

COMMON FIXED POINTS OF A PAIR OF GENERALIZED RATIONAL CONTRACTION MAPS IN EXTENDED RECTANGULAR B-METRIC SPACES

D M K KIRAN AND M V R KAMESWARI[†]

Date of Receiving : 21. 08. 2020
 Date of Revision : 06. 10. 2020
 Date of Acceptance : 11. 10. 2020

Abstract. In this paper, we obtain common fixed points for two self maps satisfying a general rational contraction conditions in the frame work of extended rectangular B-metric spaces. In order to justify our results, we have provided some examples. Our results improve and extend the results of Alqahtani *et al.* [2] and Asim *et al.* [3]. We apply our results to examine the existence of common fixed points in extended rectangular B-metric spaces equipped with a directed graph.

1. Introduction

Generalization of metric spaces in diverse structures have drawn the attention of scientists due to the development and generalization of many results in fixed point theory. In continuation, Branciari [6] introduced the concept of generalized metric space (rectangular space) as a generalization of metric space and obtained Banach contraction mapping principle in the setting of generalized metric spaces.

Definition 1.1. [6] Let A be a non-empty set. A mapping $v : A \times A \rightarrow [0, \infty)$ is said to be a *rectangular metric on A* if, v satisfies the following (for all $\mu, \eta \in A$ and all distinct $r, s \in A \setminus \{\mu, \eta\}$):

- (1) $v(\mu, \eta) = 0$ if and only if $\mu = \eta$,
- (2) $v(\mu, \eta) = v(\eta, \mu)$,
- (3) $v(\mu, \eta) \leq v(\mu, r) + v(r, s) + v(s, \eta)$.

Then the pair (A, v) is said to be a rectangular metric space.

2010 *Mathematics Subject Classification.* 46T99, 47H10, 54H25.

Key words and phrases. Extended rectangular B-metric spaces, generalized rational contractions, common fixed points .

Communicated by Radenović Stojan N.

[†]Corresponding author.

George *et al.* [9] introduced the notion of rectangular B-metric space as a generalization of rectangular metric space and they presented some fixed point results for contractive mappings.

Definition 1.2. [9] Let A be a non-empty set with the parameter $B \geq 1$. A mapping $\sigma_\zeta : A \times A \rightarrow [0, \infty)$ is said to be a *rectangular B-metric on A* if there exists $B \geq 1$ such that v_b satisfies the following (for all $\mu, \eta \in A$ and all distinct $r, s \in A \setminus \{\mu, \eta\}$):

- (1) $v_b(\mu, \eta) = 0$ if and only if $\mu = \eta$,
- (2) $v_b(\mu, \eta) = v_b(\eta, \mu)$,
- (3) $v_b(\mu, \eta) \leq B[v_b(\mu, r) + v_b(r, s) + v_b(s, \eta)]$.

Then the pair (A, v_b) is said to be a rectangular B-metric space.

Recently, new type of generalized B-metric space namely extended B-metric space is introduced Kamran *et al.* [13].

Definition 1.3. [13] Let A be a non-empty set and $\zeta : A \times A \rightarrow [1, \infty)$. A mapping $\sigma_\zeta : A \times A \rightarrow [0, \infty)$ is said to be an *extended B-metric space*, if σ_ζ satisfies the following for all μ, η and $\varrho \in A$

- (1) $\sigma_\zeta(\mu, \eta) = 0$ if and only if $\mu = \eta$,
- (2) $\sigma_\zeta(\mu, \eta) = \sigma_\zeta(\eta, \mu)$,
- (3) $\sigma_\zeta(\mu, \eta) \leq \zeta(\mu, \eta)[\sigma_\zeta(\mu, \varrho) + \sigma_\zeta(\varrho, \eta)]$.

Then the pair (A, σ_ζ) is said to be an extended B-metric space.

Using the notation of extended B-metric spaces Alqahtan *et al.* [2] proved the following theorems.

Theorem 1.4. [2] Let (A, σ_ζ) be an extended B-metric space with the function $\zeta : A \times A \rightarrow [1, \infty)$ and $T : A \rightarrow A$ be a continuous mapping such that, for all distinct $\mu, \eta \in A$

$$\sigma_\zeta(T\mu, T\eta) \leq kR(\mu, \eta) \quad (1.4.1)$$

where $k \in [0, 1)$ and

$$R(\mu, \eta) = \max\left\{\frac{\sigma_\zeta(\mu, T\mu)[1+\sigma_\zeta(\eta, T\eta)]}{1+\sigma_\zeta(\mu, \eta)}, \frac{\sigma_\zeta(\eta, T\eta)[1+\sigma_\zeta(\mu, T\mu)]}{1+\sigma_\zeta(\mu, \eta)}, \frac{\sigma_\zeta(\eta, T\eta)\sigma_\zeta(\mu, T\mu)}{\sigma_\zeta(\mu, \eta)}, \sigma_\zeta(\mu, \eta)\right\}.$$

Also suppose that, for each $\tilde{h}_0 \in A$, $\lim_{n, m \rightarrow +\infty} \theta(\tilde{h}_n, \tilde{h}_m) < \frac{1}{k}$ where $\tilde{h}_n = T^n \tilde{h}_0$, $n \in \mathbb{N}$. Then T has a fixed point u . Moreover, for each $\tilde{h}_0 \in A$, we have $T^n \tilde{h}_0 \rightarrow u$.

Theorem 1.5. [2] Let (A, σ_ζ) be an extended B-metric space with the function $\zeta : A \times A \rightarrow [1, \infty)$ and $T : A \rightarrow A$ be a mapping such that, for all distinct $\mu, \eta \in A$

$$\sigma_\zeta(T\mu, T\eta) \leq k \max\left\{\frac{\sigma_\zeta(\mu, T\mu)\sigma_\zeta(\mu, T\eta) + \sigma_\zeta(\eta, T\eta)\sigma_\zeta(\eta, T\mu)}{\sigma_\zeta(\mu, T\eta) + \sigma_\zeta(\eta, T\mu)}, \sigma_\zeta(\mu, \eta)\right\} \quad (1.5.1)$$

where $k \in [0, 1)$.

Also suppose that, for each $\tilde{h}_0 \in A$, $\lim_{n,m \rightarrow +\infty} \theta(\tilde{h}_n, \tilde{h}_m) < \frac{1}{k}$ where $\tilde{h}_n = T^n \tilde{h}_0$, $n \in \mathbb{N}$. Then T has a fixed point u . Moreover, for each $\tilde{h}_0 \in A$, we have $T^n \tilde{h}_0 \rightarrow u \in A$.

Influenced by the concepts of rectangular B-metric spaces and extended B-metric spaces, Asim, Imdad and Radenovic [3] introduced the concept of extended rectangular B-metric spaces and obtained fixed points.

Definition 1.6. [3] Let $\aleph$ be a non-empty set and $\varsigma : \aleph \times \aleph \rightarrow [1, \infty)$. A mapping $\Omega_\varsigma : \aleph \times \aleph \rightarrow [0, \infty)$ is said to be an *extended rectangular B-metric* on $\aleph$ if Ω_ς satisfies the following (for all $\mu, \eta \in \aleph$ and all distinct $p, q \in \aleph \setminus \{\mu, \eta\}$):

- (1) $\Omega_\varsigma(\mu, \eta) = 0$ if and only if $\mu = \eta$,
- (2) $\Omega_\varsigma(\mu, \eta) = \Omega_\varsigma(\eta, \mu)$,
- (3) $\Omega_\varsigma(\mu, \eta) \leq \varsigma(\mu, \eta)[\Omega_\varsigma(\mu, p) + \Omega_\varsigma(p, q) + \Omega_\varsigma(q, \eta)]$.

Then the pair $(\aleph, \Omega_\varsigma)$ is said to be a *extended rectangular B-metric space*.

Definition 1.7. [3] Let $(\aleph, \Omega_\varsigma)$ be an extended rectangular B-metric space with ς .

- (1) A sequence $\{\tilde{h}_n\}$ in $(\aleph, \Omega_\varsigma)$ is called Ω_ς *convergent* to $\tilde{h}$ if $\lim_{n \rightarrow +\infty} \Omega_\varsigma(\tilde{h}_n, \tilde{h}) = 0$. In this case we write $\lim_{n \rightarrow +\infty} \tilde{h}_n = \tilde{h}$.
- (2) A sequence $\{\tilde{h}_n\}$ in $(\aleph, \Omega_\varsigma)$ is called Ω_ς *Cauchy* if $\lim_{m,n \rightarrow +\infty} \Omega_\varsigma(\tilde{h}_n, \tilde{h}_m) = 0$.
- (3) $(\aleph, \Omega_\varsigma)$ is said to be a Ω_ς *complete* if every Ω_ς Cauchy in $(\aleph, \Omega_\varsigma)$ is Ω_ς convergent to some point in $\aleph$.

Lemma 1.8. [3] Let $(\aleph, \Omega_\varsigma)$ be an extended rectangular B-metric space with respect to ς , $\{\tilde{h}_n\}$ in $\aleph$ such that $\tilde{h}_n \neq \tilde{h}_m$ whenever $m \neq n$. Then $\{\tilde{h}_n\}$ converges to at most one point.

The authors Asim, Imdad and Radenovic [3] proved the analog of Banach contraction type principle in the context of extended rectangular B-metric space.

Theorem 1.9. [3] Let $(\aleph, \Omega_\varsigma)$ be a extended rectangular B-metric space and $T : \aleph \rightarrow \aleph$. Suppose that the following conditions:

- (i) for all $\mu, \eta \in \aleph$, we have

$$\Omega_\varsigma(T\mu, T\eta) \leq \lambda \Omega_\varsigma(\mu, \eta) \quad (1.9.1)$$

where $\lambda \in [0, 1)$

- (ii) $\lim_{m,n \rightarrow +\infty} \varsigma(\tilde{h}_n, \tilde{h}_m) < \frac{1}{\lambda}$

- (iii) $(\aleph, \Omega_\varsigma)$ is T -orbitally complete.

- (iv) T is orbitally continuous.

Then T has a unique fixed point in $\aleph$.

The following lemma is useful to prove our main results.

Lemma 1.10. [14] Let $(\aleph, \Omega_\varsigma)$ be an extended rectangular B -metric space and if there exists $\ell \in [0, 1)$ such that for $\tilde{h}_0 \in \aleph$, the sequence $\{\tilde{h}_n\}$ satisfies $\lim_{m,n \rightarrow +\infty} \varsigma(\tilde{h}_n, \tilde{h}_m) < \frac{1}{\ell}$ and $0 < \Omega_\varsigma(\tilde{h}_n, \tilde{h}_{n+1}) \leq \ell \Omega_\varsigma(\tilde{h}_n, \tilde{h}_{n-1})$ then for any $n \in \mathbb{N}$, the sequence $\{\tilde{h}_n\}$ is a Ω_ς Cauchy sequence in $\aleph$.

Inspired, the results of Alqahtan *et al.* [2] and Asim, Imdad, Radenovic [3], Alqahtani, Fulga, and Karapinar [1] and some other similar results, we obtain common fixed point theorems for a pair of selfmaps in the frame work of extended rectangular B -metric spaces by employing more general rational contraction conditions than (1.4.1) and (1.5.1). In order to justify our results, we have provided some examples. Our results improve and extend the results of Alqahtani *et al.* [2] and Asim *et al.* [3]. We apply our results to examine the existence of common fixed points in extended rectangular B -metric spaces equipped with a directed graph.

2. Main Results

In this section we prove common fixed point theorems for a pair of selfmaps by employing general rational contraction conditions in rectangular B -metric spaces as follows.

Theorem 2.1. Let $(\aleph, \Omega_\varsigma)$ be an extended rectangular B -metric space with the function $\varsigma : \aleph \times \aleph \rightarrow [1, \infty)$. Suppose that $S, T : \aleph \rightarrow \aleph$ satisfies:

(i) $\forall \mu, \eta \in \aleph, \exists r \in [0, 1)$ and $L \geq 0$ such that

$$\Omega_\varsigma(S\mu, T\eta) \leq rP(\mu, \eta) + LQ(\mu, \eta) \quad (2.1.1)$$

where

$$P(\mu, \eta) = \max\left\{\Omega_\varsigma(\mu, \eta), \frac{\Omega_\varsigma(\mu, S\mu)[1+\Omega_\varsigma(\eta, T\eta)]}{1+\Omega_\varsigma(\mu, \eta)}, \frac{\Omega_\varsigma(\eta, T\eta)[1+\Omega_\varsigma(\mu, S\mu)]}{1+\Omega_\varsigma(\mu, \eta)}, \frac{\Omega_\varsigma(\mu, S\mu)\Omega_\varsigma(\eta, T\eta)}{1+\Omega_\varsigma(\mu, \eta)}, \frac{\Omega_\varsigma(\eta, S\mu)\Omega_\varsigma(\mu, T\eta)}{1+\Omega_\varsigma(\mu, \eta)}, \frac{\Omega_\varsigma(\mu, S\mu)\Omega_\varsigma(\eta, T\eta)}{1+\Omega_\varsigma(\mu, S\mu)+\Omega_\varsigma(\eta, T\eta)}\right\}$$

and $Q(\mu, \eta) = \min\{\Omega_\varsigma(\mu, S\mu), \Omega_\varsigma(\mu, T\eta), \Omega_\varsigma(\eta, S\mu)\}$.

(ii) Suppose that, for each $\tilde{h}_0 \in \aleph$, $\lim_{m,n \rightarrow +\infty} \varsigma(\tilde{h}_n, \tilde{h}_m) < \frac{1}{r}$.

(iii) One of S and T is continuous.

Then S and T have a unique common fixed point in $\aleph$.

Proof. Let $\tilde{h}_0 \in \aleph$ we define a sequence $\{\tilde{h}_n\}$ in $\aleph$ such that

$$\tilde{h}_{2n+1} = S\tilde{h}_{2n} \text{ and } T\tilde{h}_{2n+1} = \tilde{h}_{2n+2} \quad (2.1.2)$$

for all $n \in \mathbb{N}$.

First, we suppose that $\tilde{h}_n \neq \tilde{h}_{n+1}$ for all $n \in \mathbb{N}$. Inversly, if $\tilde{h}_{n_0} = \tilde{h}_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then $k = \tilde{h}_{n_0}$ is a common fixed point of S and T which completes the proof. To show k is common fixed point of S and T , we will consider the following two cases.

If $k = 2m$, we have

$$\tilde{h}_{2m} = \tilde{h}_{2m+1} \implies \tilde{h}_{2m} = S\tilde{h}_{2m},$$

$\tilde{h}_{2m}$ is a fixed point of S . We assert that

$$T\tilde{h}_{2m+1} = S\tilde{h}_{2m}.$$

Suppose that $\Omega_\varsigma(S\tilde{h}_{2m}, T\tilde{h}_{2m+1}) > 0$.

By letting $\mu = \tilde{h}_{2m}$ and $\eta = \tilde{h}_{2m+1}$ in (2.1.1), we have

$$\Omega_\varsigma(S\tilde{h}_{2m}, T\tilde{h}_{2m+1}) \leq rP(\tilde{h}_{2m}, \tilde{h}_{2m+1}) + LQ(\tilde{h}_{2m}, \tilde{h}_{2m+1}) \quad (2.1.3)$$

$$\begin{aligned} P(\tilde{h}_{2m}, \tilde{h}_{2m+1}) &= \max\{\Omega_\varsigma(\tilde{h}_{2m}, \tilde{h}_{2m+1}), \frac{\Omega_\varsigma(\tilde{h}_{2m}, S\tilde{h}_{2m})[1 + \Omega_\varsigma(\tilde{h}_{2m+1}, T\tilde{h}_{2m+1})]}{1 + \Omega_\varsigma(\tilde{h}_{2m}, \tilde{h}_{2m+1})}, \\ &\quad \frac{\Omega_\varsigma(\tilde{h}_{2m+1}, T\tilde{h}_{2m+1})[1 + \Omega_\varsigma(\tilde{h}_{2m}, S\tilde{h}_{2m})]}{1 + \Omega_\varsigma(\tilde{h}_{2m}, \tilde{h}_{2m+1})}, \frac{\Omega_\varsigma(\tilde{h}_{2m}, S\tilde{h}_{2m})\Omega_\varsigma(\tilde{h}_{2m+1}, T\tilde{h}_{2m+1})}{1 + \Omega_\varsigma(\tilde{h}_{2m}, \tilde{h}_{2m+1})}, \\ &\quad \frac{\Omega_\varsigma(\tilde{h}_{2m+1}, S\tilde{h}_{2m})\Omega_\varsigma(\tilde{h}_{2m}, T\tilde{h}_{2m+1})}{1 + \Omega_\varsigma(\tilde{h}_{2m}, \tilde{h}_{2m+1})}, \frac{\Omega_\varsigma(\tilde{h}_{2m}, S\tilde{h}_{2m})\Omega_\varsigma(\tilde{h}_{2m+1}, T\tilde{h}_{2m+1})}{1 + \Omega_\varsigma(\tilde{h}_{2m}, S\tilde{h}_{2m}) + \Omega_\varsigma(\tilde{h}_{2m+1}, T\tilde{h}_{2m+1})}\} \\ &= \{0, \Omega_\varsigma(\tilde{h}_{2m+1}, \tilde{h}_{2m+2})\} \end{aligned} \quad (2.1.4)$$

$$\begin{aligned} \text{and } Q(\tilde{h}_{2m}, \tilde{h}_{2m+1}) &= \min\{\Omega_\varsigma(\tilde{h}_{2m}, S\tilde{h}_{2m}), \Omega_\varsigma(\tilde{h}_{2m}, T\tilde{h}_{2m+1}), \Omega_\varsigma(\tilde{h}_{2m+1}, S\tilde{h}_{2m})\} \\ &= \min\{0, \Omega_\varsigma(\tilde{h}_{2m}, \tilde{h}_{2m+2}), \Omega_\varsigma(\tilde{h}_{2m+1}, \tilde{h}_{2m+1})\} = 0. \end{aligned} \quad (2.1.5)$$

Therefore from (2.1.3) and (2.1.4), we have

$$\Omega_\varsigma(\tilde{h}_{2m+1}, \tilde{h}_{2m+2}) \leq r\Omega_\varsigma(\tilde{h}_{2m+1}, \tilde{h}_{2m+2}) < \Omega_\varsigma(\tilde{h}_{2m+1}, \tilde{h}_{2m+2}),$$

a contradiction. Therefore

$$\tilde{h}_{2m+1} = \tilde{h}_{2m+2}. \quad (2.1.6)$$

This in turn yields

$$\tilde{h}_{2m} = S\tilde{h}_{2m} = \tilde{h}_{2m+1} = T\tilde{h}_{2m+2}.$$

Thus $\tilde{h}_{2m} = \tilde{h}_{2m+1} = k$ is a common fixed point of T and S . Similarly, when $n_0 = 2m - 1$ then the following on the same steps of above case we get $k = \tilde{h}_{2k-1} = \tilde{h}_{2k}$ is a fixed point of S and T .

Thus $\tilde{h}_n \neq \tilde{h}_{n+1}$ for all $n \in \mathbb{N}$. We now show that the sequence $\{\tilde{h}_n\}$ is a Cauchy. For this purpose, we consider the two cases as below.

Case(1): Let $\mu = \tilde{h}_{2n}$ and $\eta = \tilde{h}_{2n+1}$ in (2.1.1), we have

$$\Omega_\varsigma(\tilde{h}_{2n+1}, \tilde{h}_{2n+2}) = \Omega_\varsigma(S\tilde{h}_{2n}, T\tilde{h}_{2n+1}) \leq rP(\tilde{h}_{2n}, \tilde{h}_{2n+1}) + LQ(\tilde{h}_{2n}, \tilde{h}_{2n+1}) \quad (2.1.7)$$

where

$$\begin{aligned}
P(\tilde{h}_{2n}, \tilde{h}_{2n+1}) &= \max\{\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1}), \frac{\Omega_\zeta(\tilde{h}_{2n}, S\tilde{h}_{2n})(1+\Omega_\zeta(\tilde{h}_{2n+1}, T\tilde{h}_{2n+1}))}{1+\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1})}, \\
&\quad \frac{\Omega_\zeta(\tilde{h}_{2n+1}, T\tilde{h}_{2n+1})(1+\Omega_\zeta(\tilde{h}_{2n}, S\tilde{h}_{2n}))}{1+\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1})}, \frac{\Omega_\zeta(\tilde{h}_{2n}, S\tilde{h}_{2n})\Omega_\zeta(\tilde{h}_{2n+1}, T\tilde{h}_{2n+1})}{1+\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1})}, \\
&\quad \frac{\Omega_\zeta(\tilde{h}_{2n+1}, S\tilde{h}_{2n})\Omega_\zeta(\tilde{h}_{2n}, T\tilde{h}_{2n+1})}{1+\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1})}, \frac{\Omega_\zeta(\tilde{h}_{2n}, S\tilde{h}_{2n})\Omega_\zeta(\tilde{h}_{2n+1}, T\tilde{h}_{2n+1})}{1+\Omega_\zeta(\tilde{h}_{2n}, S\tilde{h}_{2n})+\Omega_\zeta(\tilde{h}_{2n+1}, T\tilde{h}_{2n+1})}\} \\
&= \max\{\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1}), \frac{\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1})[1+\Omega_\zeta(\tilde{h}_{2n+1}, \tilde{h}_{2n+2})]}{1+\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1})}, \Omega_\zeta(\tilde{h}_{2n+1}, \tilde{h}_{2n+2}), \\
&\quad \frac{\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1})\Omega_\zeta(\tilde{h}_{2n+1}, \tilde{h}_{2n+2})}{1+\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1})}, 0, \frac{\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1})\Omega_\zeta(\tilde{h}_{2n+1}, \tilde{h}_{2n+2})}{1+\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1})+\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1})}\} \\
&\leq \max\{\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1}), \Omega_\zeta(\tilde{h}_{2n+1}, \tilde{h}_{2n+2}), \\
&\quad \frac{\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1})[1+\Omega_\zeta(\tilde{h}_{2n+1}, \tilde{h}_{2n+2})]}{1+\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1})}\} \quad (2.1.8)
\end{aligned}$$

and $Q(\tilde{h}_{2n}, \tilde{h}_{2n+1})$

$$\begin{aligned}
&= \min\{\Omega_\zeta(\tilde{h}_{2n}, S\tilde{h}_{2n}), \Omega_\zeta(\tilde{h}_{2n}, T\tilde{h}_{2n+1}), \Omega_\zeta(\tilde{h}_{2n+1}, S\tilde{h}_{2n})\} \\
&= \min\{\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1}), \Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+2}), 0\} = 0. \quad (2.1.9)
\end{aligned}$$

Hence from (2.1.7), (2.1.8) and (2.1.9), we conclude that

$$\begin{aligned}
\Omega_\zeta(\tilde{h}_{2n+1}, \tilde{h}_{2n+2}) &\leq r\max\{\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1}), \Omega_\zeta(\tilde{h}_{2n+1}, \tilde{h}_{2n+2}), \\
&\quad \frac{\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1})[1+\Omega_\zeta(\tilde{h}_{2n+1}, \tilde{h}_{2n+2})]}{1+\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1})}\}. \quad (2.1.10)
\end{aligned}$$

Case(a): If $\Omega_\zeta(\tilde{h}_{2n+1}, \tilde{h}_{2n+2})$ is maximum in (2.1.10), we have

$$\Omega_\zeta(\tilde{h}_{2n+1}, \tilde{h}_{2n+2}) \leq r\Omega_\zeta(\tilde{h}_{2n+1}, \tilde{h}_{2n+2}) < \Omega_\zeta(\tilde{h}_{2n+1}, \tilde{h}_{2n+2}),$$

contradiction.

Case(b): If $\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1})$ is maximum in (2.1.10), we have

$$\Omega_\zeta(\tilde{h}_{2n+1}, \tilde{h}_{2n+2}) \leq r\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1}). \quad (2.1.11)$$

Case(c): If $\frac{\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1})[1+\Omega_\zeta(\tilde{h}_{2n+1}, \tilde{h}_{2n+2})]}{1+\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1})}$ is maximum in (2.1.7), then

$$\max\{\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1}), \Omega_\zeta(\tilde{h}_{2n+1}, \tilde{h}_{2n+2})\} \leq \frac{\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1})[1+\Omega_\zeta(\tilde{h}_{2n+1}, \tilde{h}_{2n+2})]}{1+\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1})} \quad (2.1.12)$$

Subcase(i): If $\max\{\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1}), \Omega_\zeta(\tilde{h}_{2n+1}, \tilde{h}_{2n+2})\} = \Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1})$ then from (2.1.12), we have

$$\Omega_\zeta(\tilde{h}_{2n}, \tilde{h}_{2n+1}) \leq \Omega_\zeta(\tilde{h}_{2n+1}, \tilde{h}_{2n+2}), \quad (2.1.13)$$

a contradiction to our assumption.

Subcase(ii): If $\max\{\Omega_\varsigma(\tilde{h}_{2n}, \tilde{h}_{2n+1}), \Omega_\varsigma(\tilde{h}_{2n+1}, \tilde{h}_{2n+2})\} = \Omega_\varsigma(\tilde{h}_{2n+1}, \tilde{h}_{2n+2})$ then from (2.1.12), we have

$$\Omega_\varsigma(\tilde{h}_{2n+1}, \tilde{h}_{2n+2}) \leq \Omega_\varsigma(\tilde{h}_{2n}, \tilde{h}_{2n+1}), \quad (2.1.14)$$

a contradiction to our assumption.

Thus from (2.1.10), Case(a), Case(b) and Case(c), we conclude that

$$\Omega_\varsigma(\tilde{h}_{2n+1}, \tilde{h}_{2n+2}) \leq r\Omega_\varsigma(\tilde{h}_{2n}, \tilde{h}_{2n+1}). \quad (2.1.15)$$

Similarly by choosing $\mu = \tilde{h}_{2n}$, $\eta = \tilde{h}_{2n-1}$ in (2.1.1), we have

$$\Omega_\varsigma(\tilde{h}_{2n}, \tilde{h}_{2n+1}) \leq r\Omega_\varsigma(\tilde{h}_{2n}, \tilde{h}_{2n-1}), \quad (2.1.16)$$

Thus from (2.1.15) and (2.1.16), we can conclude that

$$\Omega_\varsigma(\tilde{h}_n, \tilde{h}_{n+1}) \leq r\Omega_\varsigma(\tilde{h}_n, \tilde{h}_{n-1}) \leq r^n\Omega_\varsigma(\tilde{h}_0, \tilde{h}_1) \quad (2.1.17)$$

hence

$$\lim_{n \rightarrow +\infty} \Omega_\varsigma(\tilde{h}_n, \tilde{h}_{n+1}) = 0. \quad (2.1.18)$$

In view of Lemma 1.10, condition (ii) of our assumption and condition (2.1.18), we have $\{\tilde{h}_n\}$ Cauchy sequence in $\mathfrak{N}$. By considering the assumption of completeness of the space $\mathfrak{N}$, we can conclude that

$$\lim_{n \rightarrow +\infty} \tilde{h}_n = \tilde{\tilde{h}}. \quad (2.1.19)$$

By virtue of (iii) of our assumption suppose that S is continuous. Since $\tilde{h}_n \rightarrow \tilde{\tilde{h}}$ as $n \rightarrow +\infty$, we have

$$S\tilde{\tilde{h}} = S(\lim_{n \rightarrow +\infty} \tilde{h}_{2n}) = \lim_{n \rightarrow +\infty} S\tilde{h}_{2n} = \lim_{n \rightarrow +\infty} \tilde{h}_{2n+1} = \tilde{\tilde{h}}.$$

We now show that $T\tilde{\tilde{h}} = \tilde{\tilde{h}}$. On using (2.1.1) with $\mu = \tilde{\tilde{h}}$ and $\eta = \tilde{\tilde{h}}$

$$\Omega_\varsigma(S\tilde{\tilde{h}}, T\tilde{\tilde{h}}) \leq rP(\tilde{\tilde{h}}, \tilde{\tilde{h}}) + LQ(\tilde{\tilde{h}}, \tilde{\tilde{h}}) \quad (2.1.20)$$

where

$$P(\tilde{\tilde{h}}, \tilde{\tilde{h}}) = \max\left\{\Omega_\varsigma(\tilde{\tilde{h}}, \tilde{\tilde{h}}), \frac{\Omega_\varsigma(\tilde{\tilde{h}}, S\tilde{\tilde{h}})[1+\Omega_\varsigma(\tilde{\tilde{h}}, T\tilde{\tilde{h}})]}{1+\Omega_\varsigma(\tilde{\tilde{h}}, \tilde{\tilde{h}})}, \frac{\Omega_\varsigma(\tilde{\tilde{h}}, T\tilde{\tilde{h}})[1+\Omega_\varsigma(\tilde{\tilde{h}}, S\tilde{\tilde{h}})]}{1+\Omega_\varsigma(\tilde{\tilde{h}}, \tilde{\tilde{h}})}, \frac{\Omega_\varsigma(\tilde{\tilde{h}}, S\tilde{\tilde{h}})\Omega_\varsigma(\tilde{\tilde{h}}, T\tilde{\tilde{h}})}{1+\Omega_\varsigma(\tilde{\tilde{h}}, \tilde{\tilde{h}})}, \frac{\Omega_\varsigma(\tilde{\tilde{h}}, T\tilde{\tilde{h}})\Omega_\varsigma(\tilde{\tilde{h}}, S\tilde{\tilde{h}})}{1+\Omega_\varsigma(\tilde{\tilde{h}}, \tilde{\tilde{h}})}, \frac{\Omega_\varsigma(\tilde{\tilde{h}}, S\tilde{\tilde{h}})\Omega_\varsigma(\tilde{\tilde{h}}, T\tilde{\tilde{h}})}{1+\Omega_\varsigma(\tilde{\tilde{h}}, S\tilde{\tilde{h}})+\Omega_\varsigma(\tilde{\tilde{h}}, T\tilde{\tilde{h}})}\right\} = \Omega_\varsigma(\tilde{\tilde{h}}, T\tilde{\tilde{h}}) \quad (2.1.21)$$

and

$$Q(\tilde{\tilde{h}}, \tilde{\tilde{h}}) = \min\{\Omega_\varsigma(\tilde{\tilde{h}}, S\tilde{\tilde{h}}), \Omega_\varsigma(\tilde{\tilde{h}}, T\tilde{\tilde{h}}), \Omega_\varsigma(\tilde{\tilde{h}}, S\tilde{\tilde{h}})\} = 0. \quad (2.1.22)$$

Thus from (2.1.20), (2.1.21) and (2.1.22), we have

$$\Omega_\varsigma(\tilde{\tilde{h}}, T\tilde{\tilde{h}}) \leq r\Omega_\varsigma(\tilde{\tilde{h}}, T\tilde{\tilde{h}}) < \Omega_\varsigma(\tilde{\tilde{h}}, T\tilde{\tilde{h}}), \text{ since } r < 1, \text{ yielding } \tilde{\tilde{h}} = T\tilde{\tilde{h}}$$

Hence $T\tilde{h} = S\tilde{h} = \tilde{h}$. Thus S and T have a common fixed point $\tilde{h} \in \aleph$.

Finally, we prove that $\tilde{h}$ is a unique common fixed point of S and T .

For, let $\tilde{h} = S\tilde{h} = T\tilde{h}$ and $\tilde{\nu} = S\tilde{\nu} = T\tilde{\nu}$. Now, let $\mu = \tilde{h}$ and $\nu = \tilde{h}$, in (2.1.1) then

$$\begin{aligned} \Omega_{\varsigma}(\tilde{h}, \tilde{\nu}) &= \Omega_{\varsigma}(S\tilde{h}, T\tilde{\nu}) \leq rP(\tilde{h}, \tilde{\nu}) + LQ(\tilde{h}, \tilde{\nu}) \\ &\leq r \max\{0, 0, \frac{\Omega_{\varsigma}(\tilde{h}, \tilde{\nu})\Omega_{\varsigma}(\tilde{h}, \tilde{\nu})}{1+\Omega_{\varsigma}(\tilde{h}, \tilde{\nu})}, \Omega_{\varsigma}(\tilde{h}, \tilde{\nu})\} + L \min\{0, 0, \Omega_{\varsigma}(\tilde{h}, \tilde{\nu})\} \\ &\leq r\Omega_{\varsigma}(\tilde{h}, \tilde{\nu}) < \Omega_{\varsigma}(\tilde{h}, \tilde{\nu}), \text{ a contraction, thus } \tilde{h} = \tilde{\nu}. \end{aligned}$$

Thus S and T have a unique common fixed point in $\aleph$. This completes the proof of this theorem. $\square$

By relaxing the condition (iii) i.e., one of S and T is continuous of Theorem 2.1, we have the following theorem.

Theorem 2.2. Let $(\aleph, \Omega_{\varsigma})$ be an extended rectangular B -metric space. Suppose two selfmaps S and T on $\aleph$ satisfy the conditions (i), (ii) of Theorem 2.1. Then S and T have a unique common fixed point in $\aleph$.

Proof. By Theorem 2.1., for any $\tilde{h}_0 \in \aleph$ the sequence $\{\tilde{h}_n\}$ in $\aleph$ defined by (2.1.2) is a Cauchy sequence in $\aleph$ and converges to $\tilde{h} \in \aleph$.

We now show that $T\tilde{h} = \tilde{h}$.

Indeed,

$$\Omega_{\varsigma}(S\tilde{h}_{2n}, T\tilde{h}) \leq rP(\tilde{h}_{2n}, \tilde{h}) + LQ(\tilde{h}_{2n}, \tilde{h}) \quad (2.2.1)$$

with

$$\begin{aligned} P(\tilde{h}_{2n}, \tilde{h}) &\leq \max\{\Omega_{\varsigma}(\tilde{h}_{2n}, \tilde{h}), \frac{\Omega_{\varsigma}(\tilde{h}_{2n}, S\tilde{h}_{2n})[1+\Omega_{\varsigma}(\tilde{h}, T\tilde{h})]}{1+\Omega_{\varsigma}(\tilde{h}_{2n}, \tilde{h})}, \frac{\Omega_{\varsigma}(\tilde{h}, T\tilde{h})[1+\Omega_{\varsigma}(\tilde{h}_{2n}, S\tilde{h}_{2n})]}{1+\Omega_{\varsigma}(\tilde{h}_{2n}, \tilde{h})}, \\ &\quad \frac{\Omega_{\varsigma}(\tilde{h}_{2n}, S\tilde{h}_{2n})\Omega_{\varsigma}(\tilde{h}, T\tilde{h})}{1+\Omega_{\varsigma}(\tilde{h}_{2n}, \tilde{h})}, \frac{\Omega_{\varsigma}(\tilde{h}, S\tilde{h}_{2n})\Omega_{\varsigma}(\tilde{h}_{2n}, T\tilde{h})}{1+\Omega_{\varsigma}(\tilde{h}_{2n}, \tilde{h})}, \frac{\Omega_{\varsigma}(\tilde{h}_{2n}, S\tilde{h}_{2n})\Omega_{\varsigma}(\tilde{h}, T\tilde{h})}{1+\Omega_{\varsigma}(\tilde{h}_{2n}, S\tilde{h}_{2n})+\Omega_{\varsigma}(\tilde{h}, T\tilde{h})}\} \\ &= \max\{\Omega_{\varsigma}(\tilde{h}_{2n}, \tilde{h}), \frac{\Omega_{\varsigma}(\tilde{h}_{2n}, \tilde{h}_{2n+1})[1+\Omega_{\varsigma}(\tilde{h}, T\tilde{h})]}{1+\Omega_{\varsigma}(\tilde{h}_{2n}, \tilde{h})}, \frac{[1+\Omega_{\varsigma}(\tilde{h}_{2n}, \tilde{h}_{2n+1})]\Omega_{\varsigma}(\tilde{h}, T\tilde{h})}{1+\Omega_{\varsigma}(\tilde{h}_{2n}, \tilde{h})}, \\ &\quad \frac{\Omega_{\varsigma}(\tilde{h}_{2n}, \tilde{h}_{2n+1})\Omega_{\varsigma}(\tilde{h}, T\tilde{h})}{1+\Omega_{\varsigma}(\tilde{h}_{2n}, \tilde{h})}, \frac{\Omega_{\varsigma}(\tilde{h}, \tilde{h}_{2n+1})\Omega_{\varsigma}(\tilde{h}_{2n}, T\tilde{h})}{1+\Omega_{\varsigma}(\tilde{h}_{2n}, \tilde{h})}, \frac{\Omega_{\varsigma}(\tilde{h}_{2n}, \tilde{h}_{2n+1})\Omega_{\varsigma}(\tilde{h}, T\tilde{h})}{1+\Omega_{\varsigma}(\tilde{h}_{2n}, \tilde{h}_{2n+1})+\Omega_{\varsigma}(\tilde{h}, T\tilde{h})}\}. \end{aligned}$$

On taking limits as $n \rightarrow +\infty$, using (2.1.19) and (2.2.1), we have

$$\lim_{n \rightarrow +\infty} P(\tilde{h}_{2n}, \tilde{h}) \leq \Omega_{\varsigma}(\tilde{h}, T\tilde{h}) \quad (2.2.2)$$

and

$$Q(\tilde{h}_{2n}, \tilde{h}) = \min\{\Omega_{\varsigma}(\tilde{h}_{2n}, S\tilde{h}_{2n}), \Omega_{\varsigma}(\tilde{h}_{2n}, T\tilde{h}), \Omega_{\varsigma}(\tilde{h}, S\tilde{h}_{2n})\},$$

which on making as $n \rightarrow +\infty$, and using (2.1.19), we have

$$\lim_{n \rightarrow +\infty} Q(\tilde{h}_{2n}, \tilde{h}) = 0. \quad (2.2.3)$$

Thus from (2.2.1), (2.2.2) and (2.2.3), we have

$$\lim_{n \rightarrow +\infty} \Omega_\varsigma(\tilde{S}\tilde{h}_{2n}, T\tilde{h}) \leq r\Omega_\varsigma(\tilde{h}, T\tilde{h}). \quad (2.2.4)$$

By condition (2.2.4), we conclude that

$$\Omega_\varsigma(\tilde{h}, T\tilde{h}) = \lim_{n \rightarrow +\infty} \Omega_\varsigma(\tilde{h}_{2n+1}, T\tilde{h}) \leq r\Omega_\varsigma(\tilde{h}, T\tilde{h}) < r\Omega_\varsigma(\tilde{h}, T\tilde{h}),$$

which implies that $\Omega_\varsigma(\tilde{h}, T\tilde{h}) = 0$. Hence we obtain $T\tilde{h} = \tilde{h}$.

We now show that $\tilde{S}\tilde{h} = \tilde{h}$.

$$\Omega_\varsigma(\tilde{S}\tilde{h}, T\tilde{h}_{2n+1}) \leq rP(\tilde{h}, \tilde{h}_{2n+1}) + LQ(\tilde{h}, \tilde{h}_{2n+1}) \quad (2.2.5)$$

with

$$P(\tilde{h}, \tilde{h}_{2n+1}) \leq \max\left\{\Omega_\varsigma(\tilde{h}, \tilde{h}_{2n+1}), \frac{\Omega_\varsigma(\tilde{h}_{2n+1}, \tilde{h}_{2n+2})[1+\Omega_\varsigma(\tilde{h}, \tilde{S}\tilde{h})]}{1+\Omega_\varsigma(\tilde{h}, \tilde{h}_{2n+1})}, \frac{[1+\Omega_\varsigma(\tilde{h}_{2n+1}, \tilde{h}_{2n+2})]\Omega_\varsigma(\tilde{h}, \tilde{S}\tilde{h})}{1+\Omega_\varsigma(\tilde{h}, \tilde{h}_{2n+1})}, \right. \\ \left. \frac{\Omega_\varsigma(\tilde{h}, \tilde{S}\tilde{h})\Omega_\varsigma(\tilde{h}_{2n+1}, \tilde{h}_{2n+2})}{1+\Omega_\varsigma(\tilde{h}, \tilde{h}_{2n+1})}, \frac{\Omega_\varsigma(\tilde{h}_{2n+1}, \tilde{S}\tilde{h})\Omega_\varsigma(\tilde{h}, \tilde{h}_{2n+2})}{1+\Omega_\varsigma(\tilde{h}, \tilde{h}_{2n+1})}, \frac{\Omega_\varsigma(\tilde{h}, \tilde{S}\tilde{h})\Omega_\varsigma(\tilde{h}_{2n+1}, \tilde{h}_{2n+2})}{1+\Omega_\varsigma(\tilde{h}_{2n+1}, \tilde{h}_{2n+2})+\Omega_\varsigma(\tilde{h}, \tilde{S}\tilde{h})}\right\}$$

which on making limits as $n \rightarrow +\infty$, using (2.1.18) and (2.1.19), we have

$$\lim_{n \rightarrow +\infty} P(\tilde{h}, \tilde{h}_{2n+1}) \leq \Omega_\varsigma(\tilde{h}, \tilde{S}\tilde{h}) \quad (2.2.6)$$

and

$$Q(\tilde{h}, \tilde{h}_{2n+1}) = \min\{\Omega_\varsigma(\tilde{h}, \tilde{S}\tilde{h}), \Omega_\varsigma(\tilde{h}_{2n+1}, \tilde{S}\tilde{h}), \Omega_\varsigma(\tilde{h}, \tilde{h}_{2n+2})\}.$$

On taking limits as $n \rightarrow +\infty$, we get

$$\lim_{n \rightarrow +\infty} Q(\tilde{h}_{2n}, \tilde{h}) = 0. \quad (2.2.7)$$

Thus from (2.2.5), (2.2.6) and (2.2.7), we have

$$\lim_{n \rightarrow +\infty} \Omega_\varsigma(\tilde{S}\tilde{h}, T\tilde{h}_{2n+1}) \leq r\Omega_\varsigma(\tilde{h}, \tilde{S}\tilde{h}). \quad (2.2.8)$$

Hence from (2.2.8), we conclude that

$$\Omega_\varsigma(\tilde{h}, \tilde{S}\tilde{h}) = \lim_{n \rightarrow +\infty} \Omega_\varsigma(\tilde{S}\tilde{h}, T\tilde{h}_{2n+1}) \leq r\Omega_\varsigma(\tilde{h}, \tilde{S}\tilde{h}) < r\Omega_\varsigma(\tilde{h}, \tilde{S}\tilde{h}),$$

which implies $\Omega_\varsigma(\tilde{h}, \tilde{S}\tilde{h}) = 0$. Hence

$$\tilde{S}\tilde{h} = \tilde{h}. \quad (2.2.9)$$

Thus S and T have a common fixed point $\tilde{h} \in \mathfrak{X}$. Hence

$$\tilde{h} = T\tilde{h} = \tilde{S}\tilde{h}. \quad (2.2.10)$$

Uniqueness of common fixed point of S and T follows on the same lines of Theorem 2.1.

This completes the proof of this theorem. $\square$

By taking $L = 0$ in Theorem 2.2, we have the following corollary.

Corollary 2.3. Let $(\aleph, \Omega_\varsigma)$ be an extended rectangular B-metric space and $S, T : \aleph \rightarrow \aleph$. Suppose that the following conditions hold.

(i) for all $\mu, \eta \in \aleph$, there exists $r \in [0, 1)$ we have

$$\Omega_\varsigma(S\mu, T\eta) \leq rP(\mu, \eta) \quad (2.3.1)$$

where

$$P(\mu, \eta) = \max\left\{\Omega_\varsigma(\mu, \eta), \frac{\Omega_\varsigma(\mu, S\mu)[1+\Omega_\varsigma(\eta, T\eta)]}{1+\Omega_\varsigma(\mu, \eta)}, \frac{\Omega_\varsigma(\eta, T\eta)[1+\Omega_\varsigma(\mu, S\mu)]}{1+\Omega_\varsigma(\mu, \eta)}, \frac{\Omega_\varsigma(\mu, S\mu)\Omega_\varsigma(\eta, T\eta)}{1+\Omega_\varsigma(\mu, \eta)}, \frac{\Omega_\varsigma(\eta, S\mu)\Omega_\varsigma(\mu, T\eta)}{1+\Omega_\varsigma(\mu, \eta)}, \frac{\Omega_\varsigma(\mu, S\mu)\Omega_\varsigma(\eta, T\eta)}{1+\Omega_\varsigma(\mu, S\mu)+\Omega_\varsigma(\eta, T\eta)}\right\}.$$

(ii) Suppose that, for each $\mu_0 \in \aleph$, $\lim_{m, n \rightarrow +\infty} \varsigma(\tilde{h}_n, \tilde{h}_m) < \frac{1}{r}$.

Then S and T have a unique common fixed point in $\aleph$.

By choosing $S = T$ in Theorem 2.2, we have the following corollary.

Corollary 2.4. Let $(\aleph, \Omega_\varsigma)$ be an extended rectangular B-metric space and $T : \aleph \rightarrow \aleph$. Suppose that the following conditions hold.

(i) for all $\mu, \eta \in \aleph$, there exists $r \in [0, 1)$ and $L \geq 0$ we have

$$\Omega_\varsigma(T\mu, T\eta) \leq rP(\mu, \eta) + LQ(\mu, \eta) \quad (2.4.1)$$

where

$$P(\mu, \eta) = \max\left\{\Omega_\varsigma(\mu, \eta), \frac{\Omega_\varsigma(\mu, T\mu)[1+\Omega_\varsigma(\eta, T\eta)]}{1+\Omega_\varsigma(\mu, \eta)}, \frac{\Omega_\varsigma(\eta, T\eta)[1+\Omega_\varsigma(\mu, T\mu)]}{1+\Omega_\varsigma(\mu, \eta)}, \frac{\Omega_\varsigma(\mu, T\mu)\Omega_\varsigma(\eta, T\eta)}{1+\Omega_\varsigma(\mu, \eta)}, \frac{\Omega_\varsigma(\eta, T\mu)\Omega_\varsigma(\mu, T\eta)}{1+\Omega_\varsigma(\mu, \eta)}, \frac{\Omega_\varsigma(\mu, T\mu)\Omega_\varsigma(\eta, T\eta)}{1+\Omega_\varsigma(\mu, T\mu)+\Omega_\varsigma(\eta, T\eta)}\right\}$$

and $Q(\mu, \eta) = \min\{\Omega_\varsigma(\mu, T\mu), \Omega_\varsigma(\mu, T\eta), \Omega_\varsigma(\eta, T\mu)\}$

(ii) Suppose that, for each $\tilde{h}_0 \in \aleph$, $\lim_{m, n \rightarrow +\infty} \varsigma(\tilde{h}_n, \tilde{h}_m) < \frac{1}{r}$.

Then T has a unique fixed point in $\aleph$.

By choosing $\varsigma(\mu, \eta) = B \geq 1$ in Theorem 2.2, we have the following corollary.

Corollary 2.5. Let $(\aleph, \Omega_\varsigma)$ be a complete rectangular B-metric space and $S, T : \aleph \rightarrow \aleph$. Suppose that the following conditions hold.

(i) for all $\mu, \eta \in \aleph$, there exists $r \in [0, \frac{1}{B})$ and $L \geq 0$ we have

$$\Omega_\varsigma(S\mu, T\eta) \leq rP(\mu, \eta) + LQ(\mu, \eta) \quad (2.5.1)$$

where

$$P(\mu, \eta) = \max\left\{\Omega_\varsigma(\mu, \eta), \frac{\Omega_\varsigma(\mu, S\mu)[1+\Omega_\varsigma(\eta, T\eta)]}{1+\Omega_\varsigma(\mu, \eta)}, \frac{\Omega_\varsigma(\eta, T\eta)[1+\Omega_\varsigma(\mu, S\mu)]}{1+\Omega_\varsigma(\mu, \eta)}, \frac{\Omega_\varsigma(\mu, S\mu)\Omega_\varsigma(\eta, T\eta)}{1+\Omega_\varsigma(\mu, \eta)}, \frac{\Omega_\varsigma(\eta, S\mu)\Omega_\varsigma(\mu, T\eta)}{1+\Omega_\varsigma(\mu, \eta)}, \frac{\Omega_\varsigma(\mu, S\mu)\Omega_\varsigma(\eta, T\eta)}{1+\Omega_\varsigma(\mu, S\mu)+\Omega_\varsigma(\eta, T\eta)}\right\}$$

and $Q(\mu, \eta) = \min\{\Omega_\varsigma(\mu, S\mu), \Omega_\varsigma(\mu, T\eta), \Omega_\varsigma(\eta, S\mu)\}$.

Then S and T have a unique fixed point in $\aleph$.

The following is an example in support of Theorem 2.1.

Example 2.6. Let $\aleph = [0, 1]$. We define the functions $\varsigma : \aleph \times \aleph \rightarrow [1, \infty)$ by $\varsigma(\mu, \eta) = 2(\mu + \eta) + 3$ and $\Omega_\varsigma : \aleph \times \aleph \rightarrow [0, \infty)$ by

$$\Omega_\varsigma(\mu, \eta) = \begin{cases} 0 & \text{if } \mu = \eta \\ (\mu + \eta)^3 & \text{if } \mu \neq \eta, \end{cases}$$

Clearly, $\Omega_\varsigma(\mu, \eta)$ forms an extended rectangular B-metric space. But $(\aleph, \Omega)$ is not a rectangular metric space at $\mu = 1, \eta = \frac{1}{2}, \tilde{h} = \frac{1}{4}$ and $v = \frac{1}{14}$, since

$$\Omega_\varsigma(1, \frac{1}{2}) = \frac{27}{8} \not\leq \Omega_\varsigma(1, \frac{1}{4}) + \Omega_\varsigma(\frac{1}{4}, \frac{1}{16}) + \Omega_\varsigma(\frac{1}{16}, \frac{1}{2}) = \frac{4424}{2048}.$$

We now define

$$T, S : \aleph \rightarrow \aleph \text{ by } T\mu = \frac{\mu}{4} \text{ for all } \mu \in [0, 1] \text{ and } S\mu = \begin{cases} 0 & \text{if } \mu \in [0, 1] \setminus \{\frac{1}{2}\} \\ \frac{1}{8} & \text{if } \mu = \frac{1}{2}. \end{cases}$$

When $\tilde{h}_0 = \frac{1}{2}$ or $\tilde{h}_0 \in [0, 1] \setminus \{\frac{1}{2}\}$, we have $\tilde{h}_n = 0$, so

$$\lim_{m, n \rightarrow +\infty} \varsigma(\tilde{h}_n, \tilde{h}_m) = 3 < \frac{1}{r} = 64,$$

so condition (ii) of our assumption is satisfied. We now verify the inequality (2.1.1) with $r = \frac{1}{64}$ and $L = 0$.

Case (1): When $\mu = \frac{1}{2}, \eta = \frac{1}{2}$ then $\Omega_\varsigma(S\mu, T\eta) = 0$, so that the inequality (2.1.1) holds.

Case (2): When $\mu = \frac{1}{2}, \eta \neq \frac{1}{2}$, we have

$$\Omega_\varsigma(S\mu, T\eta) = (\frac{1}{8} + \frac{\eta}{4})^3 \leq \frac{1}{64}(\frac{1}{2} + \eta)^3 \leq r\Omega_\varsigma(\mu, \eta) \leq rP(\mu, \eta) + LQ(\mu, \eta),$$

so that the inequality (2.1.1) holds.

Case (3): When $\mu \neq \frac{1}{2}, \eta = \frac{1}{2}$, we have

$$\Omega_\varsigma(S\mu, T\eta) = (\frac{1}{8})^3 \leq \frac{1}{64}(\frac{1}{2} + \mu)^3 \leq r\Omega_\varsigma(\mu, \eta) \leq rP(\mu, \eta) + LQ(\mu, \eta),$$

so that the inequality (2.1.1) holds.

Case (4): When $\mu \neq \frac{1}{2}, \eta \neq \frac{1}{2}$, we have

$$\Omega_\varsigma(S\mu, T\eta) = (\frac{\eta}{4})^3 \leq \frac{1}{64}(\mu + \eta)^3 \leq r\Omega_\varsigma(\mu, \eta) \leq rP(\mu, \eta) + LQ(\mu, \eta),$$

so that the inequality (2.1.1) holds. Also, we note that 0 is a unique fixed point of S and T .

Now, we provide an example which shows the validity of our results.

Example 2.7. Let $R = [0, \frac{1}{3}]$, $S = [\frac{2}{3}, 1]$ and $\aleph = R \cup S$. We define $\Omega_\varsigma : \aleph \times \aleph \rightarrow [0, \infty)$ by

$$\Omega_\varsigma(\mu, \eta) = \begin{cases} 0 & \text{if } \mu = \eta \\ \frac{1}{4} & \text{if } \mu \in R, \\ 1 & \text{if } \mu \in S \\ (\mu - \eta)^2 & \text{otherwise} \end{cases}$$

and $\varsigma : \aleph \times \aleph \rightarrow [1, \infty)$ by $\varsigma(\mu, \eta) = 2(\mu + \eta) + 1$. Clearly, $\Omega_\varsigma(\mu, \eta)$ forms an extended rectangular B-metric space with the function ς .

We define S and T on $\aleph$ by

$$S\mu = \begin{cases} \frac{\mu}{8} & \text{if } \mu \in R \\ \frac{\mu}{4} & \mu \in S, \end{cases} \quad \text{and } T\mu = \begin{cases} 0 & \text{if } \mu \in R \\ \frac{\mu}{3} & \mu \in S. \end{cases}$$

If $\tilde{h}_0 \in R$ then $\tilde{h}_{2n} = 0$ and $\tilde{h}_{2n} = \tilde{h}_{2n-1}$, thus $\tilde{h}_n = 0$.
Thus $\lim_{n \rightarrow +\infty} \varsigma(\tilde{h}_n, \tilde{h}_m) = 1 < \frac{3}{2}$ when $r = \frac{2}{3}$.

If $\tilde{h}_0 \in S$ then $\tilde{h}_{2n} = 0$ and $\tilde{h}_{2n} = \tilde{h}_{2n-1}$, then $\tilde{h}_n = 0$.
Therefore $\lim_{n \rightarrow +\infty} \varsigma(\tilde{h}_n, \tilde{h}_m) = 1 < \frac{3}{2}$.

Case (1): If $\mu \in R$ and $\eta \in R$ then

$$\begin{aligned} \Omega_\varsigma(S\mu, T\eta) &= \Omega_\varsigma(\frac{\mu}{8}, 0) \\ &\leq r \max\left\{\Omega_\varsigma(\mu, \eta), \frac{\Omega_\varsigma(\mu, \frac{\mu}{8})[1+\Omega_\varsigma(\eta, 0)]}{1+\Omega_\varsigma(\mu, \eta)}, \frac{\Omega_\varsigma(\eta, 0)[1+\Omega_\varsigma(\mu, \frac{\mu}{8})]}{1+\Omega_\varsigma(\mu, \eta)}, \frac{\Omega_\varsigma(\mu, \frac{\mu}{8})\Omega_\varsigma(\eta, 0)}{1+\Omega_\varsigma(\mu, \eta)}, \right. \\ &\quad \left. \frac{\Omega_\varsigma(\mu, 0)\Omega_\varsigma(\eta, \frac{\mu}{8})}{1+\Omega_\varsigma(\mu, \eta)}, \frac{\Omega_\varsigma(\mu, \frac{\mu}{8})\Omega_\varsigma(\eta, 0)}{1+\Omega_\varsigma(\mu, \frac{\mu}{8})+\Omega_\varsigma(\eta, 0)}\right\} \\ &\quad + L \min\{\Omega_\varsigma(\mu, \frac{\mu}{8}), \Omega_\varsigma(\eta, \frac{\mu}{8}), \Omega_\varsigma(\mu, 0)\} \end{aligned}$$

Subcase(a): When $\mu = 0$, then $\Omega_\varsigma(S\mu, T\eta) = 0 \leq rP(\mu, \eta) + LQ(\mu, \eta)$,

so inequality (2.1.1) holds for any $L \geq 0$.

Subcase(b): When $\mu = \eta \neq 0$ then

$$\Omega_\varsigma(\frac{\mu}{8}, 0) = \frac{1}{4} \leq \frac{2}{3}(\frac{1}{4} + \frac{1}{16}) + L\frac{1}{4},$$

so inequality (2.1.1) holds for any $L \geq \frac{1}{2}$.

Subcase (c): When $\mu \neq \eta \neq 0$

$$\Omega_\varsigma(\frac{\mu}{8}, 0) = \frac{1}{4} \leq \frac{2}{3}\frac{1}{4} + L\frac{1}{4},$$

so inequality (2.1.1) holds for any $L \geq \frac{1}{2}$

Case (2): If $\mu \in [\frac{2}{3}, 1]$ and $\eta \in [\frac{2}{3}, 1]$ then

$$\begin{aligned}
\Omega_{\varsigma}(S\mu, T\eta) &= \Omega_{\varsigma}\left(\frac{\mu}{4}, \frac{\eta}{3}\right) \\
&\leq rmax\left\{\Omega_{\varsigma}(\mu, \eta), \frac{\Omega_{\varsigma}(\mu, \frac{\mu}{4})[1+\Omega_{\varsigma}(\eta, \frac{\eta}{3})]}{1+\Omega_{\varsigma}(\mu, \eta)}, \frac{\Omega_{\varsigma}(\eta, \frac{\eta}{3})[1+\Omega_{\varsigma}(\mu, \frac{\mu}{4})]}{1+\Omega_{\varsigma}(\mu, \eta)}, \frac{\Omega_{\varsigma}(\mu, \frac{\mu}{4})\Omega_{\varsigma}(\eta, \frac{\eta}{3})}{1+\Omega_{\varsigma}(\mu, \eta)}, \right. \\
&\quad \left. \frac{\Omega_{\varsigma}(\mu, \frac{\eta}{3})\Omega_{\varsigma}(\eta, \frac{\mu}{4})}{1+\Omega_{\varsigma}(\mu, \eta)}, \frac{\Omega_{\varsigma}(\mu, \frac{\mu}{4})\Omega_{\varsigma}(\eta, \frac{\eta}{3})}{1+\Omega_{\varsigma}(\mu, \frac{\mu}{4})+\Omega_{\varsigma}(\eta, \frac{\eta}{3})}\right\} \\
&\quad + Lmin\{\Omega_{\varsigma}(\mu, \frac{\mu}{4}), \Omega_{\varsigma}(\eta, \frac{\mu}{4}), \Omega_{\varsigma}(\mu, \frac{\eta}{3})\}
\end{aligned}$$

Subcase(a): When $\mu = \eta$, we have

$$\begin{aligned}
\frac{1}{4} &\leq rmax\left\{0, \frac{9\mu^2}{16}\left[1 + \frac{(2\eta)^2}{9}\right], \frac{(2\eta)^2}{9}\left[1 + \frac{9\mu^2}{16}\right], \frac{(2\eta)^2}{9} \frac{9\mu^2}{16}, \frac{(3\mu-\eta)^2}{9} \frac{(4\eta-\mu)^2}{16}, \frac{9\mu^2(3\eta-\eta)^2}{144+(3\eta-\eta)^2+9\mu^2}\right\} \\
&\quad + Lmin\left\{\frac{9\mu^2}{16}, \frac{(3\mu-\eta)^2}{9}, \frac{(4\eta-\mu)^2}{16}\right\} \\
&= \frac{2}{3}\left[\frac{9\mu^2}{16} + \frac{\mu^4}{4}\right] + L\frac{4\mu^2}{9},
\end{aligned}$$

the minimum value of RHS attains at $\mu = \frac{2}{3}$, therefore

$$\frac{1}{4} \leq \frac{97}{324} + L\frac{16}{81},$$

so inequality (2.1.1) holds for any $L \geq 0$.

Subcase(b): When $\mu \neq \eta$, we have

$$\frac{1}{4} \leq \frac{2}{3} + L\frac{3}{4},$$

so inequality (2.1.1) holds for any $L \geq 0$.

Case(3): When $\mu \in [0, \frac{1}{3}]$ and $\eta \in [\frac{2}{3}, 1]$, we have

$$\begin{aligned}
\Omega_{\varsigma}(S\mu, T\eta) &= \Omega_{\varsigma}\left(\frac{\mu}{8}, \frac{\eta}{3}\right) \\
&\leq rmax\left\{\Omega_{\varsigma}(\mu, \eta), \frac{\Omega_{\varsigma}(\mu, \frac{\mu}{8})[1+\Omega_{\varsigma}(\eta, \frac{\eta}{3})]}{1+\Omega_{\varsigma}(\mu, \eta)}, \frac{\Omega_{\varsigma}(\eta, \frac{\eta}{3})[1+\Omega_{\varsigma}(\mu, \frac{\mu}{8})]}{1+\Omega_{\varsigma}(\mu, \eta)}, \frac{\Omega_{\varsigma}(\mu, \frac{\mu}{8})\Omega_{\varsigma}(\eta, \frac{\eta}{3})}{1+\Omega_{\varsigma}(\mu, \eta)}, \right. \\
&\quad \left. \frac{\Omega_{\varsigma}(\mu, \frac{\eta}{3})\Omega_{\varsigma}(\eta, \frac{\mu}{8})}{1+\Omega_{\varsigma}(\mu, \eta)}, \frac{\Omega_{\varsigma}(\mu, \frac{\mu}{8})\Omega_{\varsigma}(\eta, \frac{\eta}{3})}{1+\Omega_{\varsigma}(\mu, \frac{\mu}{8})+\Omega_{\varsigma}(\eta, \frac{\eta}{3})}\right\} \\
&\quad + Lmin\{\Omega_{\varsigma}(\mu, \frac{\mu}{8}), \Omega_{\varsigma}(\eta, \frac{\mu}{8}), \Omega_{\varsigma}(\mu, \frac{\eta}{3})\}
\end{aligned}$$

Subcase(a): When $\mu = 0$, we have

$$\Omega_{\varsigma}(S\mu, T\eta) = \frac{1}{4} \leq \frac{2}{3}\eta^2,$$

so inequality of (2.1.1) holds for any $L \geq 0$.

Subcase(b): When $\mu \neq 0$, we have

$$\begin{aligned}
\Omega_{\varsigma}(S\mu, T\eta) &= \frac{1}{4} \leq rmax\left\{(\mu - \eta)^2, \frac{9+4\eta^2}{36(1+(\mu-\eta)^2)}, \frac{20\eta^2}{36(1+(\mu-\eta)^2)}, \frac{\eta^2}{(1+(\mu-\eta)^2)}, \right. \\
&\quad \left. \frac{(8\eta-\mu)^2}{4096(1+(\mu-\eta)^2)}, \frac{36\eta^2}{37+4\eta^2}\right\} + Lmin\left\{\frac{1}{4}, (\eta - \frac{\mu}{8})^2, (\mu - \frac{\eta}{3})^2\right\}
\end{aligned}$$

the minimum value of RHS attains at $\mu = \frac{1}{3}$ and $\eta = \frac{2}{3}$, therefore

$\Omega_\varsigma(S\mu, T\eta) = \frac{1}{4} \leq \frac{2}{3} \frac{97}{360} + L\frac{1}{4}$, thus inequality (2.1.1) holds for any $L \geq \frac{1}{2}$.

Case(4): When $\mu \in [\frac{2}{3}, 1]$ and $\eta \in [0, \frac{1}{3}]$, we have

$$\begin{aligned} \Omega_\varsigma(S\mu, T\eta) &= \Omega_\varsigma(\frac{\mu}{4}, 0) = \frac{1}{4} \\ &\leq rmax\{\Omega_\varsigma(\mu, \eta), \frac{\Omega_\varsigma(\mu, \frac{\mu}{4})[1+\Omega_\varsigma(\eta, 0)]}{1+\Omega_\varsigma(\mu, \eta)}, \frac{\Omega_\varsigma(\eta, 0)[1+\Omega_\varsigma(\mu, \frac{\mu}{4})]}{1+\Omega_\varsigma(\mu, \eta)}, \frac{\Omega_\varsigma(\mu, \frac{\mu}{4})\Omega_\varsigma(\eta, 0)}{1+\Omega_\varsigma(\mu, \eta)}, \\ &\quad \frac{\Omega_\varsigma(\mu, 0)\Omega_\varsigma(\eta, \frac{\mu}{4})}{1+\Omega_\varsigma(\mu, \eta)}, \frac{\Omega_\varsigma(\mu, \frac{\mu}{4})\Omega_\varsigma(\eta, 0)}{1+\Omega_\varsigma(\mu, \frac{\mu}{4})+\Omega_\varsigma(\eta, 0)}\} \\ &\quad + Lmin\{\Omega_\varsigma(\mu, \frac{\mu}{4}), \Omega_\varsigma(\mu, 0), \Omega_\varsigma(\eta, \frac{\mu}{4})\}. \end{aligned}$$

The minimum value of RHS is at $\mu = \frac{2}{3}$ and $\eta = \frac{1}{3}$, in this case we have

$$\frac{1}{4} \leq \frac{3}{4} + L\frac{1}{36},$$

hence inequality (2.1.1) holds for any $L \geq 0$.

Here we observe that inequality (2.3.1) fails to hold for any $r \in (0, 1)$ at $\mu = 1/3$, $\eta = 1/3$ since

$$\Omega_\varsigma(\frac{1}{24}, 0) = \frac{1}{4} \not\leq \frac{2}{3} \frac{1}{4}.$$

Hence Theorem (2.2) generalizes Corollary(2.3).

The following example shows that when $S = T$ satisfies contractive condition (2.1.1) but fails to satisfy contractive conditions (1.4.1) and (1.5.1) (which is that of [2]).

Example 2.8. Let $R = \{0, \frac{1}{2}, 1, \frac{11}{10}\}$, $S = [\frac{3}{2}, 2]$ and $\aleph = R \cup S$. We define the functions $\varsigma : \aleph \times \aleph \rightarrow [1, \infty)$ by $\varsigma(\mu, \eta) = \mu + \eta + 1$ and $\Omega_\varsigma : \aleph \times \aleph \rightarrow [0, \infty)$ as follows:

$$\begin{aligned} \Omega_\varsigma(\mu, \eta) &= \Omega_\varsigma(\eta, \mu) \text{ for all } \mu, \eta \in \aleph, \\ \Omega_\varsigma(\mu, \eta) &= \begin{cases} \Omega_\varsigma(0, 1) = \Omega_\varsigma(\frac{11}{10}, \frac{1}{2}) = 2 \\ \Omega_\varsigma(\frac{11}{10}, 0) = \Omega_\varsigma(\frac{11}{10}, 1) = 3 \\ \Omega_\varsigma(\frac{1}{2}, 1) = 7 : \Omega_\varsigma(0, \frac{1}{2}) = \frac{4}{5} \\ 8 \text{ if } \mu, \eta \in S \\ (\mu - \eta)^2 \text{ otherwise.} \end{cases} \end{aligned}$$

Then clearly, $(\aleph, \Omega_\varsigma)$ forms an extended rectangular B-metric space. But, it is not rectangular metric space, at $\mu = 1$ $\eta = \frac{1}{2}$, since

$$\Omega_\varsigma(1, \frac{1}{2}) = 7 \not\leq \Omega_\varsigma(1, \frac{11}{10}) + \Omega_\varsigma(\frac{11}{10}, 0) + (0, \frac{1}{2}) = 6.8$$

We now define $T : \aleph \rightarrow \aleph$ by $T0 = T1 = T\frac{1}{2} = 0$, $T\frac{11}{10} = 1$ and $T\mu = \frac{1}{2}$ for all $\mu \in S$.

If $\tilde{h}_0 \in R$ then $\tilde{h}_{2n} = 0$ and $\tilde{h}_{2n} = \tilde{h}_{2n-1}$, thus $\tilde{h}_n = 0$.

Thus $\lim_{n \rightarrow +\infty} \varsigma(\tilde{h}_n, \tilde{h}_m) = 1 < \frac{5}{4}$ when $r = \frac{4}{5}$.

If $\tilde{h}_0 \in S$ then $\tilde{h}_{2n} = 0$ and $\tilde{h}_{2n} = \tilde{h}_{2n-1}$, then $\tilde{h}_n = 0$.
Therefore $\lim_{n \rightarrow +\infty} \varsigma(\tilde{h}_n, \tilde{h}_m) = 1 < \frac{5}{4}$.

We now verify the inequality (2.5.1) with $r = \frac{4}{5}$.

Case (1): If $\mu \in \{0, \frac{1}{2}, 1\}$ and $\eta \in \{0, \frac{1}{2}, 1\}$ we have $\Omega_\varsigma(T\mu, T\eta) = 0$, so the condition (2.1.1) is satisfied.

Case (2): If $\mu = 0$ and $\eta = \frac{11}{10}$, we have

$\Omega_\varsigma(0, 1) = 2 \leq 3r = 2.4$
so condition (2.1.1) is satisfied for any $L \geq 0$.

Case (3): If $\mu = 0$ or $[\frac{3}{2}, 2]$ and $\eta = [\frac{3}{2}, 2]$ or $\eta = 0$, we have

$$\Omega_\varsigma(0, \frac{1}{2}) \leq rmax\{\Omega_\varsigma(0, \eta), \frac{\Omega_\varsigma(0, \frac{1}{2})\Omega_\varsigma(\eta, 0)}{1+\Omega_\varsigma(0, \eta)}, 0, \frac{\Omega_\varsigma(\eta, \frac{1}{2})}{1+\Omega_\varsigma(0, \eta)}\}$$

the minimum value of RHS is at $\eta = \frac{3}{2}$, therefore

$$\frac{4}{5} \leq r\frac{9}{4} = 1.8, \text{ so the inequality (2.1.1) is satisfied for any } L \geq 0.$$

Case (4): If $\mu = \frac{1}{2}$ and $\eta = \frac{11}{10}$, we have

$$\Omega_\varsigma(0, 1) = 2 \leq \frac{14}{5} + L\frac{4}{5},$$

so condition (2.1.1) is satisfied for any $L \geq 0$.

Case (5): If $\mu = \frac{1}{2}$ or $\mu = [\frac{3}{2}, 2]$ and $\eta = [\frac{3}{2}, 2]$ or $\eta = \frac{1}{2}$, we have

$$\begin{aligned} & \Omega_\varsigma(0, \frac{1}{2}) \\ & \leq rmax\{\Omega_\varsigma(\frac{1}{2}, \eta), \frac{\Omega_\varsigma(0, \frac{1}{2})[1+\Omega_\varsigma(\eta, \frac{1}{2})]}{1+\Omega_\varsigma(\frac{1}{2}, \eta)}, \frac{\Omega_\varsigma(\eta, \frac{1}{2})[1+\Omega_\varsigma(0, \frac{1}{2})]}{1+\Omega_\varsigma(\frac{1}{2}, \eta)}, \frac{\Omega_\varsigma(\eta, \frac{1}{2})\Omega_\varsigma(0, \frac{1}{2})}{1+\Omega_\varsigma(\frac{1}{2}, \eta)} \frac{\Omega_\varsigma(\eta, \frac{1}{2})\Omega_\varsigma(\frac{1}{2}, 0)}{1+\Omega_\varsigma(\frac{1}{2}, \eta)+\Omega_\varsigma(\frac{1}{2}, 0)}\} \end{aligned}$$

the minimum value of RHS is at $\eta = \frac{3}{2}$, therefore

$$\frac{4}{5} \leq \frac{4}{5},$$

so the inequality (2.1.1) is satisfied for any $L \geq 0$.

Case (6): If $\mu = 1$ and $\eta = \frac{11}{10}$, we have

$\Omega_\varsigma(0, 1) = 2 \leq 3r = 2.4$,
so condition (2.1.1) is satisfied for any $L \geq 0$.

Case (7): If $\mu = 1$, $\eta = [\frac{3}{2}, 2]$, we have

$$\begin{aligned} & \Omega_\varsigma(0, \frac{1}{2}) \\ & \leq rmax\{\Omega_\varsigma(1, \eta), \frac{\Omega_\varsigma(\eta, \frac{1}{2})[1+\Omega_\varsigma(1, 0)]}{1+\Omega_\varsigma(1, \eta)}, \frac{\Omega_\varsigma(1, 0)[1+\Omega_\varsigma(\eta, \frac{1}{2})]}{1+\Omega_\varsigma(1, \eta)}, \frac{\Omega_\varsigma(\eta, \frac{1}{2})\Omega_\varsigma(1, 0)}{1+\Omega_\varsigma(1, \eta)} \frac{\Omega_\varsigma(1, \frac{1}{2})\Omega_\varsigma(\eta, 0)}{1+\Omega_\varsigma(\frac{1}{2}, 1)+\Omega_\varsigma(\eta, 0)}\} \end{aligned}$$

$$+Lmin\{\Omega_\varsigma(1, 0), \Omega_\varsigma(1, \frac{1}{2}), \Omega_\varsigma(\eta, 0)\},$$

the minimum value of RHS is at $\eta = \frac{3}{2}$

$\frac{4}{5} \leq r(12.6) + L\frac{4}{5}$,
so the inequality (2.1.1) is satisfied for any $L \geq 0$.

Case (8): If $\mu = \frac{11}{10}$ and $\eta = 0$, we have

$\Omega_\zeta(0, 1) = 2 \leq 2r + L2$,
so condition (2.1.1) is satisfied for any $L \geq \frac{1}{4}$.

Case (9): If $\mu = \frac{11}{10}$ and $\eta = \frac{1}{2}$, we have

$\Omega_\zeta(0, 1) = 2 \leq 2r + L2$,
so condition (2.1.1) is satisfied for any $L \geq \frac{1}{4}$.

Case (10): If $\mu = \frac{11}{10}$ and $\eta = 1$, we have

$\Omega_\zeta(0, 1) = 2 \leq 3r$,
so condition (2.1.1) is satisfied for any $L \geq 0$.

Case (11): If $\mu = \frac{11}{10}$ or $\mu = [\frac{3}{2}, 2]$ and $\eta = [\frac{3}{2}, 2]$ or $\eta = \frac{11}{10}$, we have

$$\begin{aligned} \Omega_\zeta(1, \frac{1}{2}) &= 7 \\ &\leq rmax\{\Omega_\zeta(\frac{11}{10}, \eta), \frac{\Omega_\zeta(\eta, \frac{1}{2})[1+\Omega_\zeta(1, \frac{11}{10})]}{1+\Omega_\zeta(\frac{11}{10}, \eta)}, \frac{\Omega_\zeta(1, \frac{11}{10})[1+\Omega_\zeta(\eta, \frac{1}{2})]}{1+\Omega_\zeta(\frac{11}{10}, \eta)}, \\ &\quad \frac{\Omega_\zeta(\eta, \frac{1}{2})\Omega_\zeta(1, \frac{11}{10})}{1+\Omega_\zeta(\frac{11}{10}, \eta)}, \frac{\Omega_\zeta(\eta, 1)\Omega_\zeta(\frac{11}{10}, \frac{1}{2})}{1+\Omega_\zeta(\frac{11}{10}, \eta)} \frac{\Omega_\zeta(\frac{11}{10}, \eta)\Omega_\zeta(\eta, \frac{1}{2})}{1+\Omega_\zeta(1, \frac{11}{10})+\Omega_\zeta(\eta, \frac{1}{2})}\} \\ &\quad + Lmin\{\Omega_\zeta(1, \frac{11}{10}), \Omega_\zeta(\frac{11}{10}, \frac{1}{2}), \Omega_\zeta(\eta, 1)\}, \end{aligned}$$

the minimum value of RHS is at $\eta = \frac{3}{2}$

$7 \leq \frac{100}{29} + L\frac{1}{4}$,
so the inequality (2.1.1) is satisfied for any $L \geq 17$.

Here, we note that the inequality (1.4.1) fails to hold for any $k \in [0, 1]$ at $\mu = \frac{11}{10}$, $\eta = \frac{1}{2}$, since

$$\begin{aligned} \Omega_\zeta(Tx, Ty) &= 2 \not\leq k2 \\ &= kmax\{\frac{\sigma_\zeta(\mu, T\mu)[1+\sigma_\zeta(\eta, T\eta)]}{1+\sigma_\zeta(\mu, \eta)}, \frac{\sigma_\zeta(\eta, T\eta)[1+\sigma_\zeta(\mu, T\mu)]}{1+\sigma_\zeta(\mu, \eta)}, \frac{\sigma_\zeta(\eta, T\eta)\sigma_\zeta(\mu, T\mu)}{\sigma_\zeta(\mu, \eta)}, \sigma_\zeta(\mu, \eta)\} \end{aligned}$$

Also, we note that at $\mu = \frac{11}{10}$, $\eta = \frac{1}{2}$, the condition (1.5.1) fails hold, since

$$\Omega_\zeta(Tx, Ty) = 2 \leq rmax\{\frac{\sigma_\zeta(\mu, T\mu)\sigma_\zeta(\mu, T\eta) + \sigma_\zeta(\eta, T\eta)\sigma_\zeta(\eta, T\mu)}{\sigma_\zeta(\mu, T\eta) + \sigma_\zeta(\eta, T\eta)}, \sigma_\zeta(\mu, \eta)\},$$

it follows that $r > 1$, contracting the fact that $r < 1$. Thus there does not exist $r \in [0, 1]$ such that $r < 1$.

Thus contraction condition (2.1.1) is more general than (1.4.1) and (1.5.1).

3. Applications

Let $\aleph$ be a nonempty set and $\Delta := \{(\mu, \mu) : \mu \in \aleph\}$. Let G be a directed graph such that its vertex $V(G)$ coincides with $\aleph$, and $E(G)$ contains all loops, that is, $\Delta \subseteq E(G)$.

Assume that G has no parallel edges. We can identify G with the pair $G(V(G), E(G))$. By G^{-1} we denote the conversion of the graph G , that is, the graph obtained from G by reversing the directions of edges. In 2008, Jachymski [12] opened a new direction of obtaining fixed points in metric spaces endowed with a graph. For more results on graphs we refer [4, 5, 8, 10, 11].

Recall some definitions of Chifu [7].

Definition 3.1. [7] Let $(\mathbb{N}, \Omega_\varsigma)$ be an extended rectangular B-metric space with a graph G and let S and T be a self maps on $\mathbb{N}$. Then

- (i*) The pair (T, S) is G -orbital-cyclic pair, if for any $\mu \in \mathbb{N}$,
 $(\mu, T\mu) \in E(G) \Rightarrow (T\mu, ST\mu) \in E(G)$ and $(\mu, S\mu) \in E \Rightarrow (S\mu, TS\mu) \in E(G)$.
- (ii*) If the pair (S, T) is G -orbital-cyclic pair, then
 $\mathbb{N}^T = \{\mu \in \mathbb{N} : (\mu, T\mu) \in E(G)\} \neq \phi$ and $\mathbb{N}^S = \{\mu \in \mathbb{N} : (\mu, S\mu) \in E(G)\} \neq \phi$.

We now give the existence theorem for a pair of self maps (S, T) satisfying in extended rectangular b- metric spaces equipped with a directed graph.

Theorem 3.2. Let $(\mathbb{N}, \Omega_\varsigma)$ be an extended rectangular b metric equipped with a directed graph G and $S, T : \mathbb{N} \rightarrow \mathbb{N}$ satisfying the following conditions

- (i) $\mathbb{N}^T \neq \phi$
- (ii) for all $\mu \in \mathbb{N}^T$ and $\eta \in \mathbb{N}^S$, T there exists $r \in [0, 1)$ and $L \geq 0$ such that
 $\Omega_\varsigma(S\mu, T\eta) \leq rP(\mu, \eta) + LQ(\mu, \eta)$,
 where $P(\mu, \eta)$ and $Q(\mu, \eta)$ defined as in condition (2.1.1) of Theorem 2.1
- (iii) The pair (S, T) forms a G -orbital-cyclic pair.
- (iv) For any sequence $\tilde{h}_n \subset \mathbb{N}$ with $(\tilde{h}_n, \tilde{h}_{n+1}) \in E(G)$, $\tilde{h}_0 \in \mathbb{N}$,
 $\lim_{m, n \rightarrow +\infty} \varsigma(\tilde{h}_n, \tilde{h}_m) < \frac{1}{r}$.
- (v) S or T is continuous.

Then S and T have common fixed point in $\mathbb{N}$.

Proof. Proof follows from Theorem 2.1. □

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(D M K Kiran) DEPARTMENT OF ENGINEERING MATHEMATICS, PRIME ENGINEERING COLLEGE, VISAKHAPATNAM- 531162, ANDHRA PRADESH, INDIA
E-mail address: `kiran_dmk@yahoo.com`

(M V R Kameswari) DEPARTMENT OF MATHEMATICS, GIS, GITAM(DEEMED TO BE UNIVERSITY), VISAKHAPATNAM-530045, ANDHRA PRADESH, INDIA
E-mail address, Corresponding author: `mvrkameswari@gmail.com`