

AN IMPROVED LOCAL ANALYSIS OF DEFORMED HALLEY METHOD IN BANACH SPACES

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Abstract. In this paper, we provide an improved local analysis of deformed Halley method using Hölder continuous first-order Fréchet derivative in Banach spaces. This analysis avoids the use of the extra assumption on the boundedness of the first derivative. Finally, numerical applications confirm that our analysis provides larger convergence radii in comparison with the earlier study.

1. Introduction

We present an advanced analysis of local convergence of deformed Halley scheme for solving nonlinear operator equations in the form

$$\mathcal{P}(s) = 0, \quad (1.1)$$

where $\mathcal{P} : \Omega \subseteq \mathcal{X} \rightarrow \mathcal{Y}$ is a Fréchet differentiable operator with values in the Banach space $\mathcal{Y}$ and Ω is a convex subset of the Banach space $\mathcal{X}$. Nonlinear equations play an important role in solving many application problems arise in various domains of engineering and applied sciences. Applying mathematical models many problems in Optimization, Chemistry, Economics, Physics and other applied areas can be transformed into these nonlinear equations. Due to complexity and non-availability of direct methods, iterative procedures are generally applied to address these equations [1, 2, 3, 4, 5, 7, 8, 9, 19, 21, 23, 25, 26, 27, 30, 31, 32, 37, 38, 39]. The well-known Newton's procedure for solving equations in the form (1.1) is given as:

$$s_{j+1} = s_j - [\mathcal{P}'(s_j)]^{-1} \mathcal{P}(s_j), \quad j = 0, 1, 2, \dots \quad (1.2)$$

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Third-order conventional schemes including Chebyshev's, the Halley's and Super-Halley's schemes can be produced by putting $\beta = 0$, $\beta = \frac{1}{2}$ and $\beta = 1$ respectively in

$$s_{j+1} = s_j - \left(I + \frac{1}{2}(I - \beta G_{\mathcal{P}}(s_j))^{-1} G_{\mathcal{P}}(s_j) \right) [\mathcal{P}'(s_j)]^{-1} \mathcal{P}(s_j), \quad (1.3)$$

where $G_{\mathcal{P}}(s_j) = \mathcal{P}'(s_j)^{-1} \mathcal{P}''(s_j) \mathcal{P}'(s_j)^{-1} \mathcal{P}(s_j)$. To eliminate the calculation of $\mathcal{P}''$ in the Halley method, the deformed Halley method (DHM) was derived in [39].

Convergence domain of an iterative procedure plays a key role in selecting appropriate initial points. Therefore, obtaining the radii of convergence balls as well as providing a technique to expand the convergence domain are essential parts in the study of iterative schemes. "The local convergence analysis of iterative schemes is based on the information around a solution and provides the radii of convergence balls" [7]. In [11, 12, 13, 14], analysis of local convergence of Chebyshev-Halley like algorithms including modified Halley-type, improved Chebyshev-Halley type and deformed Halley methods are provided. Also, many significant results for local analysis of various iterative schemes have been derived in [6, 7, 9, 10, 11, 13, 15, 16, 17, 18, 20, 22, 24, 29, 28, 33, 34, 35, 36]. In this paper, our main objective is to provide a better local analysis of deformed Halley scheme under weaker conditions.

In [12], the authors discussed the local convergence of the deformed Halley method (DHM). This scheme is given by

$$\begin{aligned} t_j &= s_j - [\mathcal{P}'(s_j)]^{-1} \mathcal{P}(s_j), \\ u_j &= s_j + \alpha [\mathcal{P}'(s_j)]^{-1} \mathcal{P}'(s_j), \\ H_j &= \frac{1}{\lambda} \mathcal{P}'(s_j)^{-1} [\mathcal{P}'(s_j + \lambda(u_j - s_j)) - \mathcal{P}'(s_j)], \\ s_{j+1} &= t_j + \frac{1}{2} H_j \left(I - \frac{1}{2} H_j \right)^{-1} (t_j - s_j), \quad j = 0, 1, 2, \dots, \end{aligned} \quad (1.4)$$

where s_0 is an initial point, $\lambda \in (0, 1]$ and $\alpha \in \mathbb{R}$. They proved the analysis of convergence of DHM using the condition that $\mathcal{P}'$ belongs to Hölder continuous class. However, they also used an additional condition based on the boundedness of $\mathcal{P}'$. We eliminate this extra condition and extend the convergence domain of DHM (1.4) for several problems.

Furthermore, the key features of our analysis are: This study generalizes the local analysis based on Lipschitz constants and expands the applicability of DHM. We avoid the use of the extra condition based on the boundedness of $\mathcal{P}'$. Also, we produce larger radii of convergence balls in comparison to the earlier study [12]. It should be noted that all the above are achieved without using any additional criteria.

The rest part of this article is described as follows: In Sect. 2, the local analysis of DHM (1.4) is studied. Section 3 contains the numerical implementations of our analytical findings. At last, conclusion is given.

2. Local convergence analysis

We describe the local convergence analysis of the algorithm (1.4) in this section. For $c \in \mathcal{X}$ and $\rho > 0$ let us define

$$\begin{aligned} B(c, \rho) &= \{s \in \mathcal{X} : \|c - s\| < \rho\}, \\ \bar{B}(c, \rho) &= \{s \in \mathcal{X} : \|c - s\| \leq \rho\} \end{aligned}$$

and

$$BL(\mathcal{X}, \mathcal{Y}) = \{T : \mathcal{X} \rightarrow \mathcal{Y} \text{ is bounded and linear}\}.$$

Considering $q \in (0, 1]$, $w_0 > 0$ and $w_1 > 0$ be the parameters with $w_0 \leq w_1$, we define the following functions on $\left[0, \left(\frac{1}{w_0}\right)^{\frac{1}{q}}\right)$ by

$$\mathcal{F}_1(x) = \frac{w_1 x^q}{(q+1)(1-w_0 x^q)}, \quad (2.1)$$

$$\mathcal{F}_2(x) = \mathcal{F}_1(x) + \frac{|1+\alpha|(1+w_0 x^q)}{1-w_0 x^q}, \quad (2.2)$$

$$\mathcal{F}_3(x) = \frac{w_1 |\alpha|^q |\lambda|^{q-1} (1+w_0 x^q)^q x^q}{2(1-w_0 x^q)^{q+1}}, \quad (2.3)$$

$$\begin{aligned} \mathcal{B}_3(x) &= w_1 |\alpha|^q |\lambda|^{q-1} (1+w_0 x^q)^q x^q - 2(1-w_0 x^q)^{q+1}, \\ \mathcal{F}_4(x) &= \mathcal{F}_1(x) + \frac{\mathcal{F}_3(x)(1+w_0 x^q)}{(1-w_0 x^q)(1-\mathcal{F}_3(x))}, \\ \mathcal{B}_4(x) &= \mathcal{F}_4(x) - 1 \end{aligned} \quad (2.4)$$

and the parameters

$$\Delta_1 = \left(\frac{(q+1)}{(q+1)w_0 + w_1} \right)^{\frac{1}{q}} < \left(\frac{1}{w_0} \right)^{\frac{1}{q}}$$

and

$$\Delta_2 = \left(\frac{(q+1)(1-|1+\alpha|)}{(q+1)w_0 + w_1} \right)^{\frac{1}{q}}.$$

Let us assume that $|1+\alpha| < 1$, then Δ_2 is well defined with $\Delta_2 < \Delta_1$. Observe that $0 \leq \mathcal{F}_1(x) < 1$ and $0 \leq \mathcal{F}_2(x) < 1$ for all $x \in [0, \Delta_2)$.

Now, $\mathcal{B}_3(0) = -2 < 0$ and $\mathcal{B}_3\left(\left(\frac{1}{w_0}\right)^{\frac{1}{q}}\right) = \frac{w_1 |\alpha|^q |\lambda|^{q-1} 2^q}{w_0} > 0$. So, the zeros of the function $\mathcal{B}_3(x)$ lies in $\left(0, \left(\frac{1}{w_0}\right)^{\frac{1}{q}}\right)$. We denote the smallest zero of $\mathcal{B}_3(x)$ in $\left(0, \left(\frac{1}{w_0}\right)^{\frac{1}{q}}\right)$ as Δ_3 . Then, $0 \leq \mathcal{F}_3(x) < 1$ for all $x \in [0, \Delta_3)$.

Finally, we have $\mathcal{B}_4(0) = -1 < 0$ and $\lim_{x \rightarrow \left(\left(\frac{1}{w_0}\right)^{\frac{1}{q}}\right)^-} \mathcal{B}_4(x) = +\infty$. Let Δ_4 stands for

the smallest zero of $\mathcal{B}_4(x)$ in $\left(0, \left(\frac{1}{w_0}\right)^{\frac{1}{q}}\right)$. Δ_4 is well defined due to the intermediate value theorem. We choose

$$R = \min\{\Delta_2, \Delta_3, \Delta_4\} \quad (2.5)$$

to confirm the followings.

$$0 \leq \mathcal{F}_1(x) < 1, \quad (2.6)$$

$$0 \leq \mathcal{F}_2(x) < 1, \quad (2.7)$$

$$0 \leq \mathcal{F}_3(x) < 1 \quad (2.8)$$

and

$$0 \leq \mathcal{F}_4(x) < 1, \quad (2.9)$$

for each $x \in [0, R)$. Also, we use the following assumptions on the Fréchet differentiable operator $\mathcal{P} : \Omega \subseteq \mathcal{X} \rightarrow \mathcal{Y}$.

$$\mathcal{P}(s^*) = 0, \mathcal{P}'(s^*)^{-1} \in BL(\mathcal{Y}, \mathcal{X}), \quad (2.10)$$

$$\|\mathcal{P}'(s^*)^{-1}(\mathcal{P}'(s) - \mathcal{P}'(s^*))\| \leq w_0 \|s - s^*\|^q, \forall s \in \Omega \quad (2.11)$$

and

$$\|\mathcal{P}'(s^*)^{-1}(\mathcal{P}'(s) - \mathcal{P}'(t))\| \leq w_1 \|s - t\|^q, \forall s, t \in \Omega_0, \quad (2.12)$$

where $\Omega_0 = B\left(s^*, \left(\frac{1}{w_0}\right)^{\frac{1}{q}}\right) \cap \Omega$.

Many authors [6, 16, 12] use an extra assumption

$$\|\mathcal{P}'(s^*)^{-1}\mathcal{P}'(s)\| \leq M. \quad (2.13)$$

We remove this additional condition by using the following results.

Lemma 2.1. *If $\mathcal{P}$ obeys (2.15) and $\bar{B}(s^*, R) \subseteq \Omega$, then $\forall s \in B(s^*, R)$, we get*

$$\|\mathcal{P}'(s^*)^{-1}\mathcal{P}'(s)\| \leq 1 + w_0 \|s - s^*\|^q \quad (2.14)$$

and

$$\|\mathcal{P}'(s^*)^{-1}\mathcal{P}(s)\| \leq (1 + w_0 \|s - s^*\|^q) \|s - s^*\| \quad (2.15)$$

Proof. Applying (2.15), we get

$$\|\mathcal{P}'(s^*)^{-1}\mathcal{P}'(s)\| \leq 1 + \|\mathcal{P}'(s^*)^{-1}(\mathcal{P}'(s) - \mathcal{P}'(s^*))\| \leq 1 + w_0 \|s - s^*\|^q.$$

For $\xi \in [0, 1]$,

$$\|s^* + \xi(s - s^*) - s^*\| = \xi \|s - s^*\| < R.$$

So, $s^* + \xi(s - s^*) \in B(s^*, R)$ and

$$\|\mathcal{P}'(s^*)^{-1}\mathcal{P}'(s^* + \xi(s - s^*))\| \leq 1 + w_0 \xi \|s - s^*\|^q \leq 1 + w_0 \|s - s^*\|^q. \quad (2.16)$$

We apply the mean value theorem and (2.20) to find

$$\begin{aligned} \|\mathcal{P}'(s^*)^{-1}\mathcal{P}(s)\| &= \|\mathcal{P}'(s^*)^{-1}(\mathcal{P}(s) - \mathcal{P}(s^*))\| \\ &\leq \|\mathcal{P}'(s^*)^{-1}\mathcal{P}'(s^* + \xi(s - s^*))(s - s^*)\| \\ &\leq (1 + w_0 \|s - s^*\|^q) \|s - s^*\|. \end{aligned} \quad \square$$

Theorem 2.2. *Let $s^* \in \Omega$. Suppose the Fréchet differentiable operator $\mathcal{P} : \Omega \subseteq \mathcal{X} \rightarrow \mathcal{Y}$ obeys (2.14)-(2.16) and*

$$\bar{B}(s^*, R) \subseteq \Omega, \quad (2.17)$$

where R is given in (2.9). Starting from $s_0 \in B(s^*, R)$ the scheme (1.4) produces the sequence $\{s_j\}$ which is well defined, $\{s_j\}_{j \geq 0} \in B(s^*, R)$ and converges to s^* . Also, the followings hold $\forall j \geq 0$

$$\|t_j - s^*\| \leq \mathcal{F}_1(\|s_j - s^*\|) \|s_j - s^*\| < \|s_j - s^*\| < R, \quad (2.18)$$

$$\|u_j - s^*\| \leq \mathcal{F}_2(\|s_j - s^*\|) \|s_j - s^*\| < \|s_j - s^*\| < R, \quad (2.19)$$

$$\left\| \frac{1}{2} H_j \right\| \leq \mathcal{F}_3(\|s_j - s^*\|) < 1 \quad (2.20)$$

and

$$\|s_{j+1} - s^*\| \leq \mathcal{F}_4(\|s_j - s^*\|) \|s_j - s^*\| < \|s_j - s^*\| < R, \quad (2.21)$$

where the functions $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ and $\mathcal{F}_4$ are provided in (2.5)-(2.8) respectively. For $\Delta \in \left[R, \left(\frac{q+1}{w_0} \right)^{\frac{1}{q}} \right)$, s^* is the only solution of $\mathcal{P}(s) = 0$ in $\bar{B}(s^*, \Delta) \cap \Omega$.

Proof. Since $s_0 \in B(s^*, R)$, we obtain

$$\|\mathcal{P}'(s^*)^{-1}(\mathcal{P}'(s_0) - \mathcal{P}'(s^*))\| \leq w_0 \|s_0 - s^*\|^q < w_0 R^q < 1 \quad (2.22)$$

using (2.9) and (2.15). Now, Banach Lemma on invertible operators [7, 9, 32, 37] confirms that $\mathcal{P}'(s_0)^{-1} \in BL(\mathcal{Y}, \mathcal{X})$ and

$$\|\mathcal{P}'(s_0)^{-1} \mathcal{P}'(s^*)\| \leq \frac{1}{1 - w_0 \|s_0 - s^*\|^q} < \frac{1}{1 - w_0 R^q}. \quad (2.23)$$

Therefore, t_0 is well defined. Again, we get from (1.4)

$$\begin{aligned} t_0 - s^* &= s_0 - s^* - \mathcal{P}'(s_0)^{-1} \mathcal{P}(s_0) \\ &= -[\mathcal{P}'(s_0)^{-1} \mathcal{P}'(s^*)] \left[\int_0^1 \mathcal{P}'(s^*)^{-1} (\mathcal{P}'(s^* + \xi(s_0 - s^*)) - \mathcal{P}'(s_0))(s_0 - s^*) d\xi \right]. \end{aligned} \quad (2.24)$$

Using (2.5), (2.9), (2.10), (2.27) and (2.28), we get

$$\begin{aligned} \|t_0 - s^*\| &\leq [\|\mathcal{P}'(s_0)^{-1} \mathcal{P}'(s^*)\|] \\ &\quad \left[\left\| \int_0^1 \mathcal{P}'(s^*)^{-1} (\mathcal{P}'(s^* + \xi(s_0 - s^*)) - \mathcal{P}'(s_0))(s_0 - s^*) d\xi \right\| \right] \\ &\leq \frac{w_1 \|s_0 - s^*\|^q}{(q+1)(1 - w_0 \|s_0 - s^*\|^q)} \|s_0 - s^*\| \\ &= \mathcal{F}_1(\|s_0 - s^*\|) \|s_0 - s^*\| < \|s_0 - s^*\| < R. \end{aligned} \quad (2.25)$$

So, (2.22) is true for $j = 0$. Also, we have from (1.4) that

$$u_0 - s^* = s_0 - s^* - \mathcal{P}'(s_0)^{-1} \mathcal{P}(s_0) + (1 + \alpha) \mathcal{P}'(s_0)^{-1} \mathcal{P}(s_0) \quad (2.26)$$

Then, we use (1.4), (2.6), (2.9), (2.11), (2.19), (2.27), (2.29) and (2.30)

$$\begin{aligned} \|u_0 - s^*\| &\leq \|s_0 - s^* - \mathcal{P}'(s_0)^{-1} \mathcal{P}(s_0)\| + |1 + \alpha| \|\mathcal{P}'(s_0)^{-1} \mathcal{P}'(s^*)\| \|\mathcal{P}'(s^*)^{-1} \mathcal{P}(s_0)\| \\ &\leq \|t_0 - s^*\| + |1 + \alpha| \frac{(1 + w_0 \|s - s^*\|^q) \|s - s^*\|}{1 - w_0 \|s_0 - s^*\|^q} \\ &\leq \mathcal{F}_1(\|s_0 - s^*\|) \|s_0 - s^*\| + |1 + \alpha| \frac{(1 + w_0 \|s - s^*\|^q) \|s - s^*\|}{1 - w_0 \|s_0 - s^*\|^q} \\ &= \left(\mathcal{F}_1(\|s_0 - s^*\|) + |1 + \alpha| \frac{(1 + w_0 \|s - s^*\|^q)}{1 - w_0 \|s_0 - s^*\|^q} \right) \|s_0 - s^*\| \\ &= \mathcal{F}_2(\|s_0 - s^*\|) \|s_0 - s^*\| < \|s_0 - s^*\| < R. \end{aligned} \quad (2.27)$$

Hence $u_0 \in B(s^*, R)$. Since $\lambda \in (0, 1]$, we have from (2.29) and (2.31) that

$$\|s_0 + \lambda(u_0 - s_0) - s^*\| \leq |1 - \lambda| \|s_0 - s^*\| + |\lambda| \|u_0 - s^*\| < (|1 - \lambda| + |\lambda|) R \leq R.$$

This confirms that $s_0 + \lambda(u_0 - s_0) \in B(s^*, R)$ and H_0 exists. Next, we use (1.4), (2.7), (2.9), (2.12), (2.19), (2.27) and (2.31) to get the following estimate on $\|H_0\|$.

$$\begin{aligned}
\left\| \frac{1}{2} H_0 \right\| &\leq \frac{1}{2|\lambda|} \|\mathcal{P}'(s_0)^{-1} \mathcal{P}'(s^*)\| \|\mathcal{P}'(s^*)^{-1} [\mathcal{P}'(s_0 + \lambda(u_0 - s_0)) - \mathcal{P}'(s_0)]\| \\
&\leq \frac{1}{2|\lambda|} \frac{w_1 |\lambda|^q \|u_0 - s_0\|^q}{1 - w_0 \|s_0 - s^*\|^q} \\
&\leq \frac{w_1 |\alpha|^q |\lambda|^{q-1} (\|\mathcal{P}'(s_0)^{-1} \mathcal{P}'(s^*)\| \|\mathcal{P}'(s^*)^{-1} \mathcal{P}(s_0)\|)^q}{2(1 - w_0 \|s_0 - s^*\|^q)} \\
&\leq \frac{w_1 |\alpha|^q |\lambda|^{q-1} (1 + w_0 \|s - s^*\|^q)^q \|s_0 - s^*\|^q}{2(1 - w_0 \|s_0 - s^*\|^q)^{1+q}} \\
&= \mathcal{F}_3(\|s_0 - s^*\|) < \mathcal{F}_3(R) < 1.
\end{aligned} \tag{2.28}$$

So, we get

$$\left\| \left(I - \frac{1}{2} H_0 \right)^{-1} \right\| \leq \frac{1}{1 - \mathcal{F}_3(\|s_0 - s^*\|)}. \tag{2.29}$$

Finally, we obtain with the help of (1.4), (2.8), (2.9), (2.13), (2.19), (2.27), (2.32) and (2.33) that

$$\begin{aligned}
\|s_1 - s^*\| &\leq \|t_0 - s^*\| + \left\| \frac{1}{2} H_0 \right\| \left\| \left(I - \frac{1}{2} H_0 \right)^{-1} \right\| \|t_0 - s_0\| \\
&\leq \mathcal{F}_1(\|s_0 - s^*\|) \|s_0 - s^*\| + \\
&\quad \frac{\mathcal{F}_3(\|s_0 - s^*\|)}{1 - \mathcal{F}_3(\|s_0 - s^*\|)} \|\mathcal{P}'(s_0)^{-1} \mathcal{P}'(s^*)\| \|\mathcal{P}'(s^*)^{-1} \mathcal{P}(s_0)\| \\
&\leq \mathcal{F}_1(\|s_0 - s^*\|) \|s_0 - s^*\| + \frac{\mathcal{F}_3(\|s_0 - s^*\|)}{1 - \mathcal{F}_3(\|s_0 - s^*\|)} \frac{(1 + w_0 \|s - s^*\|^q) \|s - s^*\|}{1 - w_0 \|s_0 - s^*\|^q} \\
&= \left(\mathcal{F}_1(\|s_0 - s^*\|) + \frac{\mathcal{F}_3(\|s_0 - s^*\|)}{1 - \mathcal{F}_3(\|s_0 - s^*\|)} \frac{(1 + w_0 \|s - s^*\|^q)}{1 - w_0 \|s_0 - s^*\|^q} \right) \|s_0 - s^*\| \\
&= \mathcal{F}_4(\|s_0 - s^*\|) \|s_0 - s^*\| < \|s_0 - s^*\| < R.
\end{aligned} \tag{2.31}$$

Thus, the estimate (2.25) is true for $j = 0$. We obtain the estimates (2.22)-(2.25) by substituting s_j , t_j and s_{j+1} in place of s_0 , t_0 and s_1 respectively in the preceding estimations. Since $\|s_{j+1} - s^*\| \leq \mathcal{F}_4(R) \|s_j - s^*\| < R$, we derive that $s_{j+1} \in B(s^*, R)$ and $\lim_{j \rightarrow \infty} s_j = s^*$. To prove the uniqueness part, it is supposed that

$t^* (\neq s^*) \in B(s^*, \Delta) \cap \Omega$ with $0 = \mathcal{P}(t^*)$ and $\mathcal{G} = \int_0^1 \mathcal{P}'(t^* + \xi(s^* - t^*)) d\xi$. From equation (2.15), we get

$$\begin{aligned}
\|\mathcal{P}'(s^*)^{-1}(\mathcal{G} - \mathcal{P}'(s^*))\| &\leq \int_0^1 w_0 \|t^* + \xi(s^* - t^*) - s^*\|^q d\xi \\
&\leq \frac{w_0}{q+1} \|s^* - t^*\|^q \\
&\leq \frac{w_0 \Delta^q}{q+1} < 1.
\end{aligned}$$

Applying Banach Lemma, we confirm that $\mathcal{G}^{-1} \in BL(\mathcal{Y}, \mathcal{X})$. The identity $0 = \mathcal{P}(s^*) - \mathcal{P}(t^*) = \mathcal{G}(s^* - t^*)$ implies that $s^* = t^*$. $\square$

3. Numerical Examples

In this section, we compute the radii of convergence balls of the considered method (1.4) for standard numerical problems and compare them with the convergence radii obtained by the technique suggested in [12]. We obtain bigger convergence radius for the method (1.4) using our proposed results.

Example 1 [12] Let $\mathcal{P}$ is defined on $\Omega = [1, 3]$ by

$$\mathcal{P}(s) = \frac{2}{3}x^{\frac{3}{2}} - x.$$

We have $s^* = \frac{9}{4}$, $q = 0.5$, $w_0 = 1$, $w_1 = 2$, $\alpha = -0.6585$ and $\lambda = 1$. We use “ $\mathcal{F}$ ” functions to compute the radius R .

Table 1: Parameters for example 1

<i>Proposed Analysis</i>				<i>Earlier work</i> [12]
Δ_2	Δ_3	Δ_4	R	r
0.060606	0.189936	0.048246	0.048246	0.0215

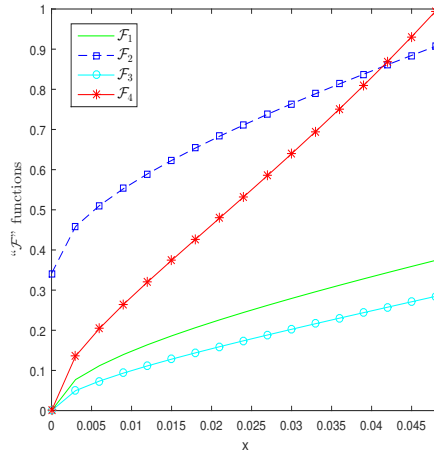


FIGURE 1. $\mathcal{F}$ functions for example 1

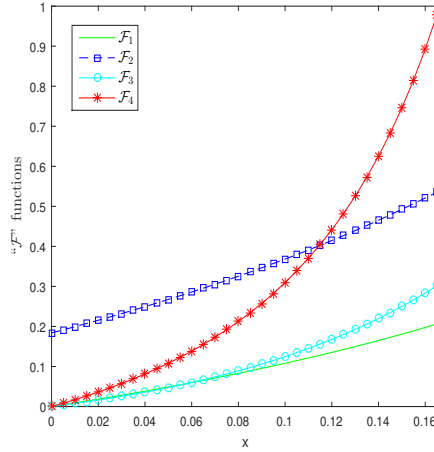
Example 2 [12] Let $\mathcal{P}$ is defined on $\bar{B}(0, 1)$ for $(s_1, s_2, s_3)^t$ by

$$\mathcal{P}(s) = (e^{s_1} - 1, \frac{e-1}{2}s_2^2 + s_2, s_3)^t.$$

We have $s^* = (0, 0, 0)^t$, $q = 1$, $w_0 = e - 1$, $w_1 = 1.789572397$, $\alpha = -0.8161$ and $\lambda = 0.5$. We use “ $\mathcal{F}$ ” functions to compute the radius R .

Table 2: Parameters for example 2

<i>Proposed Analysis</i>				<i>Earlier work [12]</i>
Δ_2	Δ_3	Δ_4	R	r
0.278622	0.269609	0.166177	0.166177	0.0819

**FIGURE 2.** $\mathcal{F}$ functions for example 2

Example 3 [12] Consider the Hammerstein type integral equation given by

$$\mathcal{P}(\phi)(s) = \phi(s) - 5 \int_0^1 s y \phi(y)^3 dy,$$

where $\phi(s) \in C[0, 1]$. We have $s^* = 0$. Also, $q = 1$, $w_0 = 7.5$, $w_1 = 15$, $\alpha = -0.9412$ and $\lambda = 0.5$. We use “ $\mathcal{F}$ ” functions to compute the radius R .

Table 3: Parameters for example 3

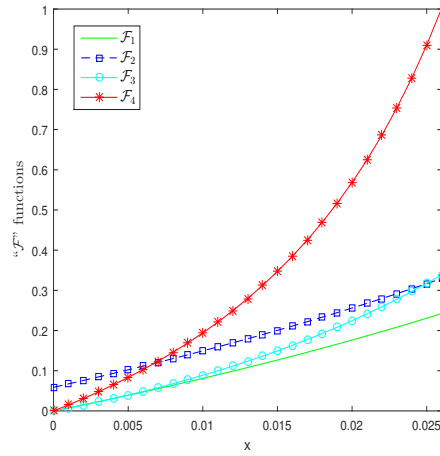
<i>Proposed Analysis</i>				<i>Earlier work [12]</i>
Δ_2	Δ_3	Δ_4	R	r
0.060955	0.045645	0.025982	0.025982	0.0065

Example 4 [12] Define $\mathcal{P}$ on $\Omega = [-\frac{1}{2}, \frac{5}{2}]$ by

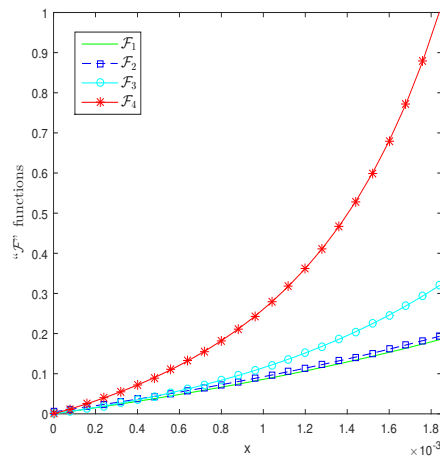
$$\mathcal{P}(s) = \begin{cases} s^3 \log(s^2) + s^5 - s^4, & \text{if } s \neq 0 \\ 0, & \text{if } s = 0 \end{cases}.$$

We have $s^* = 1$, $q = 1$, $w_0 = w_1 = 146.6629$, $\alpha = -0.9951$ and $\lambda = 0.5$. The radius R is computed using “ $\mathcal{F}$ ” functions.

Table 4: Parameters for example 4

FIGURE 3. $\mathcal{F}$ functions for example 3

<i>Proposed Analysis</i>				<i>Earlier work [12]</i>
Δ_2	Δ_3	Δ_4	R	r
0.004509	0.002995	0.001838	0.001838	0.00001

FIGURE 4. $\mathcal{F}$ functions for example 4

4. Conclusions

An improved local convergence analysis of DHM (1.4) is presented. This study generalizes the local analysis based on Lipschitz condition and also expands the

applicability of this scheme. Numerical experiments confirm that the proposed analysis provides larger radii of convergence, for the scheme (1.4), in comparison with the previous work.

Compliance with ethical standards

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Conflict of interest: The authors declare that they do not have conflict of interests.

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