

RESULTS ON UNIQUENESS OF PRODUCT OF CERTAIN TYPE OF SHIFT POLYNOMIALS

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Date of Receiving : 01. 06. 2020
 Date of Revision : 04. 08. 2020
 Date of Acceptance : 05. 08. 2020

Abstract. In this paper, using the concept of fixed point sharing with finite weight, we investigate the value distribution and uniqueness of product of certain type of difference polynomials. The results of the paper improve, extend and generalizes the results due to Renukadevi S. Dyavanal and Rajalaxmi V. Desai [19].

1. Introduction

In this article, we assume that the reader is familiar with the fundamental results and the standard notation of Nevanlinna value distribution theory (see [6],[12] and [13]). We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \rightarrow \infty$, possibly outside of a set E with finite linear or logarithmic measure, not necessarily the same at each occurrence. We say that $a(z)$ is a small function of $f(z)$ if $T(r, a) = S(r, f)$. Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions in the complex plane. We say that two meromorphic functions f and g share a IM (ignoring multiplicities) if $f(z) - a$ and $g(z) - a$ have the same zeros. If $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicities, then we say that they share a CM (counting multiplicities). The order and hyper order of meromorphic function f are defined by

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}, \quad \rho_2(f) = \limsup_{r \rightarrow \infty} \frac{\log \log T(r, f)}{\log r}.$$

In addition, we need the following definitions.

Definition 1.[21] Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f | = 1)$ the counting function of simple a points of f . For a positive integer k we denote by $N(r, a; f | \leq k)$ the counting function of those a -points of f (counted with proper multiplicities) whose multiplicities are not greater than k . By $\overline{N}(r, a; f | \leq k)$ we denote the corresponding reduced counting function. Analogously we can define $N(r, a; f | \geq k)$ and $\overline{N}(r, a; f | \geq k)$.

2010 *Mathematics Subject Classification.* 30D35.

Key words and phrases. Meromorphic function, Value distribution, Difference polynomials.

Communicated by. Indrajit Lahiri

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Definition 2.[8] Let k be a positive integer or infinity. We denote by $N_k(r, a; f)$ the counting function of a -points of f , where an a -point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f | \geq 2) + \dots + \overline{N}(r, a; f | \geq k).$$

Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 3.[2] Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p and also a 1-point of g with multiplicity q . We denote by $N_L(r, 1; f)$ the counting function of those 1-points of f and g , where $p > q$, by $N_E^{(k)}(r, 1; f)$ ($k \geq 2$ is an integer) the counting function of those 1-points of f and g , where $p = q \geq k$, where each point in these counting functions is counted only once. In the same manner we can define $N_L(r, 1; g)$ and $N_E^{(k)}(r, 1; g)$.

Definition 4.([7],[8]) Let f and g be two non-constant meromorphic functions such that f and g share the value a IM. We denote by $N_*(r, a; f, g)$ the reduced counting function of those a -points of f whose multiplicities differ from the multiplicities of the corresponding a -points of g . Clearly $N_*(r, a; f, g) = N_*(r, a; g, f)$ and $N_*(r, a; f, g) = N_L(r, a; f) + N_L(r, a; g)$.

Recently, people have raised great interest in difference analogues of Nevanlinna's theory and many articles have focused on value distribution and uniqueness of difference polynomials of entire or meromorphic functions.

In 2012, Y. H. Cao and X. B. Zhang ([3]) obtained the following theorem.

Theorem 1.1. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions, whose zeros are of multiplicities at least k , where k is a positive integer. Let $n > \max\{2k - 1; 4 + 4/k + 4\}$ be a positive integer. If $f^n(z)f^{(k)}(z)$ and $g^n(z)g^{(k)}(z)$ share z CM, and $f(z)$ and $g(z)$ share ∞ IM, then one of the following two conclusions holds

- (1) $f^n(z)f^{(k)}(z) = g^n(z)g^{(k)}(z)$;
- (2) $f(z) = c_1 e^{cz^2}, g(z) = c_2 e^{-cz^2}$, where c, c_1 and c_2 are constants such that $4(c_1 c_2)^{n+1} c_2 = -1$.

In 2014, X. B. Zhang [17] reduced the lower bound on n and relax the condition on multiplicity of zeros in Theorem 1.1 and proved the following result.

Theorem 1.2. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions and n, k two positive integers with $n > k + 6$. If $f^n(z)f^{(k)}(z)$ and $g^n(z)g^{(k)}(z)$ share z CM, and $f(z)$ and $g(z)$ share ∞ IM, then one of the following two conclusions holds

- (1) $f^n(z)f^{(k)}(z) = g^n(z)g^{(k)}(z)$;
- (2) $f(z) = c_1 e^{cz^2}, g(z) = c_2 e^{-cz^2}$, where c, c_1 and c_2 are constants such that $4(c_1 c_2)^{n+1} c_2 = -1$.

In 2016, R. S. Dyavanal and A. M. Hattikal [5] obtained the following result.

Theorem 1.3. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of hyper order $\rho_2(f) < 1$ and $\rho_2(g) < 1$. Let $k, n, d, s_j (j = 1, 2, \dots, d)$, $\lambda = \sum_{j=1}^d s_j$ be positive integers, $c_j \in \mathbb{C} - \{0\} (j = 1, 2, \dots, d)$ are distinct constants and $n > \max\{2d(k+2) + \lambda(k+3) + 7, \lambda_1; \lambda_2\}$, where $\lambda_1 = \sum_{j=1}^d \alpha_j s_j$ and $\lambda_2 = \sum_{j=1}^d \beta_j s_j$ $j = 1, 2, \dots, d$ and $f(z + c_j)$ and $g(z + c_j)$ have zeros with maximum orders α_j and β_j respectively. If $f(z)^n [\prod_{j=1}^d f(z + c_j)^{s_j}]^{(k)}$ and $g(z)^n [\prod_{j=1}^d g(z + c_j)^{s_j}]^{(k)}$ share z CM and $f(z), g(z)$ share ∞ IM, then one of the following two conclusions holds.

- (1) $f(z)^n [\prod_{j=1}^d f(z + c_j)^{s_j}]^{(k)} = g(z)^n [\prod_{j=1}^d g(z + c_j)^{s_j}]^{(k)}$
- (2) $\prod_{j=1}^d f(z + c_j)^{s_j} = c_1 e^{cz^2}$, $\prod_{j=1}^d g(z + c_j)^{s_j} = c_2 e^{-cz^2}$, where c, c_1 and c_2 are constants such that $4(c_1 c_2)^{n+1} c_2 = -1$.

In 2018, R. S. Dyavanal and R. V. Desai [19] proved the following theorem.

Theorem 1.4. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of hyper orders $\rho_2(f) < 1$ and $\rho_2(g) < 1$. Let $k, n, d, s_j (j = 1, 2, \dots, d)$, $\lambda = \sum_{j=1}^d s_j$ be positive integers, $c_j \in \mathbb{C} - \{0\} (j = 1, 2, \dots, d)$ are distinct constants. If $f(z)^n [\prod_{j=1}^d f(z + c_j)^{s_j}]^{(k)}$ and $g(z)^n [\prod_{j=1}^d g(z + c_j)^{s_j}]^{(k)}$ share (z, l) CM and $f(z), g(z)$ share ∞ IM, where $0 \leq l < \infty$. Then the conclusions of Theorem 1.3 hold provided

1. if $2 \leq l < \infty$ then $n \geq 2(k+2)d + (k+4)\lambda + 8$,
2. if $l = 1$ then $n \geq \lambda(\frac{3}{2}k + \frac{7}{2}) + d(2(k+2) + \frac{1}{2}(k+1)) + 9$,
3. if $l = 0$ then $n \geq d(5k+7) + 7\lambda + 4k\lambda + 14$.

In this paper, we assume $c_j \in \mathbb{C} \setminus \{0\} (j = 1, 2, \dots, d)$ are distinct constants, $n, k, d, s_j (j = 1, 2, \dots, d)$ are positive integers and $\lambda = s_1 + s_2 + \dots + s_d$. Let

$$F(z) = f(z)^n P(f) \left[\prod_{j=1}^d f(z + c_j)^{s_j} \right]^{(k)}$$

$$F_1(z) = f(z)^n P(f) \prod_{j=1}^d f(z + c_j)^{s_j}$$

where $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$ be a nonzero polynomial, where $a_0, \dots, a_n (\neq 0)$ are complex constants, and n is an integer, let $\Gamma_0 = m_1 + m_2$ and $\Gamma_1 = m_1 + 2m_2$, where m_1 is the number of the simple zero of $P(z)$, and m_2 is the number of multiple zeros of $P(z)$.

We consider the uniqueness problems on product of difference polynomials and obtain the following results, which improve the Theorem 1.4.

Theorem 1.5. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of hyper orders $\rho_2(f) < 1$ and $\rho_2(g) < 1$. Let $k, n, d, s_j (j = 1, 2, \dots, d)$, $\lambda = \sum_{j=1}^d s_j$ be positive integers, $c_j \in \mathbb{C} - \{0\} (j = 1, 2, \dots, d)$ are distinct constants, $P(z) = a_m z^m + a_{m-1} z^{m-1} + \dots + a_0$ be a nonzero polynomial, where $a_0, \dots, a_m (\neq 0)$ are complex constants, and m is an integer. If $f(z)^n P(f) [\prod_{j=1}^d f(z + c_j)^{s_j}]^{(k)}$ and $g(z)^n P(g) [\prod_{j=1}^d g(z + c_j)^{s_j}]^{(k)}$ share (z, l) CM and $f(z), g(z)$ share ∞ IM, where $0 \leq l < \infty$. Then the conclusions of Theorem 1.3 hold provided

1. if $2 \leq l < \infty$ then $n \geq 2(k+2)d + (k+4)\lambda + 2\Gamma_1 - m + 8$,
2. if $l = 1$ then $n \geq \lambda(\frac{3}{2}k + \frac{9}{2}) + d(2(k+2) + \frac{1}{2}(k+1)) + \frac{\Gamma_0}{2} - 2\Gamma_1 - m + 9$,
3. if $l = 0$ then $n \geq d(5k+7) + 7\lambda + 4k\lambda + 2\Gamma_1 + 3\Gamma_0 - m + 14$.

Corollary 1. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of hyper orders $\rho_2(f) < 1$ and $\rho_2(g) < 1$. Let k, n, d, s_j ($j = 1, 2, \dots, d$), $\lambda = \sum_{j=1}^d s_j$ be positive integers, $c_j \in \mathbb{C} - \{0\}$ ($j = 1, 2, \dots, d$) are distinct constants, $P(z) = a_m z^m + a_{m-1} z^{m-1} + \dots + a_0$ be a nonzero polynomial, where $a_0, \dots, a_m (\neq 0)$ are complex constants, and m is an integer. If $f(z)^n P(f) [\prod_{j=1}^d f(z + c_j)^{s_j}]^{(k)}$ and $g(z)^n P(g) [\prod_{j=1}^d g(z + c_j)^{s_j}]^{(k)}$ share $(1, l)$ CM and $f(z), g(z)$ share ∞ IM, where $0 \leq l < \infty$. Then the conclusions of Theorem 1.3 hold provided

1. if $2 \leq l < \infty$ then $n \geq 2(k+2)d + (k+4)\lambda + 2\Gamma_1 - m + 8$,
2. if $l = 1$ then $n \geq \lambda(\frac{3}{2}k + \frac{9}{2}) + d(2(k+2) + \frac{1}{2}(k+1)) + \frac{\Gamma_0}{2} - 2\Gamma_1 - m + 9$,
3. if $l = 0$ then $n \geq d(5k+7) + 7\lambda + 4k\lambda + 2\Gamma_1 + 3\Gamma_0 - m + 14$.

2. Some Lemmas

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We shall denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right).$$

Lemma 2.1.[4] Let $f(z)$ be a transcendental meromorphic function of hyper order $\rho_2(f) < 1$ and let c be a non-zero complex constant. Then we have

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

Lemma 2.2.[2] Let F, G be two non constant meromorphic functions sharing $(1, l)$, $(\infty, 0)$, where $2 \leq l < \infty$ and $H \not\equiv 0$. Then

$$\begin{aligned} T(r, F) &\leq N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + \overline{N}_*(r, \infty; F, G) \\ &\quad - m(r, 1; G) - N_E^{(3)}(r, 1; F) - \overline{N}_L(r, 1; G) + S(r, F) + S(r, G) \end{aligned}$$

$$\begin{aligned} T(r, G) &\leq N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}(r, \infty; G) + \overline{N}(r, \infty; F) + \overline{N}_*(r, \infty; F, G) \\ &\quad - m(r, 1; F) - N_E^{(3)}(r, 1; G) - \overline{N}_L(r, 1; F) + S(r, F) + S(r, G). \end{aligned}$$

Lemma 2.3.[11] Let F, G be two non constant meromorphic functions sharing $(1, l), (\infty, 0)$ and $H \not\equiv 0$. Then

$$\begin{aligned} T(r, F) &\leq N_2(r, 0; F) + N_2(r, 0; G) + \frac{3}{2}\overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + \overline{N}_*(r, \infty; F, G) \\ &\quad + \frac{1}{2}\overline{N}(r, 0; F) + S(r, F) + S(r, G) \end{aligned}$$

$$\begin{aligned} T(r, G) &\leq N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}(r, \infty; G) + \overline{N}(r, \infty; F) + \overline{N}_*(r, \infty; F, G) \\ &\quad + \frac{1}{2}\overline{N}(r, 0; G) + S(r, F) + S(r, G). \end{aligned}$$

Lemma 2.4.[11] Let F, G be two non constant meromorphic functions sharing $(1,0), (\infty, 0)$ and $H \not\equiv 0$. Then

$$T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + 3\overline{N}(r, \infty; F) + 2\overline{N}(r, \infty; G) + \overline{N}_*(r, \infty; F, G) \\ + 2\overline{N}(r, 0; F) + \overline{N}(r, 0; G) + S(r, F) + S(r, G).$$

$$T(r, G) \leq N_2(r, 0; F) + N_2(r, 0; G) + 3\overline{N}(r, \infty; G) + 2\overline{N}(r, \infty; F) + \overline{N}_*(r, \infty; F, G) \\ + 2\overline{N}(r, 0; G) + \overline{N}(r, 0; F) + S(r, F) + S(r, G).$$

Lemma 2.5.[14] Let $f(z)$ be a non-constant meromorphic function, and let $a_0(z), a_1(z), \dots, a_n(z) (\not\equiv 0)$ be a small functions with respect to f . Then

$$T(r, P(f)) = a_n f^n + a_{n-1} f^{n-1} + \dots + a_1 f + a_0 = nT(r, f) + S(r, f).$$

Lemma 2.6.[9] Let $f(z)$ be a non-constant meromorphic function, and p, k be positive integers. Then

$$T(r, f^{(k)}) \leq T(r, f) + k\overline{N}(r, f) + S(r, f), \\ N_p(r, \frac{1}{f^{(k)}}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, \frac{1}{f}) + S(r, f), \\ N_p(r, \frac{1}{f^{(k)}}) \leq N_{p+k}(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f).$$

Lemma 2.7.[10] Let $f(z)$ be a transcendental meromorphic function of hyper order $\rho_2(f) < 1$ and let c be a non-zero complex constant. Then we have

$$N(r, f(z+c)) = N(r, f(z)) + S(r, f), \\ N(r, \frac{1}{f(z+c)}) = N(r, \frac{1}{f(z)}) + S(r, f).$$

Lemma 2.8. Let $f(z)$ be a transcendental meromorphic function of hyper order $\rho_2(f) < 1$ and $F_1(z)$ be stated as in (1.2). Then

$$(n+m-\lambda)T(r, f) + S(r, f) \leq T(r, F_1(z)) \leq (n+m+\lambda)T(r, f) + S(r, f).$$

Proof. Since f is a meromorphic function with $\rho_2(f) < 1$. From Lemma 2.5 and Lemma 2.6 we have

$$T(r, F_1(z)) \leq T(r, f(z)^n) + T(r, P(f)) + T\left(r, \prod_{j=1}^d f(z+c_j)^{s_j}\right) + S(r, f) \\ \leq (n+m+\lambda)T(r, f) + S(r, f)$$

On the other hand, from Lemma 2.5 and Lemma 2.7, we have

$$\begin{aligned}
(n+m+\lambda)T(r, f) &= T(r, f^n f^m f^\lambda) + S(r, f) \\
&\leq m \left(r, \frac{F_1(z) f^\lambda}{\prod_{j=1}^d f(z+c_j)^{s_j}} \right) + N \left(r, \frac{F_1(z) f^\lambda}{\prod_{j=1}^d f(z+c_j)^{s_j}} \right) + S(r, f) \\
&\leq T(r, F_1(z)) + T \left(r, \frac{F_1(z) f^\lambda}{\prod_{j=1}^d f(z+c_j)^{s_j}} \right) + S(r, f) \\
&\leq T(r, F_1(z)) + 2\lambda T(r, f) + S(r, f). \\
(n+m+\lambda-2\lambda)T(r, f) &\leq T(r, F_1(z)) + S(r, f) \\
(n+m-\lambda)T(r, f) + S(r, f) &\leq T(r, F_1(z)).
\end{aligned}$$

Hence we get Lemma 2.8.

3. Proof of the Theorems

Proof of Theorem 1.5.

Let

$$F(z) = f(z)^n P(f) \left[\prod_{j=1}^d f(z+c_j)^{s_j} \right]^{(k)}, G(z) = g(z)^n P(g) \left[\prod_{j=1}^d g(z+c_j)^{s_j} \right]^{(k)} \quad (3.1)$$

$$F_1(z) = f(z)^n P(f) \left[\prod_{j=1}^d f(z+c_j)^{s_j} \right], G_1(z) = g(z)^n P(g) \left[\prod_{j=1}^d g(z+c_j)^{s_j} \right] \quad (3.2)$$

$$F_2(z) = \frac{f(z)^n P(f) \left[\prod_{j=1}^d f(z+c_j)^{s_j} \right]^{(k)}}{z}, G_2(z) = \frac{g(z)^n P(g) \left[\prod_{j=1}^d g(z+c_j)^{s_j} \right]^{(k)}}{z}. \quad (3.3)$$

Then $F_2(z)$, $G_2(z)$ are transcendental meromorphic functions that share $(1, l)$ and f , g share ∞ IM. Since f and g are transcendental, z is a small function with respect to both F and G .

Let us consider two cases separately.

Case 1: Assume that $H \neq 0$.

Now, we consider the following three subcases.

Subcase 1.1: Suppose that $2 \leq l < \infty$. Then using Lemma 2.2, we obtain

$$\begin{aligned}
T(r, F) &= T(r, F_2) + S(r, F_2) \\
&\leq N_2(r, 0, F_2) + N_2(r, 0, G_2) + \overline{N}(r, \infty, F_2) + \overline{N}(r, \infty, G_2) \\
&\quad + \overline{N}_*(r, \infty; F_2, G_2) - m(r, 1, G_2) - N_E^{(3)}(r, 1, F_2) - \overline{N}_L(r, 1, G_2) \\
&\quad + S(r, F_2) + S(r, G_2) \\
T(r, F) &\leq N_2(r, 0, F) + N_2(r, 0, G) + \overline{N}_*(r, \infty; F, G) + \overline{N}(r, \infty, F) + \overline{N}(r, \infty, G) \\
&\quad + S(r, F) + S(r, G). \quad (3.4)
\end{aligned}$$

Noting that

$$\begin{aligned}\overline{N}_*(r, \infty; F, G) &= \overline{N}_L(r, \infty, F) + \overline{N}_L(r, \infty, G) \\ &\leq \overline{N}(r, \infty, F) = \overline{N}(r, \infty, G)\end{aligned}\tag{3.5}$$

we obtain from (3.4), (3.5), Lemmas 2.8 and 2.6 that

$$\begin{aligned}T(r, F) &\leq N_2(r, 0, F) + N_2(r, 0, G) + 2\overline{N}(r, \infty, F) + \overline{N}(r, \infty, G) + S(r, F) + S(r, G) \\ &\leq N_2(r, 0, f^n) + N_2(r, 0, P(f)) + N_2(r, 0, (\prod_{j=1}^d f(z + c_j)^{s_j})^{(k)}) + N_2(r, 0, g^n) \\ &\quad + N_2(r, 0, P(g)) + N_2(r, 0, (\prod_{j=1}^d g(z + c_j)^{s_j})^{(k)}) \\ &\quad + 2\overline{N}(r, \infty, f^n P(f)) + 2\overline{N}(r, \infty, (\prod_{j=1}^d f(z + c_j)^{s_j})^{(k)}) + \overline{N}(r, \infty, g^n P(g)) \\ &\quad + \overline{N}(r, \infty, (\prod_{j=1}^d f(z + c_j)^{s_j})^{(k)}) + S(r, f) + S(r, g) \\ &\leq 2\overline{N}(r, 0, f^n) + N_2(r, 0, P(f)) + T(r, (\prod_{j=1}^d f(z + c_j)^{s_j})^{(k)}) \\ &\quad - T(r, \prod_{j=1}^d f(z + c_j)^{s_j}) + N_{k+2}(r, 0, \prod_{j=1}^d f(z + c_j)^{s_j}) + 2\overline{N}(r, 0, g^n) \\ &\quad + N_2(r, 0, P(g)) + T(r, (\prod_{j=1}^d g(z + c_j)^{s_j})^{(k)}) - T(r, \prod_{j=1}^d g(z + c_j)^{s_j}) \\ &\quad + N_{k+2}(r, 0, \prod_{j=1}^d g(z + c_j)^{s_j}) + 2\overline{N}(r, \infty, f) + 2\overline{N}(r, \infty, \prod_{j=1}^d f(z + c_j)^{s_j}) \\ &\quad + \overline{N}(r, \infty, g) + \overline{N}(r, \infty, \prod_{j=1}^d g(z + c_j)^{s_j}) + S(r, f) + S(r, g) \\ &\leq (2 + \Gamma_1)T(r, f) + T(r, (\prod_{j=1}^d f(z + c_j)^{s_j})^{(k)}) + T(r, f^n) + T(r, P(f)) \\ &\quad - T(r, f^n) - T(r, P(f)) \\ &\quad - T(r, \prod_{j=1}^d f(z + c_j)^{s_j}) + d(k+2)T(r, f) + (2 + \Gamma_1)T(r, g) + T(r, \prod_{j=1}^d g(z + c_j)^{s_j}) \\ &\quad + k\overline{N}(r, \infty, \prod_{j=1}^d g(z + c_j)^{s_j}) - T(r, \prod_{j=1}^d g(z + c_j)^{s_j}) + d(k+2)T(r, g) + 2T(r, f) \\ &\quad + 2\lambda T(r, f) + T(r, g) + \lambda T(r, g) + S(r, f) + S(r, g)\end{aligned}$$

$$\begin{aligned} &\leq (2 + \Gamma_1)T(r, f) + T(r, F) - T(r, F_1) + d(k + 2)T(r, f) + (2 + \Gamma_1)T(r, g) \\ &+ k\lambda T(r, g) + d(k + 2)T(r, g) + 2(\lambda + 1)T(r, f) + (\lambda + 1)T(r, g) + S(r, f) + S(r, g). \end{aligned}$$

That is,

$$\begin{aligned} T(r, F_1) &\leq (d(k + 2) + 2(\lambda + 1) + 2 + \Gamma_1)T(r, f) \\ &+ (d(k + 2) + (k + 1)\lambda + 3 + \Gamma_1)T(r, g) + S(r, f) + S(r, g) \\ (n + m - \lambda)T(r, f) &\leq (d(k + 2) + \lambda + 3 + \Gamma_1)(T(r, f) + T(r, g)) \\ &+ (\lambda + 1)T(r, f) + k\lambda T(r, g) + S(r, f) + S(r, g). \end{aligned} \quad (3.6)$$

Similarly,

$$\begin{aligned} (n + m - \lambda)T(r, g) &\leq (d(k + 2) + \lambda + 3 + \Gamma_1)(T(r, f) + T(r, g)) \\ &+ (\lambda + 1)T(r, g) + k\lambda T(r, f) + S(r, f) + S(r, g). \end{aligned} \quad (3.7)$$

From (3.6) and (3.7), we get

$$(n + m - 4\lambda - 2(k + 2)d - k\lambda - 2\Gamma_1 - 7)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g).$$

Which is contradiction to $n \geq 2(k + 2)d + (k + 4)\lambda + 2\Gamma_1 - m + 8$.

Subcase 1.2: When $l = 1$.

Then by using (3.5) and Lemma 2.3, we deduce that

$$\begin{aligned} T(r, F) &= T(r, F_2) + S(r, F_2) \\ &\leq N_2(r, 0, F_2) + N_2(r, 0, G_2) + \frac{3}{2}\overline{N}(r, \infty, F_2) + \overline{N}(r, \infty, G_2) \\ &+ \overline{N}_*(r, \infty, F_2, G_2) + \frac{1}{2}\overline{N}(r, 0, F_2) + S(r, F_2) + S(r, G_2) \\ &\leq N_2(r, 0, F) + N_2(r, 0, G) + \frac{3}{2}\overline{N}(r, \infty, F) + \overline{N}(r, \infty, G) \\ &+ \overline{N}_*(r, \infty, F, G) + \frac{1}{2}\overline{N}(r, 0, F) + S(r, F) + S(r, G) \\ &\leq N_2(r, 0, F) + N_2(r, 0, G) + \frac{5}{2}\overline{N}(r, \infty, F) + \overline{N}(r, \infty, G) + \frac{1}{2}\overline{N}(r, 0, F) \\ &+ S(r, F) + S(r, G) \\ &\leq N_2(r, 0, f^n) + N_2(r, 0, P(f)) + N_2(r, 0, (\prod_{j=1}^n f(z + c_j)^{s_j})^{(k)}) + N_2(r, 0, g^n) \\ &+ N_2(r, 0, P(g)) + N_2(r, 0, (\prod_{j=1}^n g(z + c_j)^{s_j})^{(k)}) + \frac{5}{2}\overline{N}(r, \infty, f^n P(f)) \\ &+ \frac{5}{2}\overline{N}(r, \infty, (\prod_{j=1}^n f(z + c_j)^{s_j})^{(k)}) + \overline{N}(r, \infty, g^n P(g)) + \overline{N}(r, \infty, (\prod_{j=1}^n g(z + c_j)^{s_j})^{(k)}) \\ &+ \frac{1}{2}\overline{N}(r, 0, f^n) + N_2(r, 0, P(f)) + \frac{1}{2}\overline{N}(r, 0, (\prod_{j=1}^n f(z + c_j)^{s_j})^{(k)}) + S(r, f) + S(r, g) \end{aligned}$$

$$\begin{aligned}
&\leq 2\overline{N}_{(2)}(r, 0, f^n) + N_2(r, 0, P(f)) + T(r, (\prod_{j=1}^n f(z + c_j)^{s_j})^{(k)}) - T(r, (\prod_{j=1}^n f(z + c_j)^{s_j}) \\
&+ N_{k+2}(r, 0, \prod_{j=1}^n f(z + c_j)^{s_j}) + 2\overline{N}_{(2)}(r, 0, g^n) + N_2(r, 0, P(g)) \\
&+ T(r, (\prod_{j=1}^n g(z + c_j)^{s_j})^{(k)}) - T(r, \prod_{j=1}^n g(z + c_j)^{s_j}) + N_{k+2}(r, 0, \prod_{j=1}^n g(z + c_j)^{s_j}) \\
&+ \frac{5}{2}\overline{N}(r, \infty, f) + \frac{5}{2}\overline{N}(r, \infty, \prod_{j=1}^n f(z + c_j)^{s_j}) + \overline{N}(r, \infty, g) + \overline{N}(r, \infty, \prod_{j=1}^n g(z + c_j)^{s_j}) \\
&+ \frac{1}{2}\overline{N}(r, 0, f) + \frac{1}{2}\overline{N}(r, 0, P(f)) \\
&+ \frac{1}{2}N_{k+1}(r, 0, \prod_{j=1}^n f(z + c_j)^{s_j}) + \frac{1}{2}k\overline{N}(r, \infty, \prod_{j=1}^n f(z + c_j)^{s_j}) + S(r, f) + S(r, g) \\
&\leq (2 + \Gamma_1)T(r, f) + T(r, (\prod_{j=1}^n f(z + c_j)^{s_j})^{(k)}) + T(r, f^n) + T(r, P(f)) - T(r, f^n) \\
&- T(r, P(f)) - T(r, \prod_{j=1}^n f(z + c_j)^{s_j}) + d(k+2)T(r, f) + (2 + \Gamma_1)T(r, g) \\
&+ T(r, \prod_{j=1}^n g(z + c_j)^{s_j}) + k\overline{N}(r, (\prod_{j=1}^n g(z + c_j)^{s_j}) - T(r, \prod_{j=1}^n g(z + c_j)^{s_j}) \\
&+ d(k+2)T(r, g) + \frac{5}{2}T(r, f) + \frac{5}{2}\lambda T(r, f) + T(r, g) \\
&\leq (2 + \Gamma_1)T(r, f) + T(r, F) - T(r, F_1) + d(k+2)T(r, f) + (2 + \Gamma_1)T(r, g) \\
&+ k\lambda T(r, g) + d(k+2)T(r, g) + \frac{5}{2}T(r, f) + \frac{5}{2}\lambda T(r, f) + T(r, g) + \lambda T(r, g) \\
&+ \frac{1}{2}T(r, f) + \frac{\Gamma_0}{2}T(r, f) + \frac{1}{2}d(k+1)T(r, f) + \frac{1}{2}k\lambda T(r, f) + S(r, f) + S(r, g).
\end{aligned}$$

That is,

$$\begin{aligned}
T(r, F_1) &\leq (d(k+2) + \frac{1}{2}k\lambda + 3 + \lambda + \Gamma_1)(T(r, f) + T(r, g)) \\
&+ (\frac{1}{2}d(k+1) + \frac{3}{2}\lambda + 2 + \frac{\Gamma_0}{2})T(r, f) \\
&+ \frac{1}{2}k\lambda T(r, g) + S(r, f) + S(r, g) \\
(n+m-\lambda)T(r, f) &\leq (d(k+2) + \frac{1}{2}k\lambda + 3 + \lambda + \Gamma_1)(T(r, f) + T(r, g)) \\
&+ (\frac{1}{2}d(k+1) + \frac{3}{2}\lambda + 2 + \frac{\Gamma_0}{2})T(r, f) + \frac{1}{2}k\lambda T(r, g) + S(r, f) + S(r, g).
\end{aligned} \tag{3.8}$$

Similarly

$$\begin{aligned} (n+m-\lambda)T(r, g) &\leq (d(k+2) + \frac{1}{2}k\lambda + 3 + \lambda + \Gamma_1)(T(r, f) + T(r, g)) \\ &\quad + (\frac{1}{2}d(k+1) + \frac{3}{2}\lambda + 2 + \frac{\Gamma_0}{2})T(r, g) + \frac{1}{2}k\lambda T(r, f) \\ &\quad + S(r, f) + S(r, g). \end{aligned} \quad (3.9)$$

From (3.8) and (3.9), we get,

$$(n+m-\lambda)(\frac{3}{2}k+\frac{9}{2})-d(2(k+2))+\frac{1}{2}(k+1))-8-\frac{\Gamma_0}{2}-2\Gamma_1)(T(r, f)+T(r, g)) \leq S(r, f)+S(r, g)$$

which is a contradiction to $n \geq \lambda(\frac{3}{2}k + \frac{9}{2}) + d(2(k+2) + \frac{1}{2}(k+1)) + \frac{\Gamma_0}{2} - 2\Gamma_1 - m + 9$.

Subcase 1.3: When $l = 0$.

As in Subcase 1.1 and using (3.5) and Lemma 2.4 we simplify

$$\begin{aligned} T(r, F) &= T(r, F_2) + S(r, F_2) \\ &\leq N_2(r, 0, F_2) + N_2(r, 0, G_2) + 3\overline{N}(r, \infty, F_2) + 2\overline{N}(r, \infty, G_2) + \overline{N}_*(r, \infty, F_2, G_2) \\ &\quad + 2\overline{N}(r, 0, F_2) + \overline{N}(r, 0, G_2) + S(r, F_2) + S(r, G_2) \\ &\leq N_2(r, 0, F) + N_2(r, 0, G) + 3\overline{N}(r, \infty, F) + 2\overline{N}(r, \infty, G) + \overline{N}_*(r, \infty, F, G) \\ &\quad + 2\overline{N}(r, 0, F) + \overline{N}(r, 0, G) + S(r, F) + S(r, G) \\ &\leq N_2(r, 0, F) + N_2(r, 0, G) + 4\overline{N}(r, \infty, F) + 2\overline{N}(r, \infty, G) + 2\overline{N}(r, 0, F) \\ &\quad + \overline{N}(r, 0, G) + S(r, F) + S(r, G) \\ &\leq (2 + \Gamma_1)T(r, f) + T(r, F) - T(r, F_1) + d(k+2)T(r, f) + (2 + \Gamma_1)T(r, g) \\ &\quad + k\lambda T(r, g) + d(k+2)T(r, g) + 4\overline{N}(r, \infty, f^n P(f)) + 4\overline{N}(r, \infty, (\prod_{j=1}^n f(z + c_j)^{s_j})^{(k)}) \\ &\quad + 2\overline{N}(r, \infty, g^n P(g)) + 2\overline{N}(r, \infty, (\prod_{j=1}^n g(z + c_j)^{s_j})^{(k)}) + 2\overline{N}(r, 0, f^n) \\ &\quad + 2\overline{N}(r, 0, P(f)) + 2\overline{N}(r, 0, (\prod_{j=1}^n g(z + c_j)^{s_j})^{(k)}) \\ &\quad + 2\overline{N}(r, 0, g^n) + \overline{N}(r, 0, P(g)) + \overline{N}(r, 0, (\prod_{j=1}^n g(z + c_j)^{s_j})^{(k)}) + S(r, f) + S(r, g). \end{aligned}$$

That is

$$\begin{aligned} T(r, F_1) &= (2 + \Gamma_1)T(r, f) + d(k+2)T(r, f) + (2 + \Gamma_1)T(r, g) + k\lambda T(r, g) + d(k+2)T(r, g) \\ &\quad + 4T(r, f) + 4\overline{N}(r, \infty, \prod_{j=1}^n f(z + c_j)^{s_j}) + 2T(r, g) + 2\overline{N}(r, \infty, \prod_{j=1}^n g(z + c_j)^{s_j}) \\ &\quad + (2 + \Gamma_0)T(r, f) + 2N_{k+1}(r, 0, \prod_{j=1}^n f(z + c_j)^{s_j}) + 2k\overline{N}(r, \infty, \prod_{j=1}^n f(z + c_j)^{s_j}) \\ &\quad + (1 + \Gamma_0)T(r, g) + N_{k+1}(r, 0, \prod_{j=1}^n g(z + c_j)^{s_j}) + k\overline{N}(r, \infty, \prod_{j=1}^n g(z + c_j)^{s_j}) + S(r, f) + S(r, g) \end{aligned}$$

$$(n+m-\lambda)T(r, f) \leq (d(k+2) + d(k+1) + 2k\lambda + 2\lambda + 5 + \Gamma_1 + \Gamma_0)(T(r, f) + T(r, g)) + (2\lambda + d(k+1) + 3 + \Gamma_0)T(r, f) + S(r, f) + S(r, g). \quad (3.10)$$

Similarly,

$$(n+m-\lambda)T(r, g) \leq (d(k+2) + d(k+1) + 2k\lambda + 2\lambda + 5 + \Gamma_1 + \Gamma_0)(T(r, f) + T(r, g)) + (2\lambda + d(k+1) + 3 + \Gamma_0)T(r, g) + S(r, f) + S(r, g). \quad (3.11)$$

From (3.10) and (3.11), we get,

$$(n+m-7\lambda-d(2k+7)-4k\lambda-13-2\Gamma_1-3\Gamma_0)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g),$$

which is a contradiction to $n \geq 7\lambda + d(5k+7) + 4k\lambda + 2\Gamma_1 + 3\Gamma_0 - m + 14$.

Case 2. We now assume that $H \equiv 0$. That is

$$\left(\frac{F_2''}{F_2'} - \frac{2F_2'}{F_2' - 1} \right) - \left(\frac{G_2''}{G_2'} - \frac{2G_2'}{G_2' - 1} \right) = 0.$$

Integrating both sides twice, we get

$$\frac{1}{F_2 - 1} = \frac{A}{G_2 - 1} + B \quad (3.12)$$

Where $A (\neq 0)$ and B are constants. From (3.12), we have following subcases.

Subcase 2.1. Let $B \neq 0$ and $A = B$. Then from (3.12) we get,

$$\frac{1}{F_2 - 1} = \frac{BG_2}{G_2 - 1} \quad (3.13)$$

If $B = -1$, then from (3.13) we obtain $F_2G_2 = 1$.

That is $f(z)^nP(f)[\prod_{j=1}^d f(z+c_j)^{s_j}]^{(k)} \cdot g(z)^nP(g)[\prod_{j=1}^d g(z+c_j)^{s_j}]^{(k)} = z^2$.

We proceed as in the proof of Theorem 1.4, we obtain $\prod_{j=1}^d f(z+c_j)^{s_j} = c_1 e^{cz^2}$ and $\prod_{j=1}^d g(z+c_j)^{s_j} = c_2 e^{-cz^2}$, where c_1, c_2 and c are constants such that $4(c_1c_2)^{n+1}c^2 = -1$.

If $B \neq -1$, from (3.13) we have $\frac{1}{F_2} = \frac{BG_2}{(1+B)G_2-1}$ and therefore

$$\overline{N}\left(r, \frac{1}{B+1}, G_2\right) = \overline{N}(r, 0, F_2). \quad (3.14)$$

Using Nevanlinna's Second Fundamental Theorem we get,

$$\begin{aligned} T(r, G) &= T(r, G_2) + S(r, G) \\ &\leq \overline{N}(r, 0, G_2) + \overline{N}(r, \infty, G_2) + \overline{N}\left(r, \frac{1}{B+1}, G_2\right) + S(r, G) \\ &\leq \overline{N}(r, 0, G_2) + \overline{N}(r, \infty, G_2) + \overline{N}(r, 0, F_2) + S(r, G) \\ &\leq \overline{N}(r, 0, G) + \overline{N}(r, \infty, G) + \overline{N}(r, 0, F) + S(r, G). \end{aligned}$$

Using this with Lemma 2.8, we deduce that

$$\begin{aligned}
& (n+m-\lambda)T(r, g) \\
& \leq \overline{N}(r, 0, g(z)^n) + \overline{N}(r, 0, P(g)) + \overline{N}(r, 0, \left(\prod_{j=1}^n g(z+c_j)^{s_j}\right)^{(k)}) + \overline{N}(r, \infty, g(z)^n P(g)) \\
& + \overline{N}(r, \infty, \left(\prod_{j=1}^n g(z+c_j)^{s_j}\right)^{(k)}) + \overline{N}(r, 0, f(z)^n) + \overline{N}(r, 0, P(f)) \\
& + \overline{N}(r, 0, \left(\prod_{j=1}^n f(z+c_j)^{s_j}\right)^{(k)}) + S(r, g) \\
& \leq (1+\Gamma_0)T(r, g) + k\overline{N}(r, 0, \prod_{j=1}^n g(z+c_j)^{s_j}) + N_{k+1}(r, 0, \prod_{j=1}^n g(z+c_j)^{s_j}) + T(r, g) \\
& + \overline{N}(r, \infty, \prod_{j=1}^n f(z+c_j)^{s_j}) + \overline{N}(r, 0, f(z)^n) + \overline{N}(r, 0, P(f)) + N_{k+1}(r, 0, \prod_{j=1}^n f(z+c_j)^{s_j}) \\
& + k\overline{N}(r, 0, \prod_{j=1}^n f(z+c_j)^{s_j}) + S(r, g) \\
& \leq (1+\Gamma_0)T(r, g) + k\lambda T(r, g) + d(k+1)T(r, g) + T(r, g) + \lambda T(r, g) \\
& + (1+\Gamma_0)T(r, f) + k\lambda T(r, f) + d(k+1)T(r, f) + S(r, f) + S(r, g).
\end{aligned}$$

Thus,

$$\begin{aligned}
(n+m-\lambda)T(r, g) & \leq (d(k+1) + k\lambda + 1 + \Gamma_0)(T(r, f) + T(r, g)) + (1+\lambda)T(r, g) \\
& + S(r, f) + S(r, g). \tag{3.15}
\end{aligned}$$

Similarly,

$$\begin{aligned}
(n+m-\lambda)T(r, f) & \leq (d(k+1) + k\lambda + 1 + \Gamma_0)(T(r, f) + T(r, g)) + (1+\lambda)T(r, f) \\
& + S(r, f) + S(r, g). \tag{3.16}
\end{aligned}$$

From (3.15) and (3.16), we get

$$(n+m-2\lambda-2\lambda k-2d(k+1)-2\Gamma_0-3)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g)$$

which contradicts to $n \geq 2\lambda + 2d(k+1) + 2k\lambda + 2\Gamma_0 - m + 4$.

Subcase 2.2. Let $B \neq 0$ and $A \neq B$.

Then by (3.12), we get

$$F_2 = \frac{(B+1)G_2 - (B-A+1)}{BG_2 + (A-B)}$$

and so, $\overline{N}(r, \frac{B-A+1}{B+1}, G_2) = \overline{N}(r, 0, F_2)$.

Proceeding as in Subcase 2.1, we obtain a contradiction.

Subcase 2.3. Let $B = 0$ and $A \neq 0$. Then by (3.12), we get,

$$F_2 = \frac{G_2 + A - 1}{A} \text{ and } G_2 = AF_2 - (A - 1).$$

If $A \neq 1$, we have $\overline{N}(r, \frac{A-1}{A}, F_2) = \overline{N}(r, 0, G_2)$ and $\overline{N}(r, A-1, G_2) = \overline{N}(r, 0, F_2)$. Using similar arguments as in Subcase 2.1, we obtain a contradiction. Thus $A = 1$, which implies $F = G$ and therefore

$$f(z)^n P(f) \left[\prod_{j=1}^d f(z + c_j)^{s_j} \right]^{(k)} = g(z)^n P(g) \left[\prod_{j=1}^d g(z + c_j)^{s_j} \right]^{(k)}.$$

4. Open Questions

Question 1. Can the difference polynomials in Theorems 1.5 be replaced by the difference polynomials of form $f(z)^n P(f) \left[\prod_{j=1}^d f(z + \eta_j)^{s_j} \right]^{(k)} \prod_{j=1}^k f^{(i)}(z)$?

Question 2. Is it possible to reduce the condition of the theorem?

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