (or *pre-frame operator*). The inequality (1.3) is called *retro frame inequality* for the retro Banach frame $(\{x_n\}, T)$.

Retro Banach frames were further studied by Goswami [9], Vashisht [18] and Sharma [17].

Block sequences with respect to frames in Hilbert spaces have been defined and studied by Kaushik, Singh and Virender [16]. For Banach frames Block sequences were introduced and studied by Jain, Kaushik and Raj Kumar [13]. Block sequence of Banach frames for Banach spaces were defined as follows:

Let X be a Banach space. Let $(\{f_n\}, S)$ (where $S : (X)_d \rightarrow X$ and $\{f_n\} \subset X^*$) be a Banach frame for X with respect to $(X)_d$. A sequence of finite subsets of $\mathbb{N}$, $\{D_i\}$ is such that $D_i \cap D_j = \emptyset$, for all $i, j \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} D_i = \mathbb{N}$. Then the sequence $\{g_n\}$ defined by

$$g_n = \sum_{i \in D_n} \alpha_i f_i \neq 0, \quad \text{for all } n \in \mathbb{N},$$

where α_i , $i \in D_n$, $n \in \mathbb{N}$ are any scalars, is called a *block sequence* with respect to Banach frame $(\{f_n\}, S)$.

Block sequences were further studied by Chugh, Goel and Sharma [5].

The following result stated in the form of Lemma will be referred in the subsequent work.

Lemma 1.1. [15]. *Let $(\{f_n\}, S)$ (where $\{f_n\} \subset E^*$ and $S : E_d \rightarrow E$) be a Banach frame for E with respect to E_d . Then $(\{f_n\}, S)$ is exact if and only if $f_n \notin \widetilde{[f_i]_{i \neq n}}$ for all $n \in \mathbb{N}$.*

In this article, we introduce and study Block sequences with respect to retro Banach frames and gave examples to show their existence. We give a necessary and sufficient condition for a block sequence of a retro Banach frame to be a retro Banach frame. We also give a condition under which a block sequence of an exact retro Banach frame is an exact retro Banach frame. Finally, we discuss exact retro Banach frames and prove a result related to a geometric property of the underlying space.

2. Main Results

In this section, we have introduced block sequences of retro Banach frames. We begin with following definition.

Definition 2.1. Let X be a Banach space. Let $(\{x_n\}, T)$ (where $T : (X^*)_d \rightarrow X^*$ and $\{x_n\} \subset X$) be a retro Banach frame for X^* with respect to $(X^*)_d$. A sequence of finite subsets of $\mathbb{N}$, $\{D_i\}$ is such that $D_i \cap D_j = \phi$, for all $i, j \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} D_i = \mathbb{N}$. Then the sequence $\{y_n\}$ defined by

$$y_n = \sum_{i \in D_n} \alpha_i x_i \neq 0, \text{ for all } n \in \mathbb{N},$$

where α_i , $i \in D_n$, $n \in \mathbb{N}$ are any scalars, is called a *block sequence* with respect to retro Banach frame $(\{x_n\}, T)$.

In the following examples, we show that a block sequence of a retro Banach frame is a retro Banach frame.

Example 2.2. Let $X = l^1$, $X^* = l^\infty$. Let $e_n = (0, 0, 0, \dots, 1, 0, 0, \dots)$. Define a

$\downarrow$
 n th place

sequence $\{x_n\} \subset X$ such that $x_n = e_n$. Then, there is an operator T such that $(\{x_n\}, T)$ is a retro Banach frame for X^* . Now, let $D_i = \{i\}$, $i \in \mathbb{N}$, then $D_i \cap D_j = \phi$, for all $i, j \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} D_i = \mathbb{N}$. Therefore the sequence $\{y_n\}$ defined by

$$y_n = \sum_{i \in D_n} \alpha_i x_i \neq 0, \text{ } n \in \mathbb{N},$$

where $\alpha_i = 1$, $i \in D_n$, $n \in \mathbb{N}$, is a block sequence with respect to retro Banach frame $(\{x_n\}, T)$ and $(\{y_n\}, S)$ is a retro Banach frame for X^* , where $(X^*)_{d_0}$ is an associated Banach space and $S : (X^*)_{d_0} \rightarrow X^*$ is linear and bounded.

Example 2.3. Let $e_n = (0, 0, 0, \dots, 1, 0, 0, \dots)$. Let $X = l^1$. Define a sequence

$\downarrow$
nth place

$\{x_n\} \subset X$ such that

$$x_1 = e_1, x_{2n} = e_{n+1} = x_{2n+1}, \text{ for all } n \geq 1.$$

Then $[x_n] = X$. Then, there is an operator T such that $(\{x_n\}, T)$ is a retro Banach frame for X^* . Take

$$D_1 = \{1\}, D_n = \{2n-2, 2n-1\}, \text{ for all } n \geq 2 \text{ and} \\ \alpha_i = 1, i \in D_n, n \in \mathbb{N}.$$

Define a sequence $\{y_n\}$ by

$$y_n = \sum_{i \in D_n} \alpha_i x_i \neq 0, n \in \mathbb{N}.$$

Then $\{y_n\}$ is a block sequence with respect to the retro Banach frame $(\{x_n\}, T)$ and $(\{y_n\}, S)$ is a retro Banach frame for X^* , where $(X^*)_{d_0}$ is an associated Banach Space and $S : (X^*)_{d_0} \rightarrow X^*$ is linear and bounded.

Next we give an example to show that a block sequence with respect to a retro Banach frame may not be a retro Banach frame.

Example 2.4. Let $\{x_n\}$ be the sequence of unit vectors in $X = l^1$. Let $(X^*)_d$ be an associated Banach Space. Then $T : (X^*)_d \rightarrow X^*$ be a linear and bounded operator such that $(\{x_n\}, T)$ is retro Banach frame for X^* . Take

$$D_1 = \{1, 2, 3\} \text{ and } D_n = \{2+n\}, n \geq 2.$$

Then, the sequence $\{y_n\}$ defined by

$$y_n = \sum_{i \in D_n} \alpha_i x_i \neq 0, n \in \mathbb{N}$$

where $\alpha_i = 1, i \in D_n, n \in \mathbb{N}$, is a block sequence to the retro Banach frame $(\{x_n\}, T)$. But corresponding to this sequence there is no operator $S : (X^*)_{d_0} \rightarrow X^*$ such that $(\{y_n\}, T)$ is a retro Banach frame for X^* .

In view of Examples 2.2 to 2.4, we can observe that a block sequence of the retro Banach frame may or may not be a retro Banach frame. So it is natural to find a condition under which a block sequence of a retro Banach frame is a retro Banach frame. In this direction we prove the following result:

Theorem 2.5. Let $(\{x_n\}, T)$ ($T : (X^*)_d \rightarrow X^*$; $\{x_n\} \subset X$) be a retro Banach frame for X^* with respect to $(X^*)_d$ and $\{y_n\}$ be a block sequence of the retro Banach frame $(\{x_n\}, T)$. Let $L : (X^*)_d \rightarrow (X^*)_d$ be a linear and bounded operator such that $L(\{f(y_n)\}) = \{f(x_n)\}$, $f \in X^*$. Then $(\{y_n\}, S)$ is a retro Banach frame for X^* with respect to $(X^*)_d$ such that if and only if there exists a constant $M > 1$ such that

$$\|\{f(x_n - y_n)\}\|_{(X^*)_d} \leq M \min\{\|\{f(x_n)\}\|_{(X^*)_d}, \|\{f(y_n)\}\|_{(X^*)_d}\}, \quad f \in X^*.$$

Proof. Let $A_x, B_x; A_y, B_y$ respectively, be the frame bounds for the retro Banach frames $(\{x_n\}, T)$ and $(\{y_n\}, S)$. Then by inequalities for these frames, we have

$$\|\{f(x_n - y_n)\}\|_{(X^*)_d} \leq (1 + \frac{B_y}{A_x}) \|\{f(x_n)\}\|_{(X^*)_d}, \text{ for all } f \in X^*.$$

Similarly

$$\|\{f(x_n - y_n)\}\|_{(X^*)_d} \leq (1 + \frac{B_x}{A_y}) \|\{f(y_n)\}\|_{(X^*)_d}, \text{ for all } f \in X^*.$$

Choose $M = (1 + \frac{B_y}{A_x})$ or $(1 + \frac{B_x}{A_y})$ according as $\min\{\|\{f(x_n)\}\|_{(X^*)_d}, \|\{f(y_n)\}\|_{(X^*)_d}\}$ is $\|\{f(x_n)\}\|_{(X^*)_d}$ or $\|\{f(y_n)\}\|_{(X^*)_d}$.

Conversely, by hypothesis $\{f(y_n)\} \in (X^*)_d$, $f \in X^*$. If A_x and B_x are frame bounds for the retro Banach frame $(\{x_n\}, T)$, then for each $f \in X^*$, we have

$$\begin{aligned} A_x \|f\|_{X^*} &\leq \|\{f(x_n)\}\|_{(X^*)_d} \\ &\leq \|\{f(x_n - y_n)\}\|_{(X^*)_d} + \|\{f(y_n)\}\|_{(X^*)_d} \\ &\leq (1 + M) \|\{f(y_n)\}\|_{(X^*)_d} \\ &\leq (1 + M) (\|\{f(x_n - y_n)\}\|_{(X^*)_d} + \|\{f(x_n)\}\|_{(X^*)_d}) \\ &\leq (1 + M)^2 \|\{f(x_n)\}\|_{(X^*)_d} \\ &\leq (1 + M)^2 B_x \|f\|_{X^*}. \end{aligned}$$

Let $S = TL$. Then, S is linear and bounded with $S(f(y_n)) = f$, $f \in X^*$. Hence $(\{y_n\}, S)$ is retro Banach frame for X^* with respect to $(X^*)_d$. $\square$

Recall that a retro Banach frame $(\{x_n\}, T)$ ($\{x_n\} \in X, T : (X^*)_d \rightarrow X^*$) is called exact if there exists a sequence $\{f_n\} \in X^*$ such that

$$f_i(x_j) = \delta_{ij}, \text{ for all } i, j \in \mathbb{N}.$$

The sequence $\{f_n\} \in X^*$ is called an admissible sequence to the retro Banach frame $(\{x_n\}, T)$.

If X^* has an exact retro Banach frame and L is a bounded linear operator on X^* . Then the following theorem demonstrates that one can find a block sequence of exact retro Banach frame which is an exact retro Banach frame.

Theorem 2.6. Let $(\{x_n\}, T)$ (where $T : (X^*)_d \rightarrow X^*$ and $\{x_n\} \subset X$) be an exact retro Banach frame for X^* with respect to $(X^*)_d$. Let $\{y_n\}$ be a block sequence of retro Banach frame $(\{x_n\}, T)$ and let $L : (X^*)_d \rightarrow (X^*)_d$ be a bounded linear operator such that $L(\{f(x_n)\}) = \{f(y_n)\}$, $f \in X^*$. Then there exists a reconstruction operator $S : (X^*)_d \rightarrow X^*$ such that $(\{y_n\}, S)$ is an exact retro Banach frame for X^* with respect to $(X^*)_d$ if and only if there exists a positive constant λ such that

$$\|\{f(x_n)\}\|_{(X^*)_d} \geq \lambda \|W(\{f(y_n)\})\|_{(X^*)_d}, \text{ for all } f \in X^*.$$

where $W : (X^*)_d \rightarrow (X^*)_d$ be a bounded linear operator such that $W(\{f(y_n)\}) = \{f(x_n)\}$, $f \in X^*$.

Proof. In view of Theorem 3.4 proved in [14], there exists a reconstruction operator $S : (X^*)_d \rightarrow X^*$ such that $(\{y_n\}, S)$ is retro Banach frame for X^* with respect to $(X^*)_d$. Since $(\{x_n\}, T)$ is exact, by Lemma 1.1, there exists a sequence $\{f_n\} \in X^*$ such that $f_i(x_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. Note that $x_n \neq 0$, $n \in \mathbb{N}$. Now

$$y_n = \sum_{i \in D_n} \alpha_i x_i \neq 0, \quad n \in \mathbb{N}.$$

Therefore, for each n , there exists an $\alpha_n \neq 0$. Define $\{g_n\} \in X^*$ by $g_n = \frac{1}{\alpha_n} f_n$, $n \in \mathbb{N}$. Then $g_i(y_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. Therefore, Lemma by 1.1, $(\{y_n\}, S)$ is an exact retro Banach frame for X^* with respect to $(X^*)_d$. $\square$

Finally, we prove a result for exact retro Banach space related to a geometric property of the underlying Banach space.

Theorem 2.7. *Let $(\{x_n\}, T)$ ($\{x_n\} \in X, T : (X^*)_d \rightarrow X^*$) be an exact retro Banach frame with admissible sequence $(\{f_n\} \in X^*$ and let $k \geq 0$ be an integer. Then $\text{Codim}_{X^*}[f_n] = k$ if and only if*

- (a) *every $(k+1)$ -dimension subspace A_{k+1} of X^* meets $[f_n]$ non-trivially.*
- (b) *there exists a k -dimensional subspace B_k of X^* such that $B_k \cap [f_n] = 0$.*

Proof. Suppose first that $\text{Codim}_{X^*}[f_n] = k$. Then, there exists a k -dimensional B_k of X^* such that $B_k \cap [f_n] = 0$. Let A be any $(k+1)$ -dimensional subspace. If $A \cap [f_n] = 0$, then $\dim(\frac{X^*}{[f_n]}) = k+1$, which is a contradiction. Therefore $A \cap [f_n] \neq 0$.

Conversely, suppose the conditions (a) and (b) hold. Since $B_k \cap [f_n] = 0$, we infer that $\text{Codim}_{X^*}[f_n] \geq k$.

Now, if $\text{Codim}_{X^*}[f_n] = k+1$, then there exists a $k+1$ -dimensional subspace A_{k+1} of X^* such that $A_{k+1} \cap [f_n] = 0$. This is a contradiction. Hence $\text{Codim}_{X^*}[f_n] = k$. $\square$

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