

OPERATORS OF VOLTERRA CONVOLUTION TYPE IN GENERALIZED HÖLDER SPACES

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Abstract. Zygmund type estimates are obtained for the mixed continuity modulus of some mixed convolution type integrals. These Zygmund type estimates affect the nature of the improvement of continuity modulus by mixed fractional integrals in Hölder spaces.

1. Introduction

A great number of results are known concerning boundedness of convolution type operators in the Hölder spaces of functions of one variable. In the spaces of continuous functions such as $\tilde{H}_{0,0}^\omega$, the convolution type operators are least investigated. The goal of this paper is to fill a gap to a certain extent in investigations of such kind.

Here, we consider the mixed Volterra convolution type operators

$$(\tilde{K}\varphi)(x_1, x_2) = \int_a^{x_1} \int_c^{x_2} k(x_1 - t) k(x_2 - s) \varphi(t, s) dt ds, \quad a < x_1, \quad c < x_2,$$

in generalized Hölder spaces. We assume that the kernel $k(x_1, x_2)$ is close in a sense to a power.

The result of the type

$$\tilde{K} : \tilde{H}_0^\varpi(Q) \rightarrow \tilde{H}_0^{\varpi_1}(Q) \tag{1.1}$$

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for certain characteristic functions $\omega(h, \eta)$ and $\omega_w(h, \eta)$ are already known for the case of the power kernel $k(x_1, x_2) = x_1^{\alpha-1}x_2^{\beta-1}$, $0 < \alpha, \beta < 1$ (see [6], [10]). We deal here with arbitrary kernels, i.e., they are not necessarily powers.

We introduce a certain class of kernels V_λ for which we manage to give the Zygmund type estimates which estimate the mixed continuity modulus $\tilde{\omega}^{1,1}(\tilde{K}\varphi; h, \eta)$ using the modulus of continuity $\tilde{\omega}^{1,1}(\varphi; h, \eta)$. These estimates provide the general result of the type (1.2).

In the case of purely power kernel, i.e., in the case of a mixed fractional integral operator

$$\left(I_{a+,c+}^{\alpha,\beta}\varphi\right)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^{x_1} \int_c^{x_2} (x_1-t)^{\alpha-1}(x_2-s)^{\beta-1}\varphi(t,s)dt ds, \quad (1.2)$$

where $0 < \alpha, \beta < 1$, $a < x_1$, $c < x_2$, result (1.2) is extended to isomorphism (see, [8], [11], [13], [16]):

$$I_{a+,c+}^{\alpha,\beta} \left[\tilde{H}_0^{\alpha,\beta}\right] = \tilde{H}_0^{\alpha,\beta} \quad (1.3)$$

with $\tilde{\omega}_{\alpha,\beta}^{1,1}(h, \eta) = h^\alpha \eta^\beta \tilde{\omega}^{1,1}(h, \eta)$. This is achieved by the preliminary derivation of Zygmund type estimates for fractional differentiation (see [9], [12], [15]).

$$\begin{aligned} \left(\mathbf{D}_{a+,c+}^{\alpha,\beta}\varphi\right)(x_1,x_2) &= \frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)} \left[\frac{\varphi(x_1,x_2)}{(x_1-a)^\alpha(x_2-c)^\beta} \right. \\ &\quad \left. + \alpha\beta \int_a^{x_1} \int_c^{x_2} \frac{\varphi(x_1,x_2) - \varphi(t,s)}{(x_1-t)^{1+\alpha}(x_2-s)^{1+\beta}} dt ds \right]. \end{aligned} \quad (1.4)$$

We consider the operator (1.1) in the rectangle $Q = \{(x_1, x_2) : a < x_1 < b, c < x_2 < d\}$.

2. Preliminaries

2.1. Notations and a technical Lemma. We follow the papers [3] and [4] for the definitions given below. Positive constants which can be different at different places will be denoted by C .

Definition 1. We say that a non-negative function $k(x)$ on $[0, l]$ belongs to the class V_λ if

- (a) $k(x) \not\equiv 0$, $x^\lambda k(x)$ is almost increasing and $x^\lambda k(x) \Big|_{x=0} = 0$,
- (b) there exists ε , $0 < \varepsilon < \lambda$, such that $x^{\lambda-\varepsilon} k(x)$ is almost decreasing,
- (c) there exists C such that

$$\left| \frac{k(x) - k(y)}{x - y} \right| \leq C \frac{k(x^*)}{x^*}, \quad x^* = \max(x, y). \quad (2.1)$$

In the sequel, we shall use the following inequalities:

(i) If $\omega(\varphi, h)$ is the continuity modulus, then

$$\frac{\omega(\varphi, x)}{x} \leq C \frac{\omega(\varphi, y)}{y}, \quad x \geq y. \quad (2.2)$$

(ii) If $0 < \alpha < 1$ then

$$\frac{\omega(\varphi, h)}{h^\alpha} \leq C \int_0^h \frac{\omega(\varphi, t)}{t^{1+\alpha}} dt. \quad (2.3)$$

(iii) If $k(x) \in V_\lambda$ then

$$k(x) \leq C \left(\frac{y}{x}\right)^\lambda k(y), \quad x \leq y, \quad (2.4)$$

and there exists $\varepsilon > 0$ such that

$$k(x) \leq C \left(\frac{y}{x}\right)^{\lambda-\varepsilon} k(y), \quad y \leq x. \quad (2.5)$$

Lemma 1. Let $k(x) \in V_\lambda$, $\lambda > 0$ and let $\omega(x) \geq 0$ be an almost increasing function. Then

$$\omega(x)k(x) \leq C \int_x^l \frac{\omega(t)}{t} k(t) dt, \quad \text{for } 0 < x < l/2.$$

Proof. By (2.4), we have

$$\int_x^l \frac{\omega(t)}{t} k(t) dt \geq C \omega(x) x^\lambda k(x) \int_x^l \frac{dt}{t^{1+\lambda}} \geq C \omega(x) x^\lambda k(x) \int_x^{2x} \frac{dt}{t^{1+\lambda}} = C \omega(x) k(x). \quad \square$$

Definition 2. Let $\varphi(x)$ be a bounded function on $[a, b]$. The continuity modulus of $\varphi(x)$ is the expression given by

$$\sup_{h \in [0, \delta]} |\varphi(x+h) - \varphi(x)| = \omega(\varphi; \delta), \quad 0 < \delta \leq b-a.$$

Definition 3. We denote by Φ^1 the class of functions $\omega(\delta) \in (0, b-a]$ satisfying the conditions:

(a) $\omega(\delta)$ is the continuity modulus,

$$(b) \int_0^\delta t^{-1} \omega(t) dt \leq C \omega(\delta),$$

$$(c) \delta \int_\delta^{b-a} t^{-2} \omega(t) dt \leq C \omega(\delta),$$

$$(d) \omega'(\delta) \sim \delta^{-1} \omega(\delta).$$

For a continuous function $\varphi(x, y)$ on $\mathbf{R}$, we introduce the notation

$$\left(\Delta_h^{1,0} \varphi\right)(x, y) = \varphi(x+h, y) - \varphi(x, y), \quad \left(\Delta_\eta^{0,1} \varphi\right)(x, y) = \varphi(x, y+\eta) - \varphi(x, y),$$

$$\left(\Delta_{h,\eta}^{1,1} \varphi\right)(x, y) = \varphi(x+h, y+\eta) - \varphi(x+h, y) - \varphi(x, y+\eta) + \varphi(x, y).$$

so that

$$\varphi(x+h, y+\eta) = \left(\overset{1,1}{\Delta}_{h,\eta} \varphi \right)(x, y) + \left(\overset{1,0}{\Delta}_h \varphi \right)(x, y) + \left(\overset{0,1}{\Delta}_\eta \varphi \right)(x, y) + \varphi(x, y). \quad (2.6)$$

While studying the properties of continuous functions of several variables, in particular, the following classes of functions arise:

$$\begin{aligned} H^\varpi &= H^{\omega_1, \omega_2} = \left\{ \varphi(x, y) \in \mathbf{C}_Q : \overset{1,0}{\omega}_\varphi(\delta, 0) = O(\omega_1(\delta)), \overset{0,1}{\omega}_\varphi(0, \sigma) = O(\omega_2(\sigma)) \right\}, \\ \tilde{H}^\varpi &= H^{\omega_1, \omega_2, \omega_{1,1}} = \left\{ \varphi(x, y) \in \mathbf{C}_Q : \overset{1,0}{\omega}_\varphi(\delta, 0) = O(\omega_1(\delta)), \overset{0,1}{\omega}_\varphi(0, \sigma) = O(\omega_2(\sigma)), \right. \\ &\quad \left. \overset{1,1}{\omega}_\varphi(\delta, \sigma) = O(\omega_{1,1}(\delta, \sigma)) \right\}, \end{aligned}$$

where

(a) private modulus of continuity

$$\overset{1,0}{\omega}(\varphi; \delta, 0) = \sup_{y \in [c, d]} \sup_{0 \leq h \leq \delta} \left| \left(\overset{1,0}{\Delta}_h \varphi \right)(x, y) \right| \quad \text{and}$$

$$\overset{0,1}{\omega}(\varphi; 0, \sigma) = \sup_{x \in [a, b]} \sup_{0 \leq \eta \leq \sigma} \left| \left(\overset{0,1}{\Delta}_\eta \varphi \right)(x, y) \right|,$$

(b) mixed modulus continuity of order 1, 1

$$\overset{1,1}{\omega}(\varphi; \delta, \sigma) = \sup_{x, y} \sup_{0 \leq h \leq \delta, 0 \leq \eta \leq \sigma} \left| \left(\overset{1,1}{\Delta}_{h,\eta} \varphi \right)(x, y) \right|, \quad \text{where } 0 < \delta \leq b-a; 0 < \sigma \leq d-c.$$

It follows from the definition that the function $\overset{1,1}{\omega}(\varphi; \delta, \sigma)$ belongs to Φ^1 for each of the variables. In addition, we note the following inequality

$$\overset{1,1}{\omega}(\varphi; \delta, \sigma) \leq 2 \min \left\{ \overset{1,0}{\omega}(\varphi; \delta, 0); \overset{0,1}{\omega}(\varphi; 0, \sigma) \right\}. \quad (2.7)$$

Definition 4. We denote by $\Phi^{1,1}$ the class of functions of two variables $\omega(\delta, \sigma)$ satisfying the following conditions:

- (a) $\omega(\delta, \sigma)$ in δ for any fixed σ ,
- (b) $\omega(\delta, \sigma)$ in σ for any fixed δ .

We call this as the class of mixed modulus of continuity of the first order of continuous functions of two variables.

In [1], it is shown that the properties (a) and (b) are characteristic for continuity modulus in the sense that for every $\omega(x, y) \in \Phi^{1,1}$, there exists a function $\varphi \in \mathbf{C}_Q$ such that

$$\overset{1,1}{\omega}(\varphi; \delta, \sigma) \sim \omega_{1,1}(\delta, \sigma), \quad \overset{1,0}{\omega}(\varphi; \delta, 0) \sim \omega_1(\delta), \quad \overset{0,1}{\omega}(\varphi; 0, \sigma) \sim \omega_2(\sigma).$$

Definition 5. We denote by Φ the set satisfying the conditions on $(\omega_{1,1}, \omega_1, \omega_2)$ given by

- (i) $\omega_1(\delta), \omega_2(\sigma) \in \Phi^1$;
- (ii) $\omega_{1,1}(\delta, \sigma) \in \Phi^{1,1}$;

(iii) $\omega_{1,1}(\delta, \sigma) \leq C \min\{\omega_1(\delta), \omega_2(\sigma)\}$.

Let $(\omega_1, \omega_2) \in \Phi$. Then, we introduce norm in $H^\varpi = H^{\omega_1, \omega_2}$ by

$$\|\varphi\|_{H^\varpi} = \|\varphi\|_{C(Q)} + \sup_{y \in [c, d]} \sup_{x, x+h \in [a, b]} \frac{\omega^{1,0}(\varphi; h, 0)}{\omega_1(h)} + \sup_{x \in [a, b]} \sup_{y, y+\eta \in [c, d]} \frac{\omega^{0,1}(\varphi; 0, \eta)}{\omega_2(\eta)}.$$

Let $(\omega_1, \omega_2, \omega_{1,1}) \in \Phi$. The norm for $\tilde{H}^\varpi = H^{\omega_1, \omega_2, \omega_{1,1}}$ is given by

$$\|\varphi\|_{\tilde{H}^\varpi} = \|\varphi\|_{H^\varpi} + \sup_{x, x+h \in [a, b]} \sup_{y, y+\eta \in [c, d]} \frac{\omega^{1,1}(\varphi; h, \eta)}{\omega_{1,1}(h, \eta)}.$$

We say that $\varphi \in \tilde{H}_0^\varpi(Q)$ if $\varphi \in \tilde{H}^\varpi$ and $\varphi(x, y) \Big|_{x=a, y=c} = \varphi(x, y) \Big|_{x=b, y=d} \equiv 0$.

2.2. A one-dimensional statement. The following statements are known (see [3]). We use the schemes of the proofs to make the presentation easier for two-dimensional case.

The following theorem gives estimate which might be called Zygmund type by analogy with Zygmund estimate known in the theory of singular integrals and helps in estimating the continuity modulus $\omega(H\varphi, h)$ of a conjugate function $H\varphi$ via the continuity modulus $\omega(\varphi, h)$ of a function φ itself.

Consider the one-dimensional Volterra convolution type operators

$$K\varphi = \int_a^x k(x-t)\varphi(t)dt, \quad a < x < b \quad (2.7)$$

in the generalized Hölder spaces $H_0^\omega([a, b])$.

The following theorem provides Zygmund type estimate for the integral given in (2.7).

Theorem 1. Let $k(x) \in V_\lambda$, $0 < \lambda < 1$, $\varphi \in \mathbf{C}([a, b])$, and $\varphi(a) = 0$. Then

$$\omega(K\varphi; h) \leq Chk(h)\omega(\varphi; h) + Ch \int_h^{b-a} \frac{k(t)}{t} \omega(\varphi, t)dt. \quad (2.8)$$

Proof. Let $a = 0$ for simplicity. We denote $g(x) = \varphi(x) - \varphi(0)$ and

$$f(x) = \int_0^x k(x-t)g(t)dt.$$

For all $x, x+h \in [0, b]$, we have

$$\begin{aligned} f(x+h) - f(x) &= \int_{-h}^x [g(x-t) - g(x)]k(t+h)dt - \int_0^x [g(x-t) - g(x)]k(t)dt \\ &\quad + g(x) \left[\int_{-h}^x k(t+h)dt - \int_0^x k(t)dt \right]. \end{aligned}$$

So, we get

$$\begin{aligned}
 |f(x+h) - f(x)| &\leq \left| \int_{-h}^0 [g(x-t) - g(x)]k(t+h)dt \right| + \\
 &\quad \left| \int_0^x [g(x-t) - g(x)][k(t) - k(t+h)]dt \right| + \left| g(x) \int_x^{x+h} k(t)dt \right| \\
 &= A_1 + A_2 + A_3.
 \end{aligned}$$

In view of (2.4) and the increasing of $\omega(\varphi, t)$, we have for A_1

$$A_1 \leq \int_0^h \omega(\varphi, t)k(h-t)dt \leq C\omega(\varphi, h)k(h) \int_0^h \left(\frac{h}{h-t} \right)^\lambda dt \leq Chk(h)\omega(h).$$

For A_2 , using (2.1) and (2.5) in the case $x \leq h$, we have

$$\begin{aligned}
 A_2 &\leq Ch \int_0^x \frac{k(t+h)}{t+h} \omega(\varphi, t)dt \\
 &\leq Chk(h)h^{\lambda-\varepsilon} \int_0^x \frac{\omega(\varphi, t)dt}{(t+h)^{1+\lambda-\varepsilon}} \\
 &= Chk(h) \int_0^{x/h} \frac{\omega(\varphi, ht)dt}{(t+1)^{1+\lambda+\varepsilon}} \\
 &\leq Chk(h)\omega(\varphi, h) \int_0^1 (t+1)^{\varepsilon-1-\lambda} dt \\
 &\leq Chk(h)\omega(\varphi, h). \tag{2.9}
 \end{aligned}$$

In the case $h < x$, we have

$$A_2 \leq \int_0^h \omega(\varphi, t)|k(t) - k(t+h)|dt + \int_h^x \omega(\varphi, t)|k(t) - k(t+h)|dt = B_1 + B_2.$$

For B_1 , the estimate (2.9) is valid, while for B_2 , we have

$$B_2 \leq Ch \int_h^x \omega(\varphi, t) \frac{k(h+t)}{h+t} dt \leq Ch \int_h^x \omega(\varphi, t) \frac{k(t)}{t} dt \leq Ch \int_h^b \omega(\varphi, t) \frac{k(t)}{t} dt.$$

For A_3 in the case $x \leq h$, we have

$$A_3 \leq C\omega(\varphi, h)k(x+h)(x+h)^\lambda \int_x^{x+h} t^{-\lambda} dt \leq C\omega(\varphi, h)k(h)h^\lambda \int_0^{2h} t^{-\lambda} dt \leq C\omega(\varphi, h)hk(h).$$

If $h < x$, then using Lemma 1, we obtain

$$A_3 \leq C\omega(\varphi, x)hk(x) \leq Ch \int_x^b \frac{k(t)}{t} \omega(\varphi, t) dt \leq Ch \int_h^b \frac{k(t)}{t} \omega(\varphi, t) dt.$$

Gathering all the estimates for A_i , $i = 1, 2, 3$, we arrive at (2.8). $\square$

3. Main Result

The following theorem provides Zygmund type estimates for the integral (1.1).

Theorem 2. Let $k(x_i) \in V_{\lambda_i}$, $0 < \lambda_i < 1$, $i = 1, 2$ and $\varphi(x_1, x_2) \in \mathbf{C}(Q)$, $\varphi(x_1, x_2) \Big|_{x_1=a, x_2=c} = 0$. Then

$$\tilde{\omega}^{1,0}(\tilde{K}\varphi; h, 0) \leq C_1 \left[hk(h) \tilde{\omega}^{1,0}(\varphi; h, 0) + h \int_h^{b-a} \frac{\tilde{\omega}^{1,0}(\varphi; t, 0)}{t} k(t) dt \right], \quad (3.1)$$

$$\tilde{\omega}^{0,1}(\tilde{K}\varphi; 0, \eta) \leq C_2 \left[\eta k(\eta) \tilde{\omega}^{0,1}(\varphi; 0, \eta) + \eta \int_{\eta}^{d-c} \frac{\tilde{\omega}^{0,1}(\varphi; 0, s)}{s} k(s) ds \right], \quad (3.2)$$

$$\begin{aligned} \tilde{\omega}^{1,1}(\tilde{K}\varphi; h, \eta) \leq C_3 & \left[h\eta k(h)k(\eta) \tilde{\omega}^{1,1}(\varphi; h, \eta) + h\eta k(\eta) \int_h^{b-a} \frac{k(t)}{t} \tilde{\omega}^{1,1}(\varphi; t, d-c) dt \right. \\ & + hk(h)\eta \int_{\eta}^{d-c} \frac{k(s)}{s} \tilde{\omega}^{1,1}(\varphi; b-a, s) k(s) ds + \\ & \left. h\eta \int_h^{b-a} \int_{\eta}^{d-c} \frac{k(t)k(s)}{ts} \tilde{\omega}^{1,1}(\varphi; t, s) dt ds \right]. \end{aligned} \quad (3.3)$$

Proof. Using equality (2.6), the integral (1.1) can be represented as

$$\left(\tilde{K}\varphi \right)(x_1, x_2) = x_1 k(x_1) x_2 k(x_2) \varphi(a, c) + x_2 k(x_2) \psi_1(x_1) + x_1 k(x_1) \psi_2(x_2) + \psi(x_1, x_2), \quad (3.4)$$

where

$$\psi_1(x_1) = \int_a^{x_1} [\varphi(t, c) - \varphi(a, c)] k(x_1 - t) dt, \quad \psi_2(x_2) = \int_c^{x_2} [\varphi(a, s) - \varphi(a, c)] k(x_2 - s) ds$$

and

$$\psi(x_1, x_2) = \int_a^{x_1} \int_c^{x_2} \left(\Delta_{t,s}^{1,1} \right) (a, c) k(x_1 - t) k(x_2 - s) dt ds.$$

By hypothesis $\varphi(x_1, x_2) \Big|_{x_1=0, x_2=0} = 0$. So (3.4) gives

$$f(x_1, x_2) = \psi(x_1, x_2) = \int_a^{x_1} \int_c^{x_2} \left(\Delta_{t,s}^{1,1} \right) (a, c) k(x_1 - t) k(x_2 - s) dt ds.$$

We denote $\left(\Delta_{t,s}^{1,1} \varphi \right) (a, c) = g(t, s)$ for simplicity and

$$f(x_1, x_2) = \int_a^{x_1} \int_c^{x_2} g(t, s) k(x_1 - t) k(x_2 - s) dt ds.$$

For all $x, x + h \in [a, b]$, we have

$$\begin{aligned} & f(x_1 + h, x_2) - f(x_1, x_2) \\ &= \int_{-h}^0 \int_0^{x_2-c} k(h+t) k(s) [g(x_1-t, x_2-s) - g(x_1, x_2-s)] dt ds \\ & \quad + \int_0^{x_1-a} \int_0^{x_2-c} [k(h+t) - k(t)] k(s) [g(x_1-t, x_2-s) - g(x_1, x_2-s)] dt ds \\ & \quad + \left(\int_{-h}^{x_1-a} k(h+t) dt - \int_0^{x_1-a} k(t) dt \right) \int_0^{x_2-c} g(x_1, x_2-s) ds. \end{aligned} \quad (3.5)$$

So, we get

$$\begin{aligned} |f(x_1 + h, x_2) - f(x_1, x_2)| &\leq C \left(\int_0^h \int_0^{x_2-c} k(s) \overset{1,1}{\omega}(\varphi; t, x_2-s) k(h-t) dt ds \right. \\ & \quad + \int_0^{x_1-a} \int_0^{x_2-c} k(s) \overset{1,1}{\omega}(\varphi; t, x_2-s) [k(h+t) - k(t)] dt ds \\ & \quad \left. + \int_{x-a}^{x+h-a} \int_0^{x_2-c} k(s) \overset{1,1}{\omega}(\varphi; t, x_2-s) k(t) dt ds \right). \end{aligned}$$

From (2.1), we obtain

$$\int_0^{x_2-c} k(s) \overset{1,1}{\omega}(\varphi; t, x_2-s) ds \leq \overset{1,1}{\omega}(\varphi; t, x_2-c) k(x_2-c) (x_2-c)^\lambda \int_0^{x_2-c} s^{-\lambda} ds \leq C \overset{1,0}{\omega}(\varphi; t, 0).$$

Using this estimate and the estimates A_i , $i = 1, 2, 3$ from Theorem 1, the above inequality is easily verified as

$$|f(x_1 + h, x_2) - f(x_1, x_2)| \leq C_1 \left(hk(h) \tilde{\omega}^{1,0}(\varphi; h, 0) + h \int_h^{b-a} \frac{k(t)}{t} \tilde{\omega}^{1,0}(\varphi; t, 0) dt \right)$$

which helps in obtaining the inequality (3.1).

Rearranging symmetrically the representation (3.5), we can similarly obtain

$$|f(x_1, x_2 + \eta) - f(x_1, x_2)| \leq C_2 \left(\eta k(\eta) \tilde{\omega}^{0,1}(\varphi; 0, \eta) + \eta \int_\eta^{d-c} \frac{k(s)}{s} \tilde{\omega}^{0,1}(\varphi; 0, s) ds \right).$$

The main moment in the estimations is to find the corresponding splitting which allows to derive the best information in each variable and not losing the corresponding information in another variable. The suggested splitting runs as follows

$$\begin{aligned} \left(\Delta_{h,\eta}^{1,1} f \right)(x, y) &= \left(\Delta_{h,\eta}^{1,1} \psi \right)(x, y) = \sum_{k=1}^9 T_k \\ &:= \int_{-h}^0 \int_{-\eta}^0 \left(\Delta_{-t,-s}^{1,1} g \right)(x_1, x_2) k(t+h) k(s+\eta) dt ds \\ &\quad + \int_0^{x_1-a} \int_0^{x_2-c} \left(\Delta_{-t,-s}^{1,1} g \right)(x_1, x_2) [k(t) - k(h+t)] [k(s) - k(s+\eta)] dt ds \\ &\quad + g(x_1, x_2) \int_{x_1-a}^{x_1+h-a} \int_{x_2-c}^{x_2+\eta-c} k(t) k(s) dt ds \\ &\quad + \int_{-h}^0 \int_0^{x_2-c} \left(\Delta_{-t,-s}^{1,1} g \right)(x_1, x_2) k(h+t) [k(s) - k(s+\eta)] dt ds \\ &\quad + \int_0^{x_1-a} \int_{-\eta}^0 \left(\Delta_{-t,-s}^{1,1} g \right)(x_1, x_2) [k(t) - k(h+t)] k(s+\eta) dt ds \\ &\quad + \int_{x_1-a}^{x_1+h-a} \int_{-\eta}^0 [g(x_1, x_2-s) - g(x_1, x_2)] k(t) k(s+\eta) dt ds \\ &\quad + \int_{-h}^0 \int_{x_2-c}^{x_2+\eta-c} [g(x_1-t, x_2) - g(x_1, x_2)] k(t+h) k(s) dt ds \end{aligned}$$

$$\begin{aligned}
& + \int_{x_1-a}^{x_1+h-a} \int_0^{x_2-c} [g(x_1, x_2-s) - g(x_1, x_2)] k(t) [k(s) - k(s+\eta)] dt ds \\
& + \int_0^{x_1-a} \int_{x_2-c}^{x_2+\eta-c} [g(x_1-t, x_2) - g(x_1, x_2)] [k(t) - k(t+h)] k(s) dt ds.
\end{aligned}$$

So, we have

$$\begin{aligned}
\left| \left(\overset{1,1}{\Delta}_{h,\eta} f \right) (x, y) \right| & \leq C \left\{ \int_0^h \int_0^\eta \overset{1,1}{\omega}(\varphi; t, s) k(h-t) k(\eta-s) dt ds \right. \\
& + \int_0^{x_1-a} \int_0^{x_2-c} \overset{1,1}{\omega}(\varphi; t, s) [k(t) - k(h+t)] [k(s) - k(s+\eta)] dt ds \\
& + \overset{1,1}{\omega}(\varphi; x-a, y-c) \int_{x_1-a}^{x_1+h-a} \int_{x_2-c}^{x_2+\eta-c} k(t) k(s) dt ds \\
& + \int_0^h \int_0^{x_2-c} \overset{1,1}{\omega}(\varphi; t, s) k(h-t) [k(s) - k(s+\eta)] dt ds \\
& + \int_0^{x_1-a} \int_0^\eta \overset{1,1}{\omega}(\varphi; t, s) [k(t) - k(h+t)] k(\eta-s) dt ds \\
& + \int_{x_1-a}^{x_1+h-a} \int_0^\eta \overset{1,1}{\omega}(\varphi; x_1-a, s) k(t) k(\eta-s) dt ds \\
& + \int_0^h \int_{x_2-c}^{x_2+\eta-c} \overset{1,1}{\omega}(\varphi; t, x_2-c) k(h-t) k(s) dt ds \\
& + \int_{x_1-a}^{x_1+h-a} \int_0^{x_2-c} \overset{1,1}{\omega}(\varphi; x_1-a, s) k(t) [k(s) - k(s+\eta)] dt ds \\
& \left. + \int_0^{x_1-a} \int_{x_2-c}^{x_2+\eta-c} \overset{1,1}{\omega}(\varphi; t, x_2-c) [k(t) - k(t+h)] k(s) dt ds \right\}.
\end{aligned}$$

Estimating the obtained terms in the standard way, we arrive at inequality

$$\left| \left(\overset{1,1}{\Delta}_{h,\eta} f \right) (x, y) \right| \leq C_3 \left[h\eta k(\eta) k(h) \overset{1,1}{\omega}(\varphi; h, \eta) + h\eta k(\eta) \int_h^{b-a} \frac{k(t)}{t} \overset{1,1}{\omega}(\varphi; t, d-c) dt \right]$$

$$+ hk(h)\eta \int_{\eta}^{d-c} \frac{k(s)}{s} \omega^{1,1}(\varphi; b-a, s) ds + h\eta \int_h^{b-a} \int_{\eta}^{d-c} \frac{k(t)k(s)}{ts} \omega^{1,1}(\varphi; t, s) dt ds \Big],$$

which gives (3.3). $\square$

Main Theorem. Let $k(x_i) \in V_{\lambda_i}$, $0 < \lambda_i < 1$, $i = 1, 2$ be a function satisfying

$$\int_0^{x_1-a} \int_0^{x_2} \frac{\omega^{1,1}(t, s)}{ts} dt ds \leq C \omega^{1,1}(x_1, x_2),$$

$$\int_{x_1}^{b-a} \int_{x_2}^{d-c} \frac{\omega^{1,1}(t, s)}{ts} k(t)k(s) dt ds \leq C k(x_1)k(x_2) \omega^{1,1}(x_1, x_2).$$

Then the Volterra convolution type operator $\tilde{K}\varphi$ is bounded from $\tilde{H}_0^{\varpi}$ into $\tilde{H}_0^{\varpi_k}$ with characteristics $\omega_k(h, \eta) = h\eta k(h)k(\eta)\omega(h, \eta)$.

Proof. Let $f(x_1, x_2) = (\tilde{K}\varphi)(x_1, x_2)$, where $\varphi \in \tilde{H}_0^{\varpi}(Q)$. Next, we prove that the function $f(x_1, x_2) \in \tilde{H}_0^{\varpi_k}(Q)$ if $f(x_1, x_2) \Big|_{x_i=0} = 0$, $i = 1, 2$ and

$$\sup_{0 < t \leq b-a} \frac{\omega^{1,0}(f; t, 0)}{tk(t)\omega_1(t)} < \infty, \quad \sup_{0 < s \leq d-c} \frac{\omega^{0,1}(f; 0, s)}{sk(s)\omega_2(s)} < \infty, \quad (3.6)$$

$$\sup_{0 < t \leq b-a; 0 < s \leq d-c} \frac{\omega^{1,1}(f; t, s)}{tk(t)sk(s)\omega_{1,1}(t, s)} < \infty, \quad (3.7)$$

Since $\varphi(x_1, x_2)$ belongs to the space $\tilde{H}_0^{\varpi}(Q)$, it follows that Theorem 2 is applicable. By virtue of Theorem 2, estimates of the Zygmund type (3.1), (3.2) and (3.3) hold for the integral $\tilde{K}\varphi$.

Using these estimates and the condition of the theorem, we have

$$\frac{\omega^{1,0}(f; h, 0)}{hk(h)\omega_1(h)} \leq C_1 \left(\frac{\omega^{1,0}(\varphi; h, 0)}{\omega_1(h)} + \frac{1}{k(h)\omega_1(h)} \int_h^{b-a} \frac{k(t)}{t} \omega^{1,0}(\varphi; t, 0) dt \right) \leq C \|\varphi\|_{\tilde{H}_0^{\varpi}},$$

$$\frac{\omega^{0,1}(f; 0, \eta)}{\eta k(\eta)\omega_2(\eta)} \leq C_2 \left(\frac{\omega^{0,1}(\varphi; 0, \eta)}{\omega_2(\eta)} + \frac{1}{k(\eta)\omega_2(\eta)} \int_{\eta}^{d-c} \frac{k(s)}{s} \omega^{0,1}(\varphi; 0, s) ds \right) \leq C \|\varphi\|_{\tilde{H}_0^{\varpi}},$$

$$\begin{aligned}
\frac{\omega^{1,1}(f; h, \eta)}{hk(h)\eta k(\eta)\omega_{1,1}(h, \eta)} &\leq C_3 \left(\frac{\omega^{1,1}(\varphi; h, \eta)}{\omega_{1,1}(h, \eta)} + \frac{1}{k(h)\omega_{1,1}(h, \eta)} \int_h^{b-a} \frac{k(t)}{t} \omega^{1,1}(\varphi; t, \eta) dt \right. \\
&\quad + \frac{1}{k(\eta)\omega_{1,1}(h, \eta)} \int_\eta^{d-c} \frac{k(s)}{s} \omega^{1,1}(\varphi; h, s) ds \\
&\quad \left. + \frac{1}{k(\eta)k(h)\omega_{1,1}(h, \eta)} \int_h^{b-a} \int_\eta^{d-c} \frac{k(t)k(s)}{ts} \omega^{1,1}(\varphi; t, s) dt ds \right) \\
&\leq C \|\varphi\|_{\tilde{H}_0^{\sigma}}.
\end{aligned}$$

Taking into account the inequality (2.7), it is easy to prove (3.6) and (3.7).

The equality $f(x_1, x_2) \Big|_{x_i=0} = 0$, $i = 1, 2$ follows from equality (1.1). After replacing $t = x_1 - (x_1 - a)\xi$ and $s = x_2 - (x_2 - c)\tau$, we have

$$f(x, y) = (x_1 - a)(x_2 - c) \int_0^1 \int_0^1 \varphi(x_1 - (x_1 - a)\xi, x_2 - (x_2 - c)\tau) k(\xi(x_1 - a)) k(\tau(x_2 - c)) d\xi d\tau.$$

If $x_1 \rightarrow a$ or $x_2 \rightarrow c$, then $f(x_1, x_2) \Big|_{x_i=0} = 0$, $i = 1, 2$ becomes obvious.

Further, since $\varphi(x_1, x_2) \Big|_{x_i=0} = 0$, $i = 1, 2$ from equality (3.4), we have

$$\begin{aligned}
\|f\|_{C(Q)} &= \max_{a \leq x_1 \leq b, c \leq x_2 \leq d} \left| \int_a^{x_1} \int_c^{x_2} k(x_1 - t) k(x_2 - s) [\varphi(t, c) - \varphi(a, c)] dt ds \right. \\
&\quad + \int_a^{x_1} \int_c^{x_2} k(x_1 - t) k(x_2 - s) [\varphi(a, s) - \varphi(a, c)] dt ds \\
&\quad \left. + \int_a^{x_1} \int_c^{x_2} k(x_1 - t) k(x_2 - s) \left(\Delta_{t-a, s-c}^{1,1} \right) (a, c) dt ds \right| \\
&\leq C \left(\max_{a \leq x_1 \leq b} \omega^{1,0}(\varphi; x_1 - a, 0)(x_1 - a) k(x_1 - a) \right. \\
&\quad + \max_{c \leq x_2 \leq d} \omega^{0,1}(\varphi; 0, x_2 - c)(x_2 - c) k(x_2 - c) \\
&\quad \left. + \max_{a \leq x_1 \leq b, c \leq x_2 \leq d} \omega^{1,1}(\varphi; x_1 - a, x_2 - c)(x_1 - a) k(x_1 - a)(x_2 - c) k(x_2 - c) \right) \\
&\leq C \|\varphi\|_{\tilde{H}_0^{\sigma}}.
\end{aligned} \tag{3.8}$$

It follows from inequalities (3.6), (3.7) and (3.8) that

$$\begin{aligned} \|f\|_{\tilde{H}_0^{\varpi k}} &= \|f\|_{C(Q)} + \sup_{0 < t \leq b-a} \frac{\tilde{\omega}^{1,0}(f; t, 0)}{tk(t)\omega_1(t)} + \sup_{0 < s \leq d-c} \frac{\tilde{\omega}^{0,1}(f; 0, s)}{sk(s)\omega_2(s)} \\ &+ \sup_{0 < t \leq b-a; 0 < s \leq d-c} \frac{\tilde{\omega}^{1,1}(f; t, s)}{tk(t)sk(s)\omega_{1,1}(t, s)} \leq C\|\varphi\|_{\tilde{H}_0^{\varpi}}. \end{aligned}$$

Hence, the theorem is proved. $\square$

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