

UNIQUENESS PROBLEM ON MEROMORPHIC FUNCTION CONCERNING DIFFERENTIAL POLYNOMIALS AND ITS DERIVATIVE WHICH SHARE ONE VALUE

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Abstract. In this paper, we study the uniqueness of meromorphic function concerning differential polynomials share one value with its derivative. The results extend and improve the results of Hui Huang and Bin Huang [11].

1. Introduction

In this paper, we use the standard notations and terms in the value distribution theory [1]. Let $f(z)$ be a non constant meromorphic function on the complex plane C . Define $E(a, f) = \{z | f(z) - a = 0\}$, where a zero point with multiplicity m is counted m times in the set. If there zero points are counted only once, then we denote the set by $\overline{E}(a, f)$. Let k be a positive integer. Define

$$E_k(a, f) = \{z | f(z) - a = 0, \exists i, 1 \leq i \leq k, f^{(i)}(z) \neq 0\},$$

where a zero point with multiplicity m is counted m times in the set.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. If $E(a, f) = E(a, g)$, then we say that $f(z)$ and $g(z)$ share the value CM; if $\overline{E}(a, f) = \overline{E}(a, g)$, then we say that $f(z)$ and $g(z)$ share the value IM.

Additional, we denote by $N_k(r, f)$ the counting function for poles of $f(z)$ with multiplicity $\leq k$, and by $\overline{N}_k$ the corresponding one for which multiplicity is not counted. Let $N_{(k)}(r, f)$ be the counting function for poles of $f(z)$ with multiplicity $\geq k$, and by $\overline{N}_{(k)}(r, f)$ the corresponding one for which multiplicity is not counted. Set

$$N_k(r, f) = \overline{N}(r, f) + \overline{N}_{(2)}(r, f) + \dots + \overline{N}_{(k)}(r, f).$$

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Similarly, we have the notation: $N_k(r, \frac{1}{f}), \bar{N}_k(r, \frac{1}{f}), N_{(k)}(r, \frac{1}{f}), \bar{N}_{(k)}(r, \frac{1}{f}), N_k(r, \frac{1}{f})$. If $\bar{E}(1, f) = \bar{E}(1, g)$, we denote by $N_{11}(r, \frac{1}{f-1})$ the counting function for common simple 1-points of both $f(z)$ and $g(z)$ where multiplicity is not counted. In 2011, Hui Huang and Bin Huang [11] obtained the following results.

Theorem A. Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, $k(\geq 3)$, $n(\geq 11)$ be two positive integers. If $E_k(1, f^n f') = E_k(1, g^n g')$, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2, c are three constants, satisfying $(c_1 c_2)^{n+1} c^2 = -1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.

Theorem B. Let $f(z)$ and $g(z)$ be two non constant meromorphic functions, $n(\geq 13)$ be a positive integer. If $E_2(1, f^n f') = E_2(1, g^n g')$, then the conclusion of Theorem A holds.

Theorem C. Let $f(z)$ and $g(z)$ be two non constant meromorphic functions, $n(\geq 19)$ be a positive integer. If $E_1(1, f^n f') = E_1(1, g^n g')$, then the conclusion of Theorem A holds.

2. Preliminaries

For the proof of our results, we need the following lemmas.

Lemma 2.1. Let f be a non-constant meromorphic function and let $a_0, a_1, \dots, a_n$ be finite complex numbers such that $a_n \neq 0$. Then

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \dots + a_1 f' + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.2. Let f be a non constant meromorphic function, k a positive integer, then

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\bar{N}(r, f) + S(r, f).$$

Lemma 2.3. Let f and g be two non-constant meromorphic functions, and let k be a positive integer. If $E_k(1, f) = E_k(1, g)$, then one of the following cases must occur:

$$\begin{aligned} (i) \quad T(r, f) + T(r, g) &\leq \bar{N}_2(r, f) + \bar{N}_2\left(r, \frac{1}{f}\right) + \bar{N}_2(r, g) + \bar{N}_2\left(r, \frac{1}{g}\right) \\ &\quad + \bar{N}_2\left(r, \frac{1}{f-1}\right) + \bar{N}_2\left(r, \frac{1}{g-1}\right) - N_{11}\left(r, \frac{1}{f-1}\right) \\ &\quad + N_{(k+1)}\left(r, \frac{1}{f-1}\right) + N_{(k+1)}\left(r, \frac{1}{g-1}\right) + S(r, f) + S(r, g). \end{aligned}$$

$$(ii) \quad f = \frac{(b+1)g + (a-b-1)}{bg + (a-b)}, \text{ where } a(\neq 0), b \text{ are two constants.}$$

Lemma 2.4. Let f and g be two non constant meromorphic functions, $n \geq m+6$ be a positive integer. Set $F = f^n P(f) f', G = g^n P(g) g'$, if

$$F = \frac{(b+1)G + (a-b-1)}{bG + (a-b)}, \quad (2.1)$$

where $a(\neq 0), b$ are two constants, then

(i) $f \equiv g$.

(ii) either $f \equiv tg$ for a constant such that $t^d = 1$, where $d = (n+m+1, n+m, \dots, n+1)$ or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where

$$\begin{aligned} R(w_1, w_2) &= w_1^{n+1} \left(\frac{a_m w_1^m}{n+m+1} + \frac{a_{m-1} w_1^{m-1}}{n+m} + \dots + \frac{a_0}{n+1} \right) \\ &- w_2^{n+1} \left(\frac{a_m w_2^m}{n+m+1} + \frac{a_{m-1} w_2^{m-1}}{n+m} + \dots + \frac{a_0}{n+1} \right) \end{aligned}$$

Proof. By Lemma 2.1, we get

$$\begin{aligned} T(r, F) &= T(r, f^n P(f) f') \\ &\leq T(r, f^n P(f)) + T(r, f') + S(r, f) \\ &\leq (n+m)T(r, f) + T(r, f') + S(r, f) \\ &\leq (n+m+2)T(r, f) + S(r, f) \end{aligned} \tag{2.2}$$

$$\begin{aligned} (n+m)T(r, f) &\leq T(r, f^n P(f)) + S(r, f) \\ &\leq m(r, f^n P(f)) + N(r, f^n P(f)) + S(r, f) \\ &\leq N(r, f^n P(f) f') - N(r, f') + m(r, f^n P(f) f') + m \left(r, \frac{1}{f'} \right) + S(r, f) \end{aligned} \tag{2.3}$$

Since $N(r, f^n P(f)) = N(r, f^n P(f) f') - N(r, f')$ by the first fundamental theorem, we get

$$\begin{aligned} (n+m)T(r, f) &\leq T(r, f^n P(f) f') - N(r, f') + T(r, f') - N \left(r, \frac{1}{f'} \right) + S(r, f) \\ &\leq T(r, F) + m(r, f') - N \left(r, \frac{1}{f'} \right) + S(r, f) \\ &\leq T(r, F) + m \left(r, \frac{f'}{f} \right) + m(r, f) - N \left(r, \frac{1}{f'} \right) + S(r, f) \\ &\leq T(r, F) + T(r, f) - N(r, f) - N \left(r, \frac{1}{f'} \right) + S(r, f). \\ T(r, F) &\geq (n+m-1)T(r, f) + N(r, f) + N \left(r, \frac{1}{f'} \right) + S(r, f). \end{aligned} \tag{2.4}$$

Thus, by (2.2) and (2.3), we get

$$S(r, F) = S(r, f).$$

Similarly, we get

$$T(r, G) \geq (n+m-1)T(r, g) + N(r, g) + N \left(r, \frac{1}{g'} \right) + S(r, g) \tag{2.5}$$

Without loss of generality, we suppose that $T(r, f) \leq T(r, g)$, $r \in I$, where I is a set with infinite measure. Next we consider three cases.

Case 1. $b \neq 0, -1$. If $a - b - 1 \neq 0$, then by (2.1) we have

$$\overline{N} \left(r, \frac{1}{G + \frac{(a-b-1)}{b+1}} \right) = \overline{N} \left(r, \frac{1}{F} \right)$$

By the Nevanlinna's second fundamental theorem, we have

$$\begin{aligned}
 T(r, G) &\leq \bar{N}(r, G) + \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}\left(r, \frac{1}{G + \frac{1}{\frac{a-b-1}{b+1}}}\right) + S(r, G) \\
 &= \bar{N}(r, G) + \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}\left(r, \frac{1}{F}\right) + S(r, g) \\
 &\leq \bar{N}(r, g) + \bar{N}\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{P(g)}\right) + N\left(r, \frac{1}{g'}\right) \\
 &\quad + \bar{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{P(f)}\right) + N\left(r, \frac{1}{f'}\right) + S(r, g) \\
 &\leq (2m + 7)T(r, g) + S(r, g)
 \end{aligned}$$

By $n \geq m + 6$ and (2.5), we get $T(r, g) \leq S(r, g)$, for $r \in I$, a contradiction.
If $a - b - 1 = 0$, by (2.1) we can obtain

$$F = \frac{(b+1)G}{bG+1}$$

Clearly $\bar{N}\left(r, \frac{1}{G + \frac{1}{b}}\right) = \bar{N}(r, F)$.

By the Nevanlinna's second fundamental theorem, we have

$$\begin{aligned}
 T(r, G) &\leq \bar{N}(r, G) + \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}\left(r, \frac{1}{G + \frac{1}{b}}\right) + S(r, G) \\
 &= \bar{N}(r, G) + \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}(r, F) + S(r, g) \\
 &\leq \bar{N}(r, g) + \bar{N}\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{P(g)}\right) + N\left(r, \frac{1}{g'}\right) + \bar{N}(r, f) + S(r, g) \\
 &\leq \bar{N}(r, g) + \bar{N}\left(r, \frac{1}{g}\right) + mN\left(r, \frac{1}{g}\right) + 2T(r, g) + T(r, f) + S(r, g) \\
 &\leq (m + 5)T(r, g) + S(r, g).
 \end{aligned}$$

By $n \geq m + 6$ and (2.5), we get $T(r, g) \leq S(r, g)$, $r \in I$ a contradiction.

Case 2. $a \neq b, b = -1$. So $F = \frac{a}{(a+1)-G}$. We can get $\bar{N}(r, F) = \bar{N}\left(r, \frac{1}{G-(a+1)}\right)$. Similarly as in Case 1, it is impossible.

If $a + 1 = 0$, then $FG \equiv 1$.

i.e.,

$$f^n P(f) f' g^n P(g) g' \equiv 1. \quad (2.6)$$

Let z_0 be a zero of $f(z)$ with multiplicity p . Then z_0 is a pole of g with multiplicity q such that

$$np + p - 1 = nq + q + mq + 1$$

i.e.,

$$(p - q)(n + 1) = mq + 2$$

$\Rightarrow p \geq q + 1$ and $mq + 2 \geq n + 1$. Hence, by simple computing we get

$$p \geq \frac{n + m - 1}{m} \quad (2.7)$$

Let z_1 be a zero of $P(f)$ with multiplicity p_1 , then z_1 is a pole of g with multiplicity q_1 say. So from (2.6) we get $2p_1 - 1 = (n + m)q + 1 \geq n + m + 2$ i.e.,

$$p_1 \geq \frac{n + m + 3}{2}.$$

Since a pole of f is either a zero of $g^n P(g)$ or a zero of g' , we have

$$\begin{aligned} \overline{N}(r, g) &\leq \overline{N}(r, \frac{1}{g}) + \overline{N}(r, \frac{1}{P(g)}) + \overline{N}_0(r, \frac{1}{g'}) + S(r, f) + S(r, g) \\ &\leq \frac{m}{n + m - 1} N(r, \frac{1}{g}) + \frac{2}{n + m + 3} N(r, \frac{1}{P(g)}) + \overline{N}_0(r, \frac{1}{g'}) + S(r, f) + S(r, g) \\ &\leq \left(\frac{m}{n + m - 1} + \frac{2m}{n + m + 3} \right) T(r, g) + \overline{N}_0(r, \frac{1}{g'}) + S(r, f) + S(r, g). \end{aligned}$$

Where $\overline{N}_0(r, \frac{1}{g'})$ denotes the reduced counting function of those zeros of g' which are not the zeros of $gP(g)$. As $P(f) = a_m f^m + a_{m-1} f^{m-1} + \dots + a_1 f + a_0$ where $a_m, a_{m-1}, \dots, a_0$ are m distinct complex numbers. Then by second fundamental theorem of Nevanlinna, we get

$$\begin{aligned} mT(r, f) &\leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + \sum_{j=1}^m \overline{N}(r, \frac{1}{f - a_j}) - \overline{N}_0(r, \frac{1}{f'}) + S(r, f) \\ &\leq \overline{N}(r, \frac{1}{f}) + \overline{N}(r, f) + \overline{N}(r, \frac{1}{P(f) - a}) - \overline{N}_0(r, \frac{1}{f'}) + S(r, f) \\ &\leq \left(\frac{m}{n + m - 1} + \frac{2m}{n + m + 3} \right) T(r, f) + T(r, g) + \overline{N}_0(r, \frac{1}{g'}) \\ &\quad - \overline{N}_0(r, \frac{1}{f'}) + S(r, f) + S(r, g). \end{aligned} \tag{2.8}$$

Similarly we have

$$\begin{aligned} mT(r, g) &\leq \left(\frac{m}{n + m - 1} + \frac{2m}{n + m + 3} \right) T(r, f) + T(r, g) + \overline{N}_0(r, \frac{1}{f'}) \\ &\quad - \overline{N}_0(r, \frac{1}{g'}) + S(r, f) + S(r, g). \end{aligned} \tag{2.9}$$

Adding (2.8) and (2.9) we obtain

$$\left(1 - \frac{2}{n + m - 1} - \frac{4}{n + m + 3} \right) \{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g).$$

Which is a contradiction.

Case 3. $b = 0$, $a \neq b$ so $F = \frac{G+(a-1)}{a}$. If $a - 1 \neq 0$, then

$$\overline{N}(r, \frac{1}{G + (a - 1)}) = \overline{N}(r, \frac{1}{F}).$$

Similarly, we can again deduce a contradiction in case 1.

If $a - 1 = 0$, then $F \equiv G$. i.e.,

$$f^n P(f) f' \equiv g^n P(g) g'.$$

On integrating, we obtain

$$\begin{aligned} F^* &= f^{n+1}[a_m f^m + a_{m-1} f^{m-1} + \dots + a_0] \\ F^* &= g^{n+1}[a_m g^m + a_{m-1} g^{m-1} + \dots + a_0] \quad (\text{since } F \equiv G) \\ &= G^* + c, \end{aligned} \quad (2.10)$$

where c is a constant.

If $c \neq 0$, then by Nevanlinna's second fundamental theorem we get

$$\begin{aligned} T(r, F^*) &= T(r, G^*) + S(r, g) \\ &\leq \overline{N}(r, G^*) + \overline{N}(r, \frac{1}{G^*}) + \overline{N}(r, \frac{1}{F^*}) + S(r, g) \\ &\leq \overline{N}(r, g) + \overline{N}(r, \frac{1}{g^{n+1}a_m g^m}) + \overline{N}(r, \frac{1}{f^{n+1}a_m f^m}) + S(r, g) \\ &\leq \overline{N}(r, g) + \overline{N}(r, \frac{1}{g}) + \overline{N}(r, \frac{1}{g^m - a_m}) + \overline{N}(r, \frac{1}{f}) \\ &\quad + \overline{N}(r, \frac{1}{f^m - a_m}) + S(r, g) \\ &\leq (m+2)T(r, g) + (m+1)T(r, f) + S(r, f) + S(r, g) \\ (n+m+1)T(r, g) &\leq (m+2)T(r, g) + (m+1)T(r, f) + S(r, f) + S(r, g). \end{aligned} \quad (2.11)$$

Similarly we have

$$(n+m+1)T(r, f) \leq (m+2)T(r, f) + (m+1)T(r, g) + S(r, f) + S(r, g). \quad (2.12)$$

Adding (2.12) and (2.13), we obtain

$$(n+m+1)\{T(r, f) + T(r, g)\} \leq (2m+3)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g)$$

i.e.,

$$(n-m-2)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g).$$

Which is a contradicts $n \geq m+6$. Hence $c = 0$. Therefore

$$F^* = G^*.$$

Let $h = \frac{f}{g}$. If h is a constant, then substituting $f = gh$ in (2.10) we deduce

$$a_m g^{n+m+1}(h^{n+m+1} - 1) + a_{m-1} g^{n+m-1}(h^{n+m-1} - 1) + \dots + a_0 g^{n+1}(h^{n+1} - 1) \equiv 0,$$

which implies $h^d = 1$, where

$$d = \{n+m+1, \dots, n+m+1-i, \dots, n+1\}, a_{m-i} = 0$$

for some $i = 0, 1, \dots, m$. Then $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$. If h is not a constant then we know by (2.11) that f and g satisfy the algebraic functions $R(f, g) \equiv 0$, where

$$R(w_1, w_2) = w_1^n(a_m w_1^n + a_{m-1} w_1^{m-1} + \dots + a_0) - w_2^n(a_m w_2^n + a_{m-1} w_2^{m-1} + \dots + a_0)$$

This complete the proof of Lemma 2.4. $\square$

3. Main Results

In this paper, we will extend the theorems A, B, C, D as follows.

Theorem 3.1. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let n, k and m be three positive integers satisfying $k(\geq 3)$, $n(\geq m+11)$. Suppose that $P(W) = a_m W^m + \dots a_1 W + a_0$ where $a_0(\neq 0), a_1, \dots, a_m(\neq 0)$ are complex constants. If $E_k(1, f^n P(f)f') = E_k(1, g^n P(g)g')$, then*

(i) $f \equiv g$.

(ii) either $f \equiv tg$ for a constant such that $t^d = 1$, where $d = \{n+m+1, \dots, n+m+1-i, \dots, n+1\}$, $a_{m-i} = 0$ or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where

$$\begin{aligned} R(w_1, w_2) &= w_1^{n+1} \left(\frac{a_m w_1^m}{n+m+1} + \frac{a_{m-1} w_1^{m-1}}{n+m} + \dots + \frac{a_0}{n+1} \right) \\ &- w_2^{n+1} \left(\frac{a_m w_2^m}{n+m+1} + \frac{a_{m-1} w_2^{m-1}}{n+m} + \dots + \frac{a_0}{n+1} \right) \end{aligned}$$

Proof. Noticing that $k \geq 3$, we have

$$\begin{aligned} \overline{N}(r, \frac{1}{F-1}) + \overline{N}(r, \frac{1}{G-1}) - N_{11}(r, \frac{1}{F-1}) + \overline{N}_{(k+1)}(r, \frac{1}{F-1}) \\ + \overline{N}_{(k+1)}(r, \frac{1}{G-1}) \leq \frac{1}{2} N(r, \frac{1}{F-1}) + \frac{1}{2} N(r, \frac{1}{G-1}) \\ \leq \frac{1}{2} T(r, F) + \frac{1}{2} T(r, G) + O(1). \end{aligned}$$

By Lemma 2.3, we can get

$$\begin{aligned} T(r, F) + T(r, G) &< 2 \left(N_2(r, \frac{1}{F}) + N_2(r, F) + N_2(r, \frac{1}{G}) + N_2(r, G) \right) \\ &+ S(r, F) + S(r, G) \\ &= 2 \left(N_2(r, \frac{1}{F}) + N_2(r, F) + N_2(r, \frac{1}{G}) + N_2(r, G) \right) \\ &+ S(r, f) + S(r, g). \end{aligned} \tag{3.1}$$

Because

$$\begin{aligned} N_2(r, \frac{1}{F}) + N_2(r, F) &\leq N_2(r, \frac{1}{f^n P(f)f'}) + N_2(r, f^n P(f)f') \\ &\leq 2\overline{N}(r, \frac{1}{f}) + N(r, \frac{1}{P(f)}) + N(r, \frac{1}{f'}) + 2\overline{N}(r, f) \end{aligned} \tag{3.2}$$

and

$$N_2(r, \frac{1}{G}) + N_2(r, G) \leq 2\overline{N}(r, \frac{1}{g}) + N(r, \frac{1}{P(g)}) + N(r, \frac{1}{g'}) + 2\overline{N}(r, g) \tag{3.3}$$

By (3.13)-(3.15) and Lemma 2.3, we can get

$$\begin{aligned}
T(r, F) + T(r, G) &\leq 2(2\overline{N}(r, \frac{1}{f}) + 2\overline{N}(r, f) + N(r, \frac{1}{P(f)}) + N(r, \frac{1}{f'}) + 2\overline{N}(r, \frac{1}{g})) \\
&\quad + 2(2\overline{N}(r, g) + N(r, \frac{1}{P(g)}) + N(r, \frac{1}{g'}) + S(r, f) + S(r, g)) \\
&= 4\overline{N}(r, \frac{1}{f}) + 4\overline{N}(r, f) + 2N(r, \frac{1}{P(f)}) + 4N(r, \frac{1}{f'}) \\
&\quad + 4\overline{N}(r, \frac{1}{g}) + 4\overline{N}(r, g) + 2N(r, \frac{1}{P(g)}) + 4N(r, \frac{1}{g'}) + S(r) \\
&\leq (2m + 12)T(r, f) + (2m + 12)T(r, g) + S(r)
\end{aligned}$$

This gives

$$\begin{aligned}
(n + m + 1)[T(r, f) + T(r, g)] &\leq (2m + 12)[T(r, f) + T(r, g)] + S(r, f) + S(r, g) \\
(n - 2m - 11)(T(r, f) + T(r, g)) &\leq S(r).
\end{aligned}$$

By $n \geq 2m + 11$ and (2.4) and (2.5) we obtain $T(r, f) + T(r, g) \leq S(r, f) + S(r, g)$, which is impossible. Therefore, by Lemma 2.3

$$F = \frac{(b+1)g + (a-b-1)}{bg + (a-b)},$$

where $a(\neq 0), b$ are two constants, it follows by Lemma 2.4 that $f(z) \equiv g(z)$. $\square$

Theorem 3.2. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let n , and m be two positive integers satisfying $n(\geq \frac{3m}{2} + 13)$. Suppose that $P(W) = a_m W^m + \dots a_1 W + a_0$ where $a_0(\neq 0), a_1, \dots, a_m(\neq 0)$ are complex constants. If $E_2(1, f^n P(f)f') = E_2(1, g^n P(g)g')$, then the conclusion of Theorem 3.1 holds.*

Proof. We can see easily:

$$\begin{aligned}
\overline{N}(r, \frac{1}{F-1}) + \overline{N}(r, \frac{1}{G-1}) - N_{11}(r, \frac{1}{F-1}) + \frac{1}{2}\overline{N}_{(3)}(r, \frac{1}{F-1}) \\
+ \frac{1}{2}\overline{N}_{(3)}(r, \frac{1}{G-1}) \leq \frac{1}{2}N(r, \frac{1}{F-1}) + \frac{1}{2}N(r, \frac{1}{G-1}) \\
\leq \frac{1}{2}T(r, F) + \frac{1}{2}T(r, G) + S(r, f) + S(r, g).
\end{aligned}$$

By Lemma 2.4, we can get

$$\begin{aligned}
T(r, F) + T(r, G) &\leq 2 \left[N_2(r, \frac{1}{F}) + N_2(r, F) + N_2(r, \frac{1}{G}) + N_2(r, G) \right] \\
&\quad + \overline{N}_{(3)}(r, \frac{1}{F-1}) + \overline{N}_{(3)}(r, \frac{1}{G-1}) + S(r, f) + S(r, g).
\end{aligned} \tag{3.4}$$

Considering

$$\begin{aligned}
 \overline{N}_{(3)}\left(r, \frac{1}{F-1}\right) &\leq \frac{1}{2}N\left(r, \frac{F}{F'}\right) = \frac{1}{2}N\left(r, \frac{F'}{F}\right) + S(r, f) \\
 &\leq \frac{1}{2}\overline{N}\left(r, \frac{1}{F}\right) + \frac{1}{2}\overline{N}(r, F) + S(r, f) \\
 &\leq \frac{1}{2}[\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{P(f)}\right) + N\left(r, \frac{1}{f'}\right) + \overline{N}(r, f)] + S(r, f) \\
 &\leq \left(\frac{m}{2} + 2\right)T(r, f) + S(r, f).
 \end{aligned} \tag{3.5}$$

Similarly, we can get

$$\overline{N}_{(3)}\left(r, \frac{1}{G-1}\right) \leq \left(\frac{m}{2} + 2\right)T(r, g) + S(r, g). \tag{3.6}$$

From (3.16)-(3.18) we can get

$$\begin{aligned}
 T(r, F) + T(r, G) &\leq 2[N_2\left(r, \frac{1}{f^n P(f) f'}\right) + N_2(r, f^n P(f) f') + N_2\left(r, \frac{1}{g^n P(g) g'}\right) \\
 &\quad + N_2(r, g^n P(g) g')] + \left(\frac{m}{2} + 2\right)T(r, f) + \left(\frac{m}{2} + 2\right)T(r, g) + S(r) \\
 &\leq 2[2\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{P(f)}\right) + N\left(r, \frac{1}{f'}\right) + 2\overline{N}(r, f) + 2\overline{N}\left(r, \frac{1}{g}\right) \\
 &\quad + N\left(r, \frac{1}{P(g)}\right) + N\left(r, \frac{1}{g'}\right) + 2\overline{N}(r, g)] \\
 &\quad + \left(\frac{m}{2} + 2\right)T(r, f) + \left(\frac{m}{2} + 2\right)T(r, g) + S(r).
 \end{aligned}$$

This gives

$$\begin{aligned}
 (n + m + 1)[T(r, f) + T(r, g)] &\leq (2m + 12 + \frac{m}{2} + 2)[T(r, f) + T(r, g)] + S(r, f) + S(r, g) \\
 (n - \frac{3m - 26}{2})[T(r, f) + T(r, g)] &\leq S(r).
 \end{aligned}$$

By $n \geq \frac{3m}{2} + 13$, (2.4) and (2.5), we can get $T(r, f) + T(r, g) \leq S(r, f) + S(r, g)$ it is impossible.

The proof of Theorem 3.2 is complete. $\square$

Theorem 3.3. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let n , and m be two positive integers satisfying $n(\geq 3m + 19)$. Suppose that $P(W) = a_m W^m + \dots a_1 W + a_0$ where $a_0(\neq 0), a_1, \dots, a_m(\neq 0)$ are complex constants. If $E_1(1, f^n P(f) f') = E_1(1, g^n P(g) g')$, then the conclusion of Theorem 3.1 holds.*

Proof. Since

$$\begin{aligned}
 \overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) - N_{11}\left(r, \frac{1}{F-1}\right) &\leq \frac{1}{2}N\left(r, \frac{1}{F-1}\right) + \frac{1}{2}N\left(r, \frac{1}{G-1}\right) \\
 &\leq \frac{1}{2}T(r, F) + \frac{1}{2}T(r, G) + S(r, f) + S(r, g).
 \end{aligned}$$

We can see easily from Lemma 2.4 that:

$$\begin{aligned}
& T(r, F) + T(r, G) \\
& \leq 2 \left[N_2(r, \frac{1}{F}) + N_2(r, F) + N_2(r, \frac{1}{G}) + N_2(r, G) + \overline{N}_{(2)}(r, \frac{1}{F-1}) + \overline{N}_{(2)}(r, \frac{1}{G-1}) \right] \\
& \quad + S(r, F) + S(r, G). \\
& = 2 \left[N_2(r, \frac{1}{F}) + N_2(r, F) + N_2(r, \frac{1}{G}) + N_2(r, G) + \overline{N}_{(2)}(r, \frac{1}{F-1}) + \overline{N}_{(2)}(r, \frac{1}{G-1}) \right] \\
& \quad + S(r, f) + S(r, g). \tag{3.7}
\end{aligned}$$

Considering

$$\begin{aligned}
\overline{N}_{(2)}(r, \frac{1}{F-1}) & \leq N(r, \frac{F}{F'}) = N(r, \frac{F'}{F}) + S(r, f) \\
& \leq \overline{N}(r, F) + \overline{N}(r, \frac{1}{F}) + S(r, f) \\
& \leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + mN(r, \frac{1}{f}) + 2N(r, f) + S(r, f) \\
& \leq (m+4)T(r, f) + S(r, f). \tag{3.8}
\end{aligned}$$

Similarly we can get

$$\overline{N}_{(2)}(r, \frac{1}{G-1}) \leq (m+4)T(r, g) + S(r, g). \tag{3.9}$$

from (3.19)-(3.21) we can get

$$\begin{aligned}
T(r, F) + T(r, G) & \leq 2[2\overline{N}(r, \frac{1}{f}) + N(r, \frac{1}{P(f)}) + N(r, \frac{1}{f'}) + 2\overline{N}(r, f) + 2\overline{N}(r, \frac{1}{g}) \\
& \quad + N(r, \frac{1}{P(g)}) + N(r, \frac{1}{g'}) + 2\overline{N}(r, g) \\
& \quad + (m+4)T(r, f) + (m+4)T(r, g)] + S(r, f) + S(r, g) \\
& \leq 4\overline{N}(r, \frac{1}{f}) + 2mN(r, \frac{1}{f}) + 4N(r, \frac{1}{f}) + 4\overline{N}(r, f) \\
& \quad + 4\overline{N}(r, \frac{1}{g}) + 2mN(r, \frac{1}{g}) + 4N(r, \frac{1}{g}) + 4\overline{N}(r, g) \\
& \quad + (2m+8)T(r, f) + (2m+8)T(r, g) + S(r) \\
& \leq (4m+20)T(r, f) + (4m+20)T(r, g) + S(r).
\end{aligned}$$

This gives

$$\begin{aligned}
(n+m+1)[T(r, f) + T(r, g)] & \leq (4m+20)[T(r, f) + T(r, g)] + S(r, f) + S(r, g) \\
(n-3m-19)[T(r, f) + T(r, g)] & \leq S(r).
\end{aligned}$$

Since $n \geq 3m+19$, (2.4) and (2.5), we can get $T(r, f) + T(r, g) \leq S(r, f) + S(r, g)$, it is impossible. The proof of Theorem 3.3 is complete. $\square$

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