

WAVE COUPLED SYSTEM OF THE p -LAPLACIAN TYPE

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Date of Receiving : 02. 06. 2020
 Date of Revision : 04. 08. 2020
 Date of Acceptance : 05. 08. 2020

Abstract. In this work we study the existence of solution and the asymptotic behaviour for a nonlinear coupled system of wave equation of the p -Laplacian type given by

$$\begin{cases} u'' - \Delta_p u + |u|^{r-1}u|v|^{r+1} - \Delta u' = 0 & \text{in } Q = \Omega \times (0, T), \\ v'' - \Delta_p v + |v|^{r-1}v|u|^{r+1} - \Delta v' = 0 & \text{in } Q = \Omega \times (0, T), \\ (u(x, 0), v(x, 0)) = (u_0(x), v_0(x)) & \text{in } \Omega, \\ (u'(x, 0), v'(x, 0)) = (u_1(x), v_1(x)) & \text{in } \Omega, \\ u(x, t) = v(x, t) = 0 & \text{in } \Sigma = \Gamma \times (0, T), \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set with sufficiently smooth boundary Γ . Existence of solution is made by using Faedo-Gallerkin approximations. We prove the exponential stability when $p = 2$ and polynomial decay if $p > 2$ based on M. Nakao [15, 16].

1. Introduction

Throughout this paper, we omit the space variable x of $u(x, t)$ and simply denote $u(x, t)$ by $u(t)$ when there are no conflicts in the notation. C denotes various positive constants depending on the known constants and may be different at each appearance. Let $T > 0$ be a real number, $\Omega \subset \mathbb{R}^n$ be a bounded open set with sufficiently smooth boundary Γ . We denote by $Q = \Omega \times (0, T)$ the cylinder with lateral boundary $\Sigma = \Gamma \times (0, T)$. Here we consider $2 \leq p < \infty$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$. The duality pairing between the space $W_0^{1,p}(\Omega)$ and its dual $W^{-1,q}(\Omega)$ will be denoted using the form $\langle \cdot, \cdot \rangle$. According to Poincaré's inequality, the standard norm $\|\cdot\|_{W_0^{1,p}(\Omega)}$ is equivalent to the norm $\|\nabla \cdot\|_p$ on $W_0^{1,p}(\Omega)$. Henceforth, we put $\|\cdot\|_{W_0^{1,p}(\Omega)} = \|\nabla \cdot\|_p$. We denote $\|\cdot\|_{L^2(\Omega)} = \|\cdot\|_2$ and the usual inner product by $(\cdot, \cdot)$. We denote the p -Laplacian

2010 *Mathematics Subject Classification.* 35A01, 35J57, 35M30, 35B40.

Key words and phrases. Global solution, p -Laplacian operator, asymptotic behaviour.

Communicated by. Huy Hoang Nguyen

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operator by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, which can be extended to a monotone, bounded, hemicontinuous, and coercive operator between the spaces $W_0^{1,p}(\Omega)$ and its dual by

$$-\Delta_p: W_0^{1,p}(\Omega) \rightarrow W^{-1,q}(\Omega), \quad \langle -\Delta_p u, v \rangle_p = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx.$$

The existence of a global solution for wave equation of the p -Laplacian type

$$u'' - \Delta_p u = 0 \tag{1.1}$$

without any additional dissipation term is an open problem. For $n = 1$, Derher [14] proved the local-in-time existence of solution and demonstrated by using a generic counter example that the global-in-time solution cannot be expected. Adding a strong damping $-\Delta u'$ in (1.1) the well-posedness and asymptotic behavior was studied by Greenberg [9]. Weak solutions and blow-up for wave equations of the p -Laplacian type with supercritical sources was considered in [18]. Ma and Soriano [22] gave the weak solution for the problem with a dissipative source term $g(u)$ where $g(u)u \geq 0$ has growth bound. Nevertheless, if the strong damping is replaced by a weaker damping u' , then global existence and uniqueness are only known for $n = 1, 2$, see the works of Chueshov and Lasiecka [8] and Zhijian [23]. In the manuscript [7], Gao and Ma analyzed the existence of solution with the damping $(-\Delta)^\alpha u'$ with $0 < \alpha \leq 1$ and extended the result of [22] for $g(u)$ without the sign condition $g(u)u \geq 0$. In fact, the authors deal with the following formulation

$$\begin{cases} u'' - \Delta_p u + g(x, u) - (\Delta)^\alpha u' = f(x, t) & \text{in } Q, \\ u(x, 0) = u_0(x) & \text{in } \Omega, \\ u'(x, 0) = u_1(x) & \text{in } \Omega, \\ u(x, t) = 0 & \text{in } \Sigma, \end{cases} \tag{1.2}$$

where $f \in L^2(0, T; W^{-1,q}(\Omega))$ being $g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ a continuous function satisfying the growth condition

$$|g(x, u)| \leq a|u|^{r-1} + b \text{ for all } (x, t) \in \Omega \times \mathbb{R},$$

with a, b as positive constants and

$$1 < r < \frac{np}{n-p} \text{ if } n > p \text{ and } 1 < r < +\infty \text{ if } n \leq p. \tag{1.3}$$

Coupled wave system has been considered in various contexts. In [10], both wave equations are damped on the boundary and the coupling is effected by compact operator and exponential stability is obtained when the boundary damping is linear. Boundary damping is also considered in [1]. On exact boundary controllability for linearly coupled wave equations, we cite [4]. For weak damping acting, only one equation, the optimal polynomial decay to coupled wave equations, was studied in [13]. In [5], it was proved that the energy of weakly dissipative associated coupled system decays polynomially with explicit polynomial decay rates for sufficiently smooth solutions. Exponential decay for wave equations coupled with locally distributed damping was given in [21]. Uniform exponential stability was given in [17] for wave equations coupled in parallel with coupling distributed springs and viscous dampers due to different boundary conditions and wave propagation speeds. Lack of exponential decay of a coupled system of wave equations with memory was treated in [2].

The damping term $-\Delta u'$ plays an essential role in order to obtain global solutions. For instance, the problem

$$\begin{cases} u'' - \Delta_p u + |u|^{r-1}u - \Delta u' = 0 & \text{in } Q, \\ u(x, 0) = u_0(x) & \text{in } \Omega, \\ u'(x, 0) = u_1(x) & \text{in } \Omega, \\ u(x, t) = 0 & \text{in } \Sigma, \end{cases} \quad (1.4)$$

can be derived from (1.2) and it has a global solution $u \in L^\infty(0, T; W_0^{1,p}(\Omega))$ when $u_0(x) \in W_0^{1,p}(\Omega)$, $u_1(x) \in L^2(\Omega)$ and the condition (1.3) holds. In this work we are interested in proving the existence and energy decay to the following nonlinear coupled system

$$\begin{cases} u'' - \Delta_p u + |u|^{r-1}u|v|^{r+1} - \Delta u' = 0 & \text{in } Q, \\ v'' - \Delta_p v + |v|^{r-1}v|u|^{r+1} - \Delta v' = 0 & \text{in } Q, \\ (u(x, 0), v(x, 0)) = (u_0(x), v_0(x)) & \text{in } \Omega, \\ (u'(x, 0), v'(x, 0)) = (u_1(x), v_1(x)) & \text{in } \Omega, \\ u(x, t) = v(x, t) = 0 & \text{in } \Sigma. \end{cases} \quad (1.5)$$

The mathematical structure of the paper is organized as follows. In section 2, the global solutions are constructed by means of the Galerkin approximations while in section 3 we apply the results due to M. Nakao [15, 16] to prove the exponential stability when $p = 2$ and polynomial decay if $2 < p$.

2. Existence of Solution

Theorem 2.1. *Let us assume $1 < r < \frac{np}{n-p}$ if $n > p$ and $1 < r < +\infty$ if $n \leq p$, with $r+1 < p$. Then given $(u_0, v_0) \in [W_0^{1,p}(\Omega) \cap L^{r+1}(\Omega)]^2$, $(u_1, v_1) \in [L^2(\Omega)]^2$, $p \geq 2$, there exist functions $u, v: \Omega \times (0, T) \rightarrow \mathbb{R}$ such that,*

$$(u, v) \in [L^\infty(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(0, T; L^{r+1}(\Omega))]^2, \quad (2.1)$$

$$(u', v') \in [L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))]^2, \quad (2.2)$$

$$\frac{d}{dt}(u', w) + \langle -\Delta_p u, w \rangle + (|u|^{r-1}u|v|^{r+1}, w) + (\nabla u', \nabla w) = 0, \quad (2.3)$$

$$\frac{d}{dt}(v', z) + \langle -\Delta_p v, z \rangle + (|v|^{r-1}v|u|^{r+1}, z) + (\nabla v', \nabla z) = 0, \quad (2.4)$$

$$\forall w, z \in W_0^{1,p}(\Omega) \text{ in } D'(0, T),$$

$$(u(0), v(0)) = (u_0, v_0), \quad (u'(0), v'(0)) = (u_1, v_1). \quad (2.5)$$

Proof. From Sobolev immersion, we have

$$W_0^{\nu,q}(\Omega) \hookrightarrow W_0^{\nu-k,q_k}(\Omega), \quad \frac{1}{q_k} = \frac{1}{q} - \frac{k}{n}.$$

Choosing $q_k = p$, $\nu - k = 1$, and $q = 2$, we get

$$\nu = 1 + \frac{n}{2} - \frac{n}{p} = 1 + \frac{n(p-2)}{2p} > 0$$

and we obtain a Hilbert Space $H_0^\nu(\Omega)$ such that

$$H_0^\nu(\Omega) = W_0^{\nu,2}(\Omega) \hookrightarrow W_0^{1,p}(\Omega).$$

Let s an integer for which $s > \nu$. We have

$$H_0^s(\Omega) \hookrightarrow W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega).$$

According to the Rellich-Kondrachov theorem, $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, so is also the immersion $H_0^s(\Omega) \hookrightarrow L^2(\Omega)$. From spectral theory, there exists an operator defined by

$$\{H_0^s(\Omega), L^2(\Omega), ((\cdot, \cdot))_{H_0^s(\Omega)}\}$$

and a sequence of eigenvectors $(v_j)_{j \in \mathbb{N}}$ of this operator such that

$$((v_j, v))_{H_0^s(\Omega)} = \lambda_j(v_j, v), \text{ for all } v \in H_0^s(\Omega)$$

with $\lambda_j > 0$, $\lambda_j \leq \lambda_{j+1}$, and $\lambda_j \rightarrow +\infty$ as $j \rightarrow +\infty$. Moreover $(v_j)_{j \in \mathbb{N}}$ is a complete orthonormal system in $L^2(\Omega)$ and $\left(w_j = \frac{v_j}{\sqrt{\lambda_j}}\right)_{j \in \mathbb{N}}$ is a complete orthonormal system in $H_0^s(\Omega)$. Then $(w_j)_{j \in \mathbb{N}}$ yields a ‘‘Galerkin basis’’ for both $W_0^{1,p}(\Omega)$ and $L^2(\Omega)$. For each $m \in \mathbb{N}$, let us put $V_m = \text{Span}\{w_1, w_2, \dots, w_m\}$. We search for functions

$$u_m(t) = \sum_{j=1}^m k_{jm}(t)w_j, \quad v_m(t) = \sum_{j=1}^m h_{jm}(t)w_j,$$

such that any $w, z \in V_m$, $u_m(t)$ and $v_m(t)$ satisfies the following approximate problem

$$(u_m''(t), w) + \langle -\Delta_p u_m(t), w \rangle + (|u_m(t)|^{r-1} u_m(t) |v_m(t)|^{r+1}, w) + (\nabla u_m'(t), \nabla w) = 0, \quad (2.6)$$

$$(v_m''(t), z) + \langle -\Delta_p v_m(t), z \rangle + (|v_m(t)|^{r-1} v_m(t) |u_m(t)|^{r+1}, z) + (\nabla v_m'(t), \nabla z) = 0, \quad (2.7)$$

$$(u_m(0), v_m(0)) = (u_{0m}, v_{0m}) \longrightarrow (u_0, v_0) \text{ in } [W_0^{1,p}(\Omega)]^2, \quad (2.8)$$

$$(u_m'(0), v_m'(0)) = (u_{1m}, v_{1m}) \longrightarrow (u_1, v_1) \text{ in } [L^2(\Omega)]^2. \quad (2.9)$$

Putting $w = z = w_j$, multiplying (2.6) by $k_{jm}(t)$, and (2.7) by $h_{jm}(t)$, and adding with j from 1 to m , we observe that (2.6)-(2.7) is a system of ODEs in the variable t and has a local solution $u_m(t)$, $v_m(t)$ in a interval $[0, t_m)$ by virtue of Carathéodory's theorem. In the next step, we obtain the a priori estimates for the solution $u_m(t)$, $v_m(t)$ so that it can be extended to the whole interval $[0, T)$, $T > 0$.

A priori Estimates

We consider $w = u_m'(t)$ and $z = v_m'(t)$ in (2.6) and (2.7), respectively, we obtain

$$\frac{d}{dt} \left[\frac{1}{2} |u_m'(t)|_2^2 + \frac{1}{p} \|\nabla u_m(t)\|_p^p \right] + (|u_m(t)|^{r-1} u_m(t) |v_m(t)|^{r+1}, u_m'(t)) + |\nabla u_m'(t)|_2^2 = 0 \quad (2.10)$$

and

$$\frac{d}{dt} \left[\frac{1}{2} |v_m'(t)|_2^2 + \frac{1}{p} \|\nabla v_m(t)\|_p^p \right] + (|v_m(t)|^{r-1} v_m(t) |u_m(t)|^{r+1}, v_m'(t)) + |\nabla v_m'(t)|_2^2 = 0. \quad (2.11)$$

Adding (2.10) and (2.11) and noting that

$$\begin{aligned} \frac{1}{r+1} \frac{d}{dt} (|u_m(t)|^{r+1}, |v_m(t)|^{r+1}) &= (|u_m(t)|^{r-1} u_m(t) |v_m(t)|^{r+1}, u_m'(t)) \\ &\quad + (|v_m(t)|^{r-1} v_m(t) |u_m(t)|^{r+1}, v_m'(t)) \end{aligned}$$

we obtain that

$$\begin{aligned} & \frac{d}{dt} \left[\frac{1}{2} |u'_m(t)|_2^2 + \frac{1}{2} |v'_m(t)|_2^2 + \frac{1}{p} \|\nabla u_m(t)\|_p^p + \frac{1}{p} \|\nabla v_m(t)\|_p^p \right. \\ & \left. + \frac{1}{r+1} (|u_m(t)|^{r+1}, |v_m(t)|^{r+1}) \right] + |\nabla u'_m(t)|_2^2 + |\nabla v'_m(t)|_2^2 = 0. \end{aligned}$$

Considering the approximate energy

$$\begin{aligned} E_m(t) &= \frac{1}{2} (|u'_m(t)|_2^2 + |v'_m(t)|_2^2) + \frac{1}{p} (\|\nabla u_m(t)\|_p^p + \|\nabla v_m(t)\|_p^p) \\ &+ \frac{1}{r+1} (|u_m(t)|^{r+1}, |v_m(t)|^{r+1}), \end{aligned}$$

we have

$$\frac{d}{dt} E_m(t) = -|\nabla u'_m(t)|_2^2 - |\nabla v'_m(t)|_2^2. \quad (2.12)$$

Integrating (2.12) from 0 to t , $t \leq t_m$, we obtain

$$E_m(t) + \int_0^t [|\nabla u'_m(s)|_2^2 + |\nabla v'_m(s)|_2^2] ds = E_m(0) \leq C, \quad (2.13)$$

$C > 0$ constant independent of m and t . Then we have

$$(u_m), (v_m) \text{ are bounded in } L^\infty(0, T; W_0^{1,p}(\Omega)), \quad (2.14)$$

$$(u'_m), (v'_m) \text{ are bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)), \quad (2.15)$$

$$(-\Delta_p u_m), (-\Delta_p v_m) \text{ are bounded in } L^q(0, T; W^{-1,q}(\Omega)), \quad (2.16)$$

$$(u_m v_m) \text{ is bounded in } L^\infty(0, T; L^{r+1}(\Omega)). \quad (2.17)$$

Since our Galerkin basis was taken in the Hilbert space $L^2(\Omega)$, we can use the standard projection arguments as described in Lions [12]. Then, the approximate equations and the estimates (2.14)-(2.17) we get

$$(u''_m), (v''_m) \text{ are bounded in } L^q(0, T; W^{-1,q}(\Omega)). \quad (2.18)$$

Passage to the Limit

From (2.14)-(2.17), there are subsequences of $(u_m), (v_m)$ still denoted by $(u_m), (v_m)$ such that

$$(u_m, v_m) \xrightarrow{*} (u, v) \text{ in } [L^\infty(0, T; W_0^{1,p}(\Omega))]^2, \quad (2.19)$$

$$(u'_m, v'_m) \xrightarrow{*} (u', v') \text{ in } [L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))]^2, \quad (2.20)$$

$$(-\Delta_p u_m, -\Delta_p v_m) \rightharpoonup (\chi_1, \chi_2) \text{ in } [L^q(0, T; W^{-1,q}(\Omega))]^2, \quad (2.21)$$

$$(u_m v_m) \xrightarrow{*} (uv) \text{ in } L^\infty(0, T; L^{r+1}(\Omega)). \quad (2.22)$$

Applying the Lions-Aubin compactness Lemma [12], we get from (2.19), (2.20) and (2.18), (2.20), respectively,

$$(u_m, v_m) \longrightarrow (u, v) \text{ in } [L^2(0, T; L^2(\Omega))]^2 = [L^2(Q)]^2, \quad (2.23)$$

$$(u'_m, v'_m) \longrightarrow (u', v') \text{ in } [L^2(0, T; L^2(\Omega))]^2 = [L^2(Q)]^2. \quad (2.24)$$

Whence

$$(u_m, v_m) \longrightarrow (u, v) \text{ a.e. in } [Q]^2$$

and

$$|u_m|^{r-1} u_m |v_m|^{r+1} \longrightarrow |u|^{r-1} u |v|^{r+1} \text{ a.e. in } Q,$$

$$|v_m|^{r-1} v_m |u_m|^{r+1} \longrightarrow |v|^{r-1} v |u|^{r+1} \text{ a.e. in } Q.$$

Since $p > r + 1$, we have $u_m v_m$ bounded in $L^\infty(0, T; L^{r+1}(\Omega)) \subset L^{r+1}(Q)$ and v_m bounded in $L^\infty(0, T; W_0^{1,p}(\Omega)) \subset L^{r+1}(Q)$.

Consider $\frac{r}{r+1} + \frac{1}{r+1} = 1$. Applying Hölder's inequality we obtain

$$\begin{aligned} \int_Q |u_m|^{r-1} u_m |v_m|^{r+1} dx dt &\leq \int_Q |u_m v_m|^r |v_m| dx dt \\ &\leq \left[\int_Q (|u_m v_m|^r)^{\frac{r+1}{r}} dx dt \right]^{\frac{r}{r+1}} \left[\int_Q |v_m|^{r+1} dx dt \right]^{\frac{1}{r+1}} \\ &= \left[\int_Q (|u_m v_m|)^{r+1} dx dt \right]^{\frac{r}{r+1}} \left[\int_Q |v_m|^{r+1} dx dt \right]^{\frac{1}{r+1}} \\ &= |u_m v_m|_{L^{r+1}(Q)}^r |v_m|_{L^{r+1}(Q)} \\ &\leq C. \end{aligned}$$

Therefore, from Lions [12] Lemma 1.3, we infer that

$$|u_m|^{r-1} u_m |v_m|^{r+1} \rightharpoonup |u|^{r-1} u |v|^{r+1} \text{ in } L^1(Q) \quad (2.25)$$

and

$$|v_m|^{r-1} v_m |u_m|^{r+1} \rightharpoonup |v|^{r-1} v |u|^{r+1} \text{ in } L^1(Q). \quad (2.26)$$

Finally, since we have the strong convergence (2.23), we can use a standard monotonicity argument as proposed by Lions [12] to show that $\chi_1 = -\Delta_p u$ and $\chi_2 = -\Delta_p v$.

Now, with the convergences (2.19)-(2.21) and (2.25), (2.26) we have

$$\frac{d}{dt}(u'(t), w) + \langle -\Delta_p u(t), w \rangle + (|u(t)|^{r-1} u(t) |v(t)|^{r+1}, w) + (\nabla u'(t), w) = 0 \quad (2.27)$$

and

$$\frac{d}{dt}(v'(t), z) + \langle -\Delta_p v(t), z \rangle + (|v(t)|^{r-1} v(t) |u(t)|^{r+1}, z) + (\nabla v'(t), z) = 0 \quad (2.28)$$

for all $w, z \in W_0^{1,p}(\Omega)$ in $D'(0, T)$, at the sense of distributions.

The verification of the initial data is a routine procedure. The proof of existence is complete. $\square$

3. Asymptotic Behavior

In proof of Theorem 3.2, we use the following result due to M. Nakao.

Lemma 3.1. *Suppose that $\phi(t)$ is a bounded nonnegative function on $\mathbb{R}^+$, satisfying*

$$\sup_{t \leq s \leq t+1} \phi^{1+\beta}(s) \leq C_0[\phi(t) - \phi(t+1)],$$

$\forall t \geq 0$, where $\beta \geq 0$ and C_0 are positive constants. Then,

- (a) If $\beta = 0$ then $\phi(t) \leq Ce^{-\gamma t}$, $\forall t \geq 0$,
- (b) If $\beta > 0$ then $\phi(t) \leq C(1+t)^{-\frac{1}{\beta}}$, $\forall t \geq 0$,

where C and γ are positive constants.

Proof. See, theorem 1 of [15] and lemma 2 of [16]. □

The main result of this paper is

Theorem 3.2. *Under the hypotheses of Theorem 2.1., the solution of system (1.5) satisfies:*

- (a) $E(t) \leq C(E(0))e^{-\gamma t}$, if $p = 2$,
- (b) $E(t) \leq C(E(0))(1+t)^{-\frac{p}{p-2}}$, if $p > 2$,

$\forall t \geq 0$, where $C(E(0))$ and γ are positive constants.

Proof. Replacing w, z in (2.26), (2.27) by $u'(t), v'(t)$, respectively, and adding the equations we have

$$\frac{d}{dt}E(t) + |\nabla u'(t)|_2^2 + |\nabla v'(t)|_2^2 = 0$$

where

$$E(t) = \frac{1}{2}|u'(t)|_2^2 + \frac{1}{2}|v'(t)|_2^2 + \frac{1}{p}\|\nabla u(t)\|_p^p + \frac{1}{p}\|\nabla v(t)\|_p^p + \frac{1}{r+1} \int_{\Omega} |u(t)|^{r+1}|v(t)|^{r+1} dx.$$

Integrating from t to $t+1$, $t \geq 0$, we obtain

$$E(t+1) = \int_t^{t+1} (|\nabla u'(s)|_2^2 + |\nabla v'(s)|_2^2) ds = E(t) \quad (3.1)$$

so,

$$\begin{aligned} \int_t^{t+1} (|u'(s)|_2^2 + |v'(s)|_2^2) ds &\leq C \int_t^{t+1} (|\nabla u'(s)|_2^2 + |\nabla v'(s)|_2^2) ds \\ &= C[E(t) - E(t+1)] \stackrel{\text{def}}{=} F^2(t) \end{aligned} \quad (3.2)$$

thus, there exist $t_1 \in [t, t + \frac{1}{4}]$, $t_2 \in [t + \frac{3}{4}, t+1]$ such that

$$|u'(t_i)| \leq 2F(t), \quad |v'(t_i)| \leq 2F(t), \quad i = 1; 2. \quad (3.3)$$

Replacing w, z in (2.27), (2.28) by $u(t)$ and $v(t)$ respectively, and adding the equations we have

$$\begin{aligned} \frac{d}{dt} [(u'(t), u(t)) + (v'(t), v(t))] - |u'(t)|_2^2 - |v'(t)|_2^2 + \|\nabla u(t)\|_p^p + \|\nabla v(t)\|_p^p \\ + 2 \int_{\Omega} |u(t)|^{r+1}|v(t)|^{r+1} dx + (\nabla u'(t), \nabla u(t)) + (\nabla v'(t), \nabla v(t)) = 0. \end{aligned} \quad (3.4)$$

Integrating (3.4) from t_1 to t_2 and using (3.2) and (3.3) we obtain

$$\begin{aligned} & \int_{t_1}^{t_2} \left[\|\nabla u(t)\|_p^p + \|\nabla v(t)\|_p^p + 2 \int_{\Omega} |u(s)|^{r+1} |v(s)|^{r+1} dx \right] ds \\ & \leq [|u'(t_1)|_2 \|\nabla u(t_1)\|_p + |u'(t_2)| \|\nabla u(t_2)\|_p] + F^2(t) \\ & \quad + \int_{t_1}^{t_2} [| \nabla u'(s)|_2 |\nabla u(s)|_2 + | \nabla v'(s)|_2 |\nabla v(s)|_2] ds. \end{aligned} \quad (3.5)$$

Now using Young's inequality, we have

$$\begin{aligned} & \int_{t_1}^{t_2} [| \nabla u'(s)|_2 |\nabla u(s)|_2 + | \nabla v'(s)|_2 |\nabla v(s)|_2] ds \\ & \leq C \int_{t_1}^{t_2} [| \nabla u'(s)|_2 \|\nabla u(s)\|_p + | \nabla v'(s)|_2 \|\nabla v(s)\|_p] ds \\ & \leq \frac{1}{p} \int_{t_1}^{t_2} [\|\nabla u(s)\|_p^p + \|\nabla v(s)\|_p^p] ds + C_1 \left(\int_{t_1}^{t_2} [| \nabla u'(s)|_2^2 + | \nabla v'(s)|_2^2] ds \right)^{\frac{q}{2}} \\ & \leq \frac{1}{p} \int_{t_1}^{t_2} [\|\nabla u(s)\|_p^p + \|\nabla v(s)\|_p^p] ds + C_1 F^q(t). \end{aligned} \quad (3.6)$$

From (3.5) and (3.6), we get

$$\begin{aligned} & \int_{t_1}^{t_2} \left\{ \left(1 - \frac{1}{p} \right) [\|\nabla u(s)\|_p^p + \|\nabla v(s)\|_p^p] + 2 \int_{\Omega} |u(s)|^{r+1} |v(s)|^{r+1} dx \right\} ds \\ & \leq C_2 \left[F^2(t) + F^q(t) + F(t) \sup_{t \leq s \leq t+1} \text{ess } \sqrt[p]{E(s)} \right] \stackrel{\text{def}}{=} G^2(t). \end{aligned} \quad (3.7)$$

From (3.2) and (3.7), it follows that

$$\int_{t_1}^{t_2} E(s) ds \leq F^2(t) + G^2(t).$$

Then, there exists $t^* \in [t_1, t_2]$ such that

$$E(t^*) \leq 2F^2(t) + 2G^2(t).$$

From (3.1) we have

$$\sup_{t \leq s \leq t+1} \text{ess } E(s) \leq E(t^*) + \int_{t_1}^{t_2} [| \nabla u'(s)|_2^2 + | \nabla v'(s)|_2^2] ds$$

so,

$$\begin{aligned} \sup_{t \leq s \leq t+1} \text{ess } E(s) & \leq C_3 [F^2(t) + G^2(t)] \\ & \leq C_4 \left[F^2(t) + F^q(t) + F(t) \sup_{t \leq s \leq t+1} \text{ess } \sqrt[p]{E(s)} \right] \\ & \leq C_5 [F^2(t) + F^q(t)] + \frac{1}{p} \sup_{t \leq s \leq t+1} \text{ess } E(s). \end{aligned}$$

Therefore,

$$\sup_{t \leq s \leq t+1} \text{ess } E(s) \leq C_6 F^q(s) [F^{2-q}(t) + 1]. \quad (3.8)$$

By (3.2) we have

$$\begin{aligned} F^2(t) &= \int_t^{t+1} [|\nabla u'(s)|_2^2 + |\nabla v'(s)|_2^2] \, ds \leq \int_0^{t+1} [|\nabla u'(s)|_2^2 + |\nabla v'(s)|_2^2] \, ds \\ &= E(0) - E(t+1) < E(0). \end{aligned}$$

Whence,

$$F^{2-q}(t) = (F^2(t))^{\frac{2-q}{2}} < (E(0))^{\frac{2-q}{2}} = C(E(0)). \quad (3.9)$$

From (3.8) and (3.9), we get,

$$\sup_{t \leq s \leq t+1} \text{ess } E(s) \leq C(E(0))F^q = C(E(0)) (F^2(t))^{\frac{q}{2}}$$

and

$$\sup_{t \leq s \leq t+1} \text{ess } E^{\frac{2}{q}}(s) \leq C(E(0))F^2(t).$$

Thus, by (3.2) we have

$$\sup_{t \leq s \leq t+1} \text{ess } E^{\frac{2}{q}}(s) \leq C(E(0))[E(t) - E(t+1)].$$

Therefore, applying lemma for $\phi(t) = E(t)$ we have:

(a) If $p = 2$ then $q = 2$ and

$$E(t) \leq C(E(0))e^{-\gamma t}, \quad \forall t \geq 0$$

where γ is a positive constant.

(b) If $p > 2$ then $\frac{2}{q} = 1 + \frac{p-2}{p}$ and so

$$E(t) \leq C(E(0))(1+t)^{-\frac{p}{p-2}}, \quad \forall t \geq 0.$$

The proof of Theorem is complete. $\square$

Final comments

Results on existence of solutions to problem with the p -Laplacian operator has a non-trivial history. Only the subcritical case ($r < p$) was considered in this work. In the supercritical scenario ($r > p$), we have a global solution just under a small assumption on the initial data. It should be noted here that the presence of the damping mechanism was also essential for stabilization of the system. For instance, the presence of the source $|u|^{r-1}u$ in (1.4) alone should drive the solution to blow-up in finite time, as seen, for example, in the work of Glassey [19]. However, when both damping and source are present, the analysis of their interaction and their influence on the global behavior of solutions become more difficult and have served as a motivation for this study, see [11, 3, 6, 20] and the references therein.

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