

ON THE DEGREE OF APPROXIMATION OF CONJUGATE FUNCTIONS OF PERIODIC CONTINUOUS FUNCTIONS

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Abstract. In this paper we have obtained the degree of approximation of conjugate functions of periodic continuous functions. The results are expressed in terms of modulus of continuity and via a mediate function.

1. Introduction

Let $f(x)$ be a 2π - periodic continuous function and $\omega(\delta) = \omega(\delta, f)$ its modulus of continuity defined by

$$\omega(\delta, f) := \sup_{|x-y| \leq \delta} |f(x) - f(y)|.$$

Denote by $A := (a_{n,k})$ ($k, n = 0, 1, \dots$) a lower triangular infinite matrix of real numbers and the A -transform of $\{S_n(f; x)\}$ by

$$T_{n,A}(f; x) := \sum_{k=0}^n a_{n,k} S_k(f; x), \quad (n = 0, 1, \dots),$$

where

$$S_k(f; x) = \frac{a_0}{2} + \sum_{j=1}^k (a_j \cos jx + b_j \sin jx)$$

are partial sums of the Fourier series of $f(x)$ at x ,

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).$$

We write $u = \mathcal{O}(v)$ if there exists a positive constant C such that $u \leq Cv$, and $\|f\|$ denotes the norm of f :

$$\|f\| = \max_{0 \leq x \leq 2\pi} |f(x)|.$$

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By $\mathbb{G}_n(t)$ we denote the matrix kernel

$$\mathbb{G}_n(t) = \sum_{k=0}^n a_{n,k} \frac{\cos\left(k + \frac{1}{2}\right)t}{2 \sin \frac{t}{2}},$$

and by $H(t)$ a non-negative mediate function.

Firstly, was P. Chandra [1]–[2] who estimated the deviation $\|T_{n,A}(f) - f\|$, in sup-norm, when $\{a_{n,k}\}$ is a monotonic sequences with respect to k .

Let us recall Chandra's results proved in [2].

Theorem 1.1. *Let $\{a_{n,k}\}$ satisfy the following conditions:*

$$a_{n,k} \geq 0 \quad \text{and} \quad \sum_{k=0}^n a_{n,k} = 1, \quad (1.1)$$

$$a_{n,k} \leq a_{n,k+1} \quad (k = 0, 1, \dots, n-1; n = 0, 1, \dots). \quad (1.2)$$

Suppose $\omega(t)$ is such that

$$\int_u^\pi t^{-2} \omega(t) dt = \mathcal{O}(H(u)) \quad (u \rightarrow +0), \quad (1.3)$$

where $H(u) \geq 0$ and

$$\int_0^t H(u) du = \mathcal{O}(tH(t)) \quad (t \rightarrow +0). \quad (1.4)$$

Then

$$\|T_{n,A}(f) - f\| = \mathcal{O}(a_{n,n}H(a_{n,n})).$$

Theorem 1.2. *Let (1.1), (1.2) and (1.3) hold. Then*

$$\|T_{n,A}(f) - f\| = \mathcal{O}(\omega(\pi/n)) + \mathcal{O}(a_{n,n}H(\pi/n)).$$

If, in addition, $\omega(t)$ satisfies (1.4) then

$$\|T_{n,A}(f) - f\| = \mathcal{O}(a_{n,n}H(\pi/n)).$$

Theorem 1.3. *Let us assume that (1.1) and*

$$a_{n,k} \geq a_{n,k+1} \quad (k = 0, 1, \dots, n-1; n = 0, 1, \dots) \quad (1.5)$$

hold. Then

$$\|T_{n,A}(f) - f\| = \mathcal{O}\left(\omega(\pi/n) + \sum_{k=1}^n k^{-1} \omega(\pi/k) \sum_{r=0}^{k+1} a_{n,r}\right).$$

Theorem 1.4. *Let (1.1), (1.3), (1.4) and (1.5) hold. Then*

$$\|T_{n,A}(f) - f\| = \mathcal{O}(a_{n,0}H(a_{n,0})).$$

Years later, Leindler [7] replaced the monotonicity condition with the so-called *rest bounded variation sequences* and *head bounded variation sequences*.

A sequence $\mathbf{c} := \{c_n\}$ of non-negative numbers tending to zero is called of *Rest Bounded Variation*, or briefly $\mathbf{c} \in RBVS$, if it has the property

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \leq K(\mathbf{c})c_m$$

for all natural numbers m , where $K(\mathbf{c})$ is a constant depending only on $\mathbf{c}$.

A sequence $\mathbf{c} := \{c_n\}$ of non-negative numbers will be called of *Head Bounded Variation*, or briefly $\mathbf{c} \in HBVS$, if it has the property

$$\sum_{n=0}^{m-1} |c_n - c_{n+1}| \leq K(\mathbf{c})c_m$$

for all natural numbers m , or only for all $m \leq N$ if the sequence $\mathbf{c}$ has only finite nonzero terms, and the last nonzero term is c_N .

It is clear that condition $0 < K(\mathbf{c}) \leq K$ needs to be assumed, where K is a positive constant. Supposing that for all n and $0 \leq m \leq n$

$$\sum_{k=m}^{\infty} |a_{n,k} - a_{n,k+1}| \leq K a_{n,m} \quad (1.6)$$

and

$$\sum_{k=0}^{m-1} |a_{n,k} - a_{n,k+1}| \leq K a_{n,m} \quad (1.7)$$

hold, where K is a positive constant, Leindler proved the following:

Theorem 1.5 ([7]). *The statements of Theorems 1.1, 1.2, 1.3 and 1.4 hold with (1.7) in place of (1.2), and with (1.6) in place of (1.5), respectively; naturally maintaining all the other assumptions.*

In his paper, he summed up all results in one theorem, but their proofs were given separately. Indeed, he generalized Chandra's results using a broader class of matrices $(a_{n,k})$. These results are extended further in [3], they are generalized in [4], and are treated in [14] as well, where some assumptions are removed, but the obtained results have a weaker degree of approximation. Very recently, see [5], [6], we have studied the same topic, proving similar but different results, using the so-called the generalized deferred Voronoi-Nörlund means of partial sums of their Fourier series. Also we have to mention the results of Németh [12] who, in the sense a slightly wider class of functions, generalized Chandra and Leindler's results. Moreover, he considered the limit missing case $\alpha = 0$ (giving a very good answer) using a specific modulus of continuity. Once again Leindler [9] showed, as an example, that integral-type conditions can be replaced by sequence-type assumptions to Chandra's theorems and its generalizations. So, such conditions can be assumed in the results of present paper, as well. To my best knowledge, we do not encounter so far such results (in accessible literature) regarding to a conjugate function of a 2π periodic continuous function.

The conjugate series of the above Fourier series is defined to be

$$\sum_{j=1}^{\infty} (b_j \cos jx - a_j \sin jx)$$

with their partial sums $\tilde{S}_k f$:

$$\tilde{S}_n(f; x) := \sum_{j=1}^n (b_j \cos jx - a_j \sin jx).$$

It is a well-known fact that if $f \in L$, then

$$\tilde{f}(x) := -\frac{1}{2\pi} \int_0^\pi \psi_x(t) \cot \frac{t}{2} dt = \lim_{\varepsilon \rightarrow 0} \tilde{f}(x, \varepsilon),$$

where

$$\tilde{f}(x, \varepsilon) := -\frac{1}{2\pi} \int_\varepsilon^\pi \psi_x(t) \cot \frac{t}{2} dt$$

with

$$\psi_x(t) := f(x+t) - f(x-t),$$

exists for almost x (see [15, Theorem (3.1), Chapter IV]).

Another class of sequences, important in applications, has been introduced in [8].

The sequence $\mathbf{c}$ of non-negative numbers tending to zero is called the Mean Rest Bounded Variation Sequence, or briefly $\mathbf{c} \in MRBVS$, if it has the property

$$\sum_{k=m}^{\infty} |c_k - c_{k+1}| \leq \frac{K(\mathbf{c})}{m+1} \sum_{k=\frac{m}{2}}^m c_k \quad (1.8)$$

for all natural numbers m , where $K(\mathbf{c})$ is a positive constant depending only on $\mathbf{c}$.

The sequence $\mathbf{c}$ of non-negative numbers is called Mean Head Bounded Variation Sequence (see [13]), or briefly $\mathbf{c} \in MHBVS$, if it has the property

$$\sum_{k=0}^{n-m-1} |c_k - c_{k+1}| \leq \frac{K(\mathbf{c})}{m+1} \sum_{k=n-m}^n c_k \quad (1.9)$$

for all positive integers $m < n$, the sequence $\mathbf{c}$ has only finite nonzero terms and the last nonzero term is c_n , where positive constant depending only on $\mathbf{c}$.

We see that in the above inequalities a positive constant $K(\mathbf{c})$ is appearing, and we have to assume it to be a finite one i.e., we need to assume $0 < K(\mathbf{c}) \leq K$, where $K > 0$ is a real number.

It was pointed out ([13], see also [11]) that

$$RBVS \subsetneq MRBVS \quad \text{and} \quad HBVS \subsetneq MHBVS. \quad (1.10)$$

In the sequel we suppose that for all n and $0 \leq m \leq n$

$$\sum_{k=m}^{\infty} |a_{n,k} - a_{n,k+1}| \leq \frac{K}{m+1} \sum_{k=\frac{m}{2}}^m a_{n,k} \quad (1.11)$$

and

$$\sum_{k=0}^{n-m-1} |a_{n,k} - a_{n,k+1}| \leq \frac{K}{m+1} \sum_{k=n-m}^n a_{n,k} \quad (1.12)$$

hold true, where K is a positive constant.

As we said above, there are no counterparts of Theorems 1.1–1.4 in the case of conjugate series of a Fourier series. Taking into account this gap, we will prove some corresponding theorems when condition (1.2) is replaced with condition (1.12) and condition (1.5) with that of (1.11), which indeed is the main aim of this paper. In

fact, we will estimate the deviation $\tilde{T}_{n,A}f - \tilde{f}$ in the norm $\|\cdot\|$, and for a wider range of matrices (because of the inclusions (1.10)), where

$$\tilde{T}_{n,A}f(x) := \sum_{k=0}^n a_{n,k} \tilde{S}_k(\cdot; x) \quad (n = 0, 1, \dots).$$

In the second section we are going to recall some lemmas and in last section we formulate and prove the main results.

2. Lemmas

For $m = 0, 1, \dots, n$, let

$$A_{n,m} := \sum_{k=0}^m a_{n,k}, \quad \bar{A}_{n,m} := \sum_{k=m}^n a_{n,k} \quad \text{and} \quad \tilde{D}_k^\circ(t) := \frac{\cos(k + \frac{1}{2})t}{2 \sin \frac{t}{2}}.$$

Lemma 2.1 ([2]). *If (1.3) and (1.4) hold, then*

$$\int_0^v t^{-1} \omega(t) dt = \mathcal{O}(vH(v)) \quad (v \rightarrow +0).$$

Lemma 2.2 ([11]). *If $\{a_{n,k}\} \in MHBVS$, then*

$$\left| \sum_{k=0}^n a_{n,k} \tilde{D}_k^\circ(t) \right| = \mathcal{O}(t^{-1} \bar{A}_{n,n-2\tau}),$$

and if $\{a_{n,k}\} \in MRBVS$, then

$$\left| \sum_{k=0}^n a_{n,k} \tilde{D}_k^\circ(t) \right| = \mathcal{O}(t^{-1} A_{n,\tau}),$$

for $\frac{2\pi}{n} \leq t \leq \pi$, ($n = 2, 3, \dots$), where $\tau = [\pi/t]$.

Now we pass to our main results.

3. Main Results

The first result is the following.

Theorem 3.1. *Let $\{a_{n,k}\}$ satisfy the following conditions (1.1), and (1.12). Suppose $\omega(t)$ is such that (1.3), and (1.4) are satisfied, and*

$$\bar{A}_{n,n-2\tau} = \mathcal{O}(\tau a_{n,n}) \tag{3.1}$$

holds, then

$$\|\tilde{T}_{n,A}(f) - \tilde{f}\| = \mathcal{O}(a_{n,n} H(a_{n,n})). \tag{3.2}$$

Proof. We easily find that

$$\tilde{T}_{n,A}f(x) - \tilde{f}(x) = \frac{1}{\pi} \int_0^\pi \psi_x(t) \sum_{k=0}^n a_{n,k} \tilde{D}_k^\circ(t) dt,$$

where $\psi_x(t) = f(x+t) - f(x-t)$.

Onward, we decompose the above integral into sub-integrals as follows

$$\|\tilde{T}_{n,A}(f) - \tilde{f}\| = \mathcal{O} \left(\int_0^{a_{n,n}} |\psi_x(t)| |\mathbb{G}_n(t)| dt + \int_{a_{n,n}}^{\pi} |\psi_x(t)| |\mathbb{G}_n(t)| dt \right), \quad (3.3)$$

The well-known inequality $\sin \theta \geq \frac{2}{\pi} \theta$ for $0 \leq \theta \leq \frac{\pi}{2}$, $|\cos \theta| \leq 1$, $\theta \in \mathbb{R}$, and Lemma 2.1, we have

$$\int_0^{a_{n,n}} |\psi_x(t)| |\mathbb{G}_n(t)| dt = \mathcal{O}(1) \int_0^{a_{n,n}} t^{-1} \omega(t) dt = \mathcal{O}(a_{n,n} H(a_{n,n})), \quad (3.4)$$

since $|\psi_x(t)| \leq 2\omega(|t|)$.

Using Lemma 2.2 and (1.12), we have

$$\begin{aligned} \int_{a_{n,n}}^{\pi} |\psi_x(t)| |\mathbb{G}_n(t)| dt &= \mathcal{O}(1) \int_{a_{n,n}}^{\pi} t^{-1} \bar{A}_{n,n-2\tau} \omega(t) dt \\ &= \mathcal{O}(a_{n,n}) \int_{a_{n,n}}^{\pi} t^{-2} \omega(t) dt = \mathcal{O}(a_{n,n} H(a_{n,n})). \end{aligned} \quad (3.5)$$

Inserting (3.4) and (3.5) into (3.3) we obtain (3.2).

The proof is completed. $\square$

Note that if $\{a_{n,k}\}$ satisfies condition (1.7), then condition $\bar{A}_{n,n-2\tau} = \mathcal{O}(\tau a_{n,n})$ is satisfied automatically (see the proof of Lemma 2, in [10]).

Corollary 3.2. Let $\{a_{n,k}\}$ satisfy conditions (1.1) and (1.7). Suppose $\omega(t)$ is such that (1.3) and (1.4) hold, then

$$\|\tilde{T}_{n,A}(f) - \tilde{f}\| = \mathcal{O}(a_{n,n} H(a_{n,n})).$$

Remark 3.3. We also note that if $\{a_{n,k}\}$ is a non-decreasing sequence with respect to k , ($k = 0, 1, \dots, n-1$; $n = 0, 1, \dots$), then conditions (1.7) and (1.12) are satisfied as well. So, Theorem 3.1 and Corollary 3.2 hold true even with one of these two conditions.

Theorem 3.4. Let the conditions (1.1), (1.3), (1.4), and (1.12) be satisfied. If condition (3.1) holds, then

$$\|\tilde{T}_{n,A}(f) - \tilde{f}\| = \mathcal{O} \left(\left(\frac{1}{n+1} + a_{n,n} \right) H \left(\frac{\pi}{n+1} \right) \right). \quad (3.6)$$

Otherwise, if we remove condition (3.1), then we have

$$\|\tilde{T}_{n,A}(f) - \tilde{f}\| = \mathcal{O} \left(\frac{1}{n+1} H \left(\frac{\pi}{n+1} \right) + \sum_{v=1}^n \frac{\pi}{v} \omega \left(\frac{\pi}{v} \right) A_{n,n-2v} \right), \quad (3.7)$$

with agreement $A_{n,n-2v} := 0$ whenever $n - 2v < 0$.

Proof. It is clear that

$$\begin{aligned} \|\tilde{T}_{n,A}(f) - \tilde{f}\| &= \mathcal{O}(1) \int_0^{\pi} |\psi_x(t)| |\mathbb{G}_n(t)| dt \\ &= \mathcal{O}(1) \left(\int_0^{\frac{\pi}{n+1}} |\psi_x(t)| |\mathbb{G}_n(t)| dt + \int_{\frac{\pi}{n+1}}^{\pi} |\psi_x(t)| |\mathbb{G}_n(t)| dt \right). \end{aligned} \quad (3.8)$$

Once again, using inequalities $\sin \theta \geq \frac{2}{\pi} \theta$ for $0 \leq \theta \leq \frac{\pi}{2}$, $|\cos t| \leq 1$, and (1.4), we have

$$\int_0^{\frac{\pi}{n+1}} |\psi_x(t)| |\mathbb{G}_n(t)| = \mathcal{O}(1) \int_0^{\frac{\pi}{n+1}} \frac{\omega(t)}{t} dt = \mathcal{O} \left(\frac{\pi}{n+1} H \left(\frac{\pi}{n+1} \right) \right). \quad (3.9)$$

Moreover, our assumptions, Lemma 2.2, and (1.3) imply

$$\begin{aligned} \int_{\frac{\pi}{n+1}}^{\pi} |\psi_x(t)| |\mathbb{G}_n(t)| dt &= \mathcal{O}(1) \int_{\frac{\pi}{n+1}}^{\pi} t^{-1} \omega(t) \bar{A}_{n,n-2\tau} dt \\ &= \mathcal{O}(a_{n,n}) \int_{\frac{\pi}{n+1}}^{\pi} t^{-2} \omega(t) dt = \mathcal{O} \left(a_{n,n} H \left(\frac{\pi}{n+1} \right) \right). \end{aligned} \quad (3.10)$$

Putting (3.9) and (3.10) into (3.8) we obtain (3.6).

We have proved (3.7) showing that

$$\begin{aligned} \int_{\frac{\pi}{n+1}}^{\pi} t^{-1} \omega(t) \bar{A}_{n,n-2\tau} dt &= \sum_{v=1}^n \int_{\frac{\pi}{v+1}}^{\frac{\pi}{v}} t^{-1} \omega(t) \bar{A}_{n,n-2\tau} dt \\ &= \mathcal{O}(1) \sum_{v=1}^n (v+1) \omega \left(\frac{\pi}{v} \right) \int_{\frac{\pi}{v+1}}^{\frac{\pi}{v}} \sum_{k=n-2\tau}^n a_{n,k} dt \\ &= \mathcal{O}(1) \sum_{v=1}^n \frac{\pi}{v} \omega \left(\frac{\pi}{v} \right) \sum_{k=n-2v}^n a_{n,k} \\ &= \mathcal{O}(1) \sum_{v=1}^n \frac{\pi}{v} \omega \left(\frac{\pi}{v} \right) A_{n,n-2v}, \end{aligned}$$

where we have agreed that $A_{n,n-2v} = 0$ whenever $n - 2v < 0$.

The proof is completed. $\square$

Theorem 3.5. Assume that (1.11) and $A_{n,\tau} = \mathcal{O}(\tau a_{n,n})$ hold true. Then

$$\|\tilde{T}_{n,A}(f) - \tilde{f}\| = \mathcal{O} \left(a_{n,n} H \left(\frac{\pi}{n+1} \right) \right). \quad (3.11)$$

Otherwise, if we remove condition $A_{n,\tau} = \mathcal{O}(\tau a_{n,n})$, then we have

$$\|\tilde{T}_{n,A}(f) - \tilde{f}\| = \mathcal{O} \left(a_{n,n} H \left(\frac{\pi}{n+1} \right) + \sum_{v=1}^n v^{-1} \omega \left(\frac{\pi}{v} \right) A_{n,v} \right). \quad (3.12)$$

Proof. Again, we start from (3.8). For its first integral we use (3.9) :

$$\int_0^{\frac{\pi}{n+1}} |\psi_x(t)| |\mathbb{G}_n(t)| = \mathcal{O} \left(\frac{\pi}{n+1} H \left(\frac{\pi}{n+1} \right) \right).$$

Condition (1.11) implies

$$\begin{aligned} a_{n,m} - a_{n,r} &\leq \sum_{k=m}^{r-1} |a_{n,k} - a_{n,k+1}| \\ &\leq \sum_{k=0}^{r-1} |a_{n,k} - a_{n,k+1}| \leq \frac{K}{n-r+1} \sum_{k=r}^n a_{n,k}, \quad (0 \leq m < r \leq n). \end{aligned}$$

Hence, for $r = n$, we have

$$1 = \sum_{m=0}^n a_{n,m} \leq K(n+1)a_{n,n} \implies \frac{1}{n+1} = \mathcal{O}(a_{n,n}).$$

So, we get

$$\int_0^{\frac{\pi}{n+1}} |\psi_x(t)| |\mathbb{G}_n(t)| = \mathcal{O} \left(a_{n,n} H \left(\frac{\pi}{n+1} \right) \right). \quad (3.13)$$

Our assumptions, Lemma 2.2, and $A_{n,\tau} = \mathcal{O}(\tau a_{n,n})$ imply

$$\begin{aligned} \int_{\frac{\pi}{n+1}}^{\pi} |\psi_x(t)| |\mathbb{G}_n(t)| dt &= \mathcal{O}(1) \int_{\frac{\pi}{n+1}}^{\pi} t^{-1} \omega(t) \bar{A}_{n,\tau} dt \\ &= \mathcal{O}(a_{n,n}) \int_{\frac{\pi}{n+1}}^{\pi} t^{-2} \omega(t) dt = \mathcal{O} \left(a_{n,n} H \left(\frac{\pi}{n+1} \right) \right). \end{aligned} \quad (3.14)$$

Therefore, putting (3.13) along with (3.14) into (3.8), we obtain (3.11).

Further, the use of Lemma 2.2 gives

$$\begin{aligned} \int_{\frac{\pi}{n+1}}^{\pi} |\psi_x(t)| |\mathbb{G}_n(t)| dt &= \mathcal{O} \left(\sum_{v=1}^n \int_{\frac{\pi}{v+1}}^{\frac{\pi}{v}} t^{-1} \omega(t) A_{n,\tau} dt \right) \\ &= \mathcal{O} \left(\sum_{v=1}^n (v+1) \omega \left(\frac{\pi}{v} \right) \int_{\frac{\pi}{v+1}}^{\frac{\pi}{v}} \sum_{j=0}^{\tau} a_{n,j} dt \right) \\ &= \mathcal{O} \left(\sum_{v=1}^n v^{-1} \omega \left(\frac{\pi}{v} \right) A_{n,v} \right). \end{aligned} \quad (3.15)$$

Thus, (3.13) and (3.15) imply (3.12).

The proof is completed. $\square$

Theorem 3.6. *Let (1.1), (1.3), (1.4), and (1.11) hold. If $A_{n,\tau} = \mathcal{O}(\tau a_{n,0})$ holds, then*

$$\|\tilde{T}_{n,A}(f) - \tilde{f}\| = \mathcal{O}(a_{n,0} H(a_{n,0})). \quad (3.16)$$

Proof. Firstly, we write

$$\|\tilde{T}_{n,A}(f) - \tilde{f}\| = \mathcal{O} \left(\int_0^{a_{n,0}} |\psi_x(t)| |\mathbb{G}_n(t)| dt + \int_{a_{n,0}}^{\pi} |\psi_x(t)| |\mathbb{G}_n(t)| dt \right), \quad (3.17)$$

The use of inequalities $\sin \theta \geq \frac{2}{\pi} \theta$ for $0 \leq \theta \leq \frac{\pi}{2}$, $|\cos \theta| \leq 1$, $\theta \in \mathbb{R}$, and Lemma 2.1, imply

$$\int_0^{a_{n,0}} |\psi_x(t)| |\mathbb{G}_n(t)| dt = \mathcal{O}(1) \int_0^{a_{n,0}} t^{-1} \omega(t) dt = \mathcal{O}(a_{n,0} H(a_{n,0})). \quad (3.18)$$

Lemma 2.2 and (1.11) imply

$$\begin{aligned} \int_{a_{n,0}}^{\pi} |\psi_x(t)| |\mathbb{G}_n(t)| dt &= \mathcal{O}(1) \int_{a_{n,0}}^{\pi} t^{-1} \bar{A}_{n,\tau} \omega(t) dt \\ &= \mathcal{O}(a_{n,0}) \int_{a_{n,0}}^{\pi} t^{-2} \omega(t) dt = \mathcal{O}(a_{n,0} H(a_{n,0})). \end{aligned} \quad (3.19)$$

Putting (3.18) and (3.19) into (3.17) we get (3.16).

The proof is completed. $\square$

If $\{a_{n,k}\} \in RBVS$, then condition $A_{n,\tau} = \mathcal{O}(\tau a_{n,0})$ is satisfied (see the proof of Lemma 2, in [10]). Whence, Theorem 3.6 implies the following.

Corollary 3.7. Let (1.1), (1.3), (1.4), and (1.6) hold. Then

$$\|\tilde{T}_{n,A}(f) - \tilde{f}\| = \mathcal{O}(a_{n,0}H(a_{n,0})).$$

Remark 3.8. We also note that if $\{a_{n,k}\}$ is a non-increasing sequence with respect to k , ($k = 0, 1, \dots, n-1$; $n = 0, 1, \dots$), then conditions (1.2) and (1.6) are satisfied too. So, Theorem 3.6 and Corollary 3.7 hold true even with one of these two conditions.

Remark 3.9. One have to note that our results contain seven well-known important particular cases of matrix means: $(C, 1)$ means, harmonic means, (C, α) means, (H, p) means, Nörlund means, Riesz means, and Generalized Nörlund means. So, a lot of corollaries can be derived from our results. Also, if the mediate function or "the aid-function" $H(t)$ is chosen to be of the form

$$H(u) = \begin{cases} u^{\alpha-1}, & 0 < \alpha < 1, \\ \log\left(\frac{\pi}{u}\right), & \alpha = 1, \end{cases}$$

we can obtain several specific results.

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