

A NOTE ON GENERALIZED GROWTH ANALYSIS OF COMPOSITE ENTIRE FUNCTIONS

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Abstract. In this paper we wish to prove some results relating to the growth rates of maximum modulus and maximum terms of composition of two entire functions with their corresponding left and right factors on the basis of their generalized order (α, β) and generalized lower order (α, β) where α and β are continuous non-negative functions on $(-\infty, +\infty)$.

1. Introduction, Definitions and Notations

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be an entire function defined on $\mathbb{C}$. The maximum modulus function $M_f(r)$ and the maximum term $\mu_f(r)$ of $f = \sum_{n=0}^{+\infty} a_n z^n$ on $|z| = r$ are defined as $M_f = \max_{|z|=r} |f(z)|$ and $\mu_f(r) = \max_{n \geq 0} (|a_n| r^n)$ respectively. We use the standard notations and definitions of the theory of entire functions which are available in [10] and [11], and therefore we do not explain those in details. For $x \in [0, +\infty)$ and $k \in \mathbb{N}$ where $\mathbb{N}$ be the set of all positive integers, define iterations of the exponential and logarithmic functions as $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ and $\log^{[k]} x = \log(\log^{[k-1]} x)$, with convention that $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$, and $\exp^{[-1]} x = \log x$. Now considering this, let us recall that Juneja et al. [3] defined the (p, q) -th order and (p, q) -th lower order of an entire function, respectively, as follows:

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Definition 1.1. [3] The (p, q) -th order $\rho^{(p,q)}(f)$ and (p, q) -th lower order $\lambda^{(p,q)}(f)$ of an entire function f are defined as:

$$\frac{\rho^{(p,q)}(f)}{\lambda^{(p,q)}(f)} = \lim_{r \rightarrow +\infty} \sup \frac{\log^{[p]} M_f(r)}{\log^{[q]} r},$$

where p and q always denote positive integers with $p \geq q$.

Extending the notion (p, q) -th order, recently Shen et al. [4] introduced the new concept of $[p, q]$ - φ order of an entire function where $p \geq q$. Later on, combining the definition of (p, q) -order and $[p, q]$ - φ order, Biswas (see, e.g., [1]) redefined the (p, q) -th order of an entire function without restriction $p \geq q$.

However the above definition is very useful for measuring the growth of entire functions. If $p = l$ and $q = 1$ then we write $\rho^{(l,1)}(f) = \rho^{(l)}(f)$ and $\lambda^{(l,1)}(f) = \lambda^{(l)}(f)$ where $\rho^{(l)}(f)$ and $\lambda^{(l)}(f)$ are respectively known as generalized order and generalized lower order of entire function f . For details about generalized order one may see [5]. Also for $p = 2$ and $q = 1$, we respectively denote $\rho^{(2,1)}(f)$ and $\lambda^{(2,1)}(f)$ by $\rho(f)$ and $\lambda(f)$ which are classical growth indicators such as order and lower order of entire function f .

Now let L be a class of continuous non-negative function α on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \rightarrow +\infty$ and $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \rightarrow +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha(cx) = (1+o(1))\alpha(x)$ as $x_0 \leq x \rightarrow +\infty$ for each $c \in (0, +\infty)$, i.e., α is slowly increasing function. Clearly $L^0 \subset L$.

Further we assume that throughout the present paper $\alpha, \alpha_1, \alpha_2, \beta, \beta_1$ and β_2 always denote the functions belonging to L^0 unless otherwise specifically stated. The value

$$\varrho_{(\alpha,\beta)}[f] = \limsup_{r \rightarrow +\infty} \frac{\alpha(\log M_f(r))}{\beta(\log r)} (\alpha \in L, \beta \in L)$$

is called [8] generalized order (α, β) of an entire function f .

In order to keep accordance with Definition 1.1, here we rewrite the definition of the generalized order (α, β) and generalized lower order (α, β) of an entire function after giving a minor modification to the original definition of generalized order (α, β) of an entire function (e.g. see, [8]).

Definition 1.2. The generalized order (α, β) and generalized lower order (α, β) denoted by $\rho_{(\alpha,\beta)}[f]$ and $\lambda_{(\alpha,\beta)}[f]$ respectively of an entire function f are defined as:

$$\frac{\rho_{(\alpha,\beta)}[f]}{\lambda_{(\alpha,\beta)}[f]} = \lim_{r \rightarrow +\infty} \sup \frac{\alpha(M_f(r))}{\beta(r)}.$$

Definition 1.1 is a special case of Definition 1.2 for $\alpha(r) = \log^{[p]} r$ and $\beta(r) = \log^{[q]} r$.

Since for $0 \leq r < R$,

$$\mu_f(r) \leq M_f(r) \leq \frac{R}{R-r} \mu_f(R) \{cf. [7]\},$$

it is easy to see that

$$\frac{\rho_{(\alpha,\beta)}[f]}{\lambda_{(\alpha,\beta)}[f]} = \lim_{r \rightarrow +\infty} \sup \frac{\alpha(\mu_f(r))}{\beta(r)}.$$

In this paper we wish to prove some results relating to the growth rates of maximum modulus and maximum terms of composition of two entire functions with their corresponding left and right factors on the basis of their generalized order (α, β) and generalized lower order (α, β) which considerably extend some earlier results of [9].

2. Known Results

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [2] *Let f and g be any two entire functions with $g(0) = 0$. Also let B satisfy $0 < B < 1$ and $c(B) = \frac{(1-B)^2}{4B}$. Then for all sufficiently large values of r ,*

$$M_f(c(B)M_g(Br)) \leq M_{f \circ g}(r) \leq M_f(M_g(r)).$$

In addition if $B = \frac{1}{2}$, then for all sufficiently large values of r ,

$$M_{f \circ g}(r) \geq M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right)\right).$$

Lemma 2.2. [6] *Let f and g be entire functions. Then for every $\delta > 1$ and $0 < r < R$,*

$$\mu_{f \circ g}(r) \leq \frac{\delta}{\delta - 1} \mu_f\left(\frac{\delta R}{R - r} \mu_g(R)\right).$$

Lemma 2.3. [6] *If f and g are any two entire functions. Then for all sufficiently large values of r ,*

$$\mu_{f \circ g}(r) \geq \frac{1}{2} \mu_f\left(\frac{1}{16} \mu_g\left(\frac{r}{4}\right)\right).$$

3. Main Results

In this section we present the main results of the paper.

Theorem 3.1. *Let f and g be any two entire functions such that $0 < \lambda_{(\alpha_1, \beta_1)}[f] \leq \rho_{(\alpha_1, \beta_1)}[f] < +\infty$ and $\lambda_{(\alpha_2, \beta_2)}[g] > 0$. Also let γ be a positive continuous function on $[0, +\infty)$ increasing to $+\infty$ and $A \geq 0$ be any number.*

(i) If $\beta_1(\alpha_2^{-1}(\log r)) \geq r$ and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = 0$, then

$$\lim_{r \rightarrow +\infty} \frac{\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r)))}{\{\alpha_1(M_f(\beta_1^{-1}(\gamma(r))))\}^{1+A}} = +\infty \text{ and}$$

(ii) if either $\beta_1(r) = B(\alpha_2(r))$ where B is any positive constant and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = 0$ or

$\beta_1(\alpha_2^{-1}(r)) \in L^0$ and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))} = 0$, then

$$\lim_{r \rightarrow +\infty} \frac{\exp(\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))))}{\{\alpha_1(M_f(\beta_1^{-1}(\gamma(r))))\}^{1+A}} = +\infty.$$

Proof. From the definition of $\rho_{(\alpha_1, \beta_1)}[f]$, it follows for all sufficiently large values of r that

$$\alpha_1(M_f(\beta_1^{-1}(\gamma(r)))) \leq (\rho_{(\alpha_1, \beta_1)}[f] + \varepsilon)\gamma(r). \quad (3.1)$$

Further it follows from Lemma 2.1 and for all sufficiently large values r that

$$\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))) \geq \alpha_1\left(M_f\left(\frac{1}{8}M_g\left(\frac{\beta_2^{-1}(\log r)}{2}\right)\right)\right)$$

$$\begin{aligned} i.e., \alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))) &\geq \\ (1 + o(1))(\lambda_{(\alpha_1, \beta_1)}[f] - \varepsilon)\beta_1\left(M_g\left(\frac{\beta_2^{-1}(\log r)}{2}\right)\right). \end{aligned} \quad (3.2)$$

Case I. Let $\beta_1(\alpha_2^{-1}(\log r)) \geq r$. Now from (3.2) it follows for all sufficiently large values of r that

$$\begin{aligned} \alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))) &\geq \\ (1 + o(1))(\lambda_{(\alpha_1, \beta_1)}[f] - \varepsilon)\beta_1\left(\alpha_2^{-1}\left(M_g\left(\frac{\beta_2^{-1}(\log r)}{2}\right)\right)\right) \end{aligned} \quad (3.3)$$

$$\begin{aligned} i.e., \alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))) &\geq \\ (1 + o(1))(\lambda_{(\alpha_1, \beta_1)}[f] - \varepsilon)\beta_1(\alpha_2^{-1}(\log r^{(1+o(1))(\lambda_{(\alpha_2, \beta_2)}[g] - \varepsilon)})). \end{aligned} \quad (3.4)$$

Case II. Let $\beta_1(r) = B\alpha_2(r)$ where B is any positive constant. Then from (3.2) it follows for all sufficiently large values of r that

$$\begin{aligned} \alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))) &\geq \\ B(1 + o(1))(\lambda_{(\alpha_1, \beta_1)}[f] - \varepsilon)\alpha_2\left(M_g\left(\frac{\beta_2^{-1}(\log r)}{2}\right)\right) \end{aligned} \quad (3.5)$$

$$\begin{aligned} i.e., \alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))) &\geq \log r^{B(1+o(1))(\lambda_{(\alpha_1, \beta_1)}[f] - \varepsilon)(\lambda_{(\alpha_2, \beta_2)}[g] - \varepsilon)} \\ i.e., \exp(\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r)))) &\geq r^{B(1+o(1))(\lambda_{(\alpha_1, \beta_1)}[f] - \varepsilon)(\lambda_{(\alpha_2, \beta_2)}[g] - \varepsilon)}. \end{aligned} \quad (3.6)$$

Case III. Let $\beta_1(\alpha_2^{-1}(r)) \in L^0$. Then from (3.3) we obtain for all sufficiently large values of r that

$$\begin{aligned} \alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))) &\geq (1 + o(1))(\lambda_{(\alpha_1, \beta_1)}[f] - \varepsilon)\beta_1(\alpha_2^{-1}(\log r)) \\ i.e., \exp(\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r)))) &\geq \\ \exp((1 + o(1))(\lambda_{(\alpha_1, \beta_1)}[f] - \varepsilon)\beta_1(\alpha_2^{-1}(\log r))). \end{aligned} \quad (3.7)$$

Now combining (3.1) and (3.4) of Case I it follows for all sufficiently large values of r that

$$\begin{aligned} \frac{\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r)))}{\{\alpha_1(M_f(\beta_1^{-1}(\gamma(r))))\}^{1+A}} &\geq \\ \frac{(1 + o(1))(\lambda_{(\alpha_1, \beta_1)}[f] - \varepsilon)\beta_1(\alpha_2^{-1}(\log r^{(1+o(1))(\lambda_{(\alpha_2, \beta_2)}[g] - \varepsilon)}))}{(\rho_{(\alpha_1, \beta_1)}[f] + \varepsilon)^{1+A}\{\gamma(r)\}^{1+A}}. \end{aligned}$$

Since $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = 0$ and $\beta_1(\alpha_2^{-1}(\log r)) \geq r$, therefore $\frac{\beta_1(\alpha_2^{-1}(\log r^{(1+o(1))(\lambda_{(\alpha_2, \beta_2)}[g] - \varepsilon)}))}{\{\gamma(r)\}^{1+A}} \rightarrow +\infty$ as $r \rightarrow +\infty$, then from above it follows that

$$\liminf_{r \rightarrow +\infty} \frac{\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r)))}{\{\alpha_1(M_f(\beta_1^{-1}(\gamma(r))))\}^{1+A}} = +\infty,$$

from which the first part of the theorem follows.

Again combining (3.1) and (3.6) of Case II, we get for all sufficiently large values of r that

$$\frac{\exp(\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))))}{\{\alpha_1(M_f(\beta_1^{-1}(\gamma(r))))\}^{1+A}} \geq \frac{r^{B(1+o(1))(\lambda_{(\alpha_1, \beta_1)}[f] - \varepsilon)(\lambda_{(\alpha_2, \beta_2)}[g] - \varepsilon)}}{(\rho_{(\alpha_1, \beta_1)}[f] + \varepsilon)^{1+A}\{\gamma(r)\}^{1+A}}.$$

As $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = 0$, so $\frac{r^{B(1+o(1))(\lambda_{(\alpha_1, \beta_1)}[f] - \varepsilon)(\lambda_{(\alpha_2, \beta_2)}[g] - \varepsilon)}}{\{\gamma(r)\}^{1+A}} \rightarrow +\infty$ as $r \rightarrow +\infty$.

Thus it follows from above that

$$\lim_{r \rightarrow +\infty} \frac{\exp(\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))))}{\{\alpha_1(M_f(\beta_1^{-1}(\gamma(r))))\}^{1+A}} = +\infty. \quad (3.8)$$

Further combining (3.1) and (3.7) of Case III it follows for all sufficiently large values of r that

$$\frac{\exp(\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))))}{\{\alpha_1(M_f(\beta_1^{-1}(\gamma(r))))\}^{1+A}} \geq \frac{\exp(1+o(1))(\lambda_{(\alpha_1, \beta_1)}[f] - \varepsilon)\beta_1(\alpha_2^{-1}(\log r))}{(\rho_{(\alpha_1, \beta_1)}[f] + \varepsilon)^{1+A}\{\gamma(r)\}^{1+A}}.$$

Since $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))} = 0$, so $\frac{\exp(1+o(1))(\lambda_{(\alpha_1, \beta_1)}[f] - \varepsilon)\beta_1(\alpha_2^{-1}(\log r))}{\{\gamma(r)\}^{1+A}} \rightarrow +\infty$ as $r \rightarrow +\infty$. Thus from above we obtain that

$$\lim_{r \rightarrow +\infty} \frac{\exp(\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))))}{\{\alpha_1(M_f(\beta_1^{-1}(\gamma(r))))\}^{1+A}} = +\infty. \quad (3.9)$$

Hence the second part of the theorem follows from (3.8) and (3.9).

Thus the theorem follows. $\square$

Remark 3.2. Theorem 3.1 is still valid with “limit superior” instead of “limit” if we replace the condition “ $0 < \lambda_{(\alpha_1, \beta_1)}[f] \leq \rho_{(\alpha_1, \beta_1)}[f] < +\infty$ ” by “ $0 < \lambda_{(\alpha_1, \beta_1)}[f] < +\infty$ ”.

In the line of Theorem 3.1 one may state the following theorem without proof:

Theorem 3.3. Let f and g be any two entire functions such that $0 < \lambda_{(\alpha_2, \beta_2)}[g] \leq \rho_{(\alpha_2, \beta_2)}[g] < +\infty$ and $\lambda_{(\alpha_1, \beta_1)}[f] > 0$. Also let γ be a positive continuous function on $[0, +\infty)$ increasing to $+\infty$. For any number $A \geq 0$,
(i) if $\beta_1(\alpha_2^{-1}(\log r)) \geq r$ and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = 0$, then

$$\lim_{r \rightarrow +\infty} \frac{\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r)))}{\{\alpha_2(M_g(\beta_2^{-1}(\gamma(r))))\}^{1+A}} = +\infty \text{ and}$$

(ii) if either $\beta_1(r) = B(\alpha_2(r))$ where B is any positive constant and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = 0$ or $\beta_1(\alpha_2^{-1}(r)) \in L^0$ and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))} = 0$, then

$$\lim_{r \rightarrow +\infty} \frac{\exp(\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))))}{\{\alpha_2(M_g(\beta_2^{-1}(\gamma(r))))\}^{1+A}} = +\infty,$$

Remark 3.4. In Theorem 3.3 if we take the condition $\lambda_{(\alpha_2, \beta_2)}[g] < +\infty$ instead of $0 < \lambda_{(\alpha_2, \beta_2)}[g] \leq \rho_{(\alpha_2, \beta_2)}[g] < +\infty$, then also Theorem 3.3 remains true with “limit superior” in place of “limit”.

In the line of Theorem 3.1 and with the help of Lemma 2.3, one can easily proof the following theorem and therefore its proof is omitted:

Theorem 3.5. Let f and g be any two entire functions such that $0 < \lambda_{(\alpha_1, \beta_1)}[f] \leq \rho_{(\alpha_1, \beta_1)}[f] < +\infty$ and $\lambda_{(\alpha_2, \beta_2)}[g] > 0$. Also let γ be a positive continuous function on

$[0, +\infty)$ increasing to $+\infty$ and $A \geq 0$ be any number.

(i) If $\beta_1(\alpha_2^{-1}(\log r)) \geq r$ and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = 0$, then

$$\lim_{r \rightarrow +\infty} \frac{\alpha_1(\mu_{f \circ g}(\beta_2^{-1}(\log r)))}{\{\alpha_1(\mu_f(\beta_1^{-1}(\gamma(r))))\}^{1+A}} = +\infty \text{ and}$$

(ii) if either $\beta_1(r) = B(\alpha_2(r))$ where B is any positive constant and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = 0$ or $\beta_1(\alpha_2^{-1}(r)) \in L^0$ and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))} = 0$, then

$$\lim_{r \rightarrow +\infty} \frac{\exp(\alpha_1(\mu_{f \circ g}(\beta_2^{-1}(\log r))))}{\{\alpha_1(\mu_f(\beta_1^{-1}(\gamma(r))))\}^{1+A}} = +\infty.$$

Remark 3.6. Theorem 3.5 is still valid with “limit superior” instead of “limit” if we replace the condition “ $0 < \lambda_{(\alpha_1, \beta_1)}[f] \leq \rho_{(\alpha_1, \beta_1)}[f] < +\infty$ ” by “ $0 < \lambda_{(\alpha_1, \beta_1)}[f] < +\infty$ ”.

Theorem 3.7. Let f and g be any two entire functions such that $0 < \lambda_{(\alpha_2, \beta_2)}[g] \leq \rho_{(\alpha_2, \beta_2)}[g] < +\infty$ and $\lambda_{(\alpha_1, \beta_1)}[f] > 0$. Also let γ be a positive continuous function on $[0, +\infty)$ increasing to $+\infty$. For any number $A \geq 0$,

(i) if $\beta_1(\alpha_2^{-1}(\log r)) \geq r$ and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = 0$, then

$$\lim_{r \rightarrow +\infty} \frac{\alpha_1(\mu_{f \circ g}(\beta_2^{-1}(\log r)))}{\{\alpha_2(\mu_g(\beta_2^{-1}(\gamma(r))))\}^{1+A}} = +\infty \text{ and}$$

(ii) if either $\beta_1(r) = B(\alpha_2(r))$ where B is any positive constant and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = 0$ or $\beta_1(\alpha_2^{-1}(r)) \in L^0$ and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))} = 0$, then

$$\lim_{r \rightarrow +\infty} \frac{\exp(\alpha_1(\mu_{f \circ g}(\beta_2^{-1}(\log r))))}{\{\alpha_2(\mu_g(\beta_2^{-1}(\gamma(r))))\}^{1+A}} = +\infty.$$

The proof of the above theorem can be carried out in the line of Theorem 3.3 and with the help of Lemma 2.3 and therefore its proof is omitted.

Remark 3.8. In Theorem 3.7 if we take the condition $\lambda_{(\alpha_2, \beta_2)}[g] < +\infty$ instead of $0 < \lambda_{(\alpha_2, \beta_2)}[g] \leq \rho_{(\alpha_2, \beta_2)}[g] < +\infty$, then also Theorem 3.7 remains true with “limit superior” in place of “limit”.

Theorem 3.9. Let f and g be any two entire functions such that $0 < \lambda_{(\alpha_1, \beta_1)}[f] \leq \rho_{(\alpha_1, \beta_1)}[f] < +\infty$ and $\rho_{(\alpha_2, \beta_2)}[g] < +\infty$. Also let γ be a positive continuous function on $[0, +\infty)$ increasing to $+\infty$ and $A \geq 0$ be any number.

(i) If $\beta_1(\alpha_2^{-1}(\log r)) \leq r$ and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = +\infty$, then

$$\lim_{r \rightarrow +\infty} \frac{\{\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r)))\}^{1+A}}{\alpha_1(M_f(\beta_1^{-1}(\gamma(r))))} = 0 \text{ and}$$

(ii) if either $\beta_1(r) = B\alpha_2(r)$ where B is any positive constant and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = +\infty$ or $\beta_1(\alpha_2^{-1}(r)) \in L^0$ and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))} = +\infty$, then

$$\lim_{r \rightarrow +\infty} \frac{\{\exp(\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))))\}^{1+A}}{\alpha_1(M_f(\beta_1^{-1}(\gamma(r))))} = 0.$$

Proof. From the definition of $\lambda_{(\alpha_1, \beta_1)}[f]$, we get for all sufficiently large values of r that

$$\alpha_1(M_f(\beta_1^{-1}(\gamma(r)))) \geq (\lambda_{(\alpha_1, \beta_1)}[f] - \varepsilon)\gamma(r). \quad (3.10)$$

Further it follows from Lemma 2.1 for all sufficiently large values of r that

$$\begin{aligned} \alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))) &\leq \\ &(\rho_{(\alpha_1, \beta_1)}[f] + \varepsilon)\beta_1(M_g(\beta_2^{-1}(\log r))). \end{aligned} \quad (3.11)$$

Now the following cases may arise :

Case I. Let $\beta_1(\alpha_2^{-1}(\log r)) \leq r$. Now we get from (3.11) for all sufficiently large values of r that

$$\begin{aligned} \alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))) &\leq \\ &(\rho_{(\alpha_1, \beta_1)}[f] + \varepsilon)\beta_1(\alpha_2^{-1}(\alpha_2(M_g(\beta_2^{-1}(\log r))))) \end{aligned} \quad (3.12)$$

$$\begin{aligned} i.e., \alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))) &\leq \\ &(\rho_{(\alpha_1, \beta_1)}[f] + \varepsilon)\beta_1(\alpha_2^{-1}(\log r^{(\rho_{(\alpha_2, \beta_2)}[g] + \varepsilon)})). \end{aligned} \quad (3.13)$$

Case II. Let $\beta_1(r) = B\alpha_2(r)$ where B is any positive constant. Then we have from (3.11) for all sufficiently large values of r that

$$\begin{aligned} \alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))) &\leq B(\rho_{(\alpha_1, \beta_1)}[f] + \varepsilon)\alpha_2(M_g(\beta_2^{-1}(\log r))) \\ i.e., \alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))) &\leq B(\rho_{(\alpha_1, \beta_1)}[f] + \varepsilon)(\rho_{(\alpha_2, \beta_2)}[g] + \varepsilon) \log r \\ i.e., \exp(\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r)))) &\leq r^{B(\rho_{(\alpha_1, \beta_1)}[f] + \varepsilon)(\rho_{(\alpha_2, \beta_2)}[g] + \varepsilon)}. \end{aligned} \quad (3.14)$$

Case III. Let $\beta_1(\alpha_2^{-1}(r)) \in L^0$ and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))} = +\infty$. Then we have from (3.12) for all sufficiently large values of r that

$$\begin{aligned} \alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))) &\leq (1 + o(1))(\rho_{(\alpha_1, \beta_1)}[f] + \varepsilon)\beta_1(\alpha_2^{-1}(\log r)) \\ i.e., \exp(\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r)))) &\leq \\ &\exp((1 + o(1))(\rho_{(\alpha_1, \beta_1)}[f] + \varepsilon)\beta_1(\alpha_2^{-1}(\log r))). \end{aligned} \quad (3.15)$$

Now when $\beta_1(\alpha_2^{-1}(\log r)) \leq r$ and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = +\infty$, we obtain from (3.10) and (3.13) of Case I for all sufficiently large values of r that

$$\begin{aligned} \frac{\{\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r)))\}^{1+A}}{\alpha_1(M_f(\beta_1^{-1}(\gamma(r))))} &\leq \\ &\frac{(\rho_{(\alpha_1, \beta_1)}[f] + \varepsilon)^{1+A}[\beta_1(\alpha_2^{-1}(\log r^{(\rho_{(\alpha_2, \beta_2)}[g] + \varepsilon)}))]^{1+A}}{(\lambda_{(\alpha_1, \beta_1)}[f] - \varepsilon)\gamma(r)} \\ i.e., \limsup_{r \rightarrow +\infty} \frac{\{\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r)))\}^{1+A}}{\alpha_1(M_f(\beta_1^{-1}(\gamma(r))))} &= 0, \end{aligned}$$

This proves the first part of the theorem.

Again combining (3.10) and (3.14) of Case II, we get for all sufficiently large values of r that

$$\frac{\{\exp(\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))))\}^{1+A}}{\alpha_1(M_f(\beta_1^{-1}(\gamma(r))))} \leq \frac{r^{B(\rho_{(\alpha_1, \beta_1)}[f] + \varepsilon)(\rho_{(\alpha_2, \beta_2)}[g] + \varepsilon)(1+A)}}{(\lambda_{(\alpha_1, \beta_1)}[f] - \varepsilon)\gamma(r)}$$

As $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = +\infty$, so $\frac{r^{B(\rho_{(\alpha_1, \beta_1)}[f] + \varepsilon)(\rho_{(\alpha_2, \beta_2)}[g] + \varepsilon)(1+A)}}{\gamma(r)} \rightarrow 0$ as $r \rightarrow +\infty$.

Thus it follows from above that

$$\lim_{r \rightarrow +\infty} \frac{\{\exp(\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))))\}^{1+A}}{\alpha_1(M_f(\beta_1^{-1}(\gamma(r))))} = 0. \quad (3.16)$$

Further combining (3.10) and (3.15) of Case III it follows for all sufficiently large values of r that

$$\frac{\{\exp(\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))))\}^{1+A}}{\alpha_1(M_f(\beta_1^{-1}(\gamma(r))))} \leq \frac{[\exp((1+o(1))(\rho_{(\alpha_1, \beta_1)}[f] + \varepsilon)\beta_1(\alpha_2^{-1}(\log r)))]^{1+A}}{(\lambda_{(\alpha_1, \beta_1)}[f] - \varepsilon)\gamma(r)}.$$

Since $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))} = +\infty$, so $\frac{[\exp((1+o(1))(\rho_{(\alpha_1, \beta_1)}[f] + \varepsilon)\beta_1(\alpha_2^{-1}(\log r)))]^{1+A}}{\gamma(r)} \rightarrow 0$ as $r \rightarrow +\infty$. Thus from above we obtain that

$$\lim_{r \rightarrow +\infty} \frac{\{\exp(\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))))\}^{1+A}}{\alpha_1(M_f(\beta_1^{-1}(\gamma(r))))} = 0 \quad (3.17)$$

Hence the second part of the theorem follows from (3.16) and (3.17).

Thus the theorem follows. $\square$

Remark 3.10. In Theorem 3.9 if we take the condition $\rho_{(\alpha_1, \beta_1)}[f] > 0$ instead of $0 < \lambda_{(\alpha_1, \beta_1)}[f] \leq \rho_{(\alpha_1, \beta_1)}[f] < +\infty$, the theorem remains true with “limit inferior” in place of “limit”.

Theorem 3.11. Let f and g be any two entire functions such that $0 < \lambda_{(\alpha_2, \beta_2)}[g] \leq \rho_{(\alpha_2, \beta_2)}[g] < +\infty$ and $\rho_{(\alpha_1, \beta_1)}[f] < +\infty$. Also let γ be a positive continuous function on $[0, +\infty)$ increasing to $+\infty$ and $A \geq 0$ be any number.

(i) If $\beta_1(\alpha_2^{-1}(\log r)) \leq r$ and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = +\infty$, then

$$\lim_{r \rightarrow +\infty} \frac{\{\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r)))\}^{1+A}}{\alpha_2(M_g(\beta_2^{-1}(\gamma(r))))} = 0 \text{ and}$$

(ii) if either $\beta_1(r) = B\alpha_2(r)$ where B is any positive constant and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = +\infty$ or $\beta_1(\alpha_2^{-1}(r)) \in L^0$ and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))} = +\infty$, then

$$\lim_{r \rightarrow +\infty} \frac{\{\exp(\alpha_1(M_{f \circ g}(\beta_2^{-1}(\log r))))\}^{1+A}}{\alpha_2(M_g(\beta_2^{-1}(\gamma(r))))} = 0.$$

The proof of Theorem 3.11 would run parallel to that of Theorem 3.9. We omit the details.

Remark 3.12. In Theorem 3.11, if we take the condition $\rho_{(\alpha_2, \beta_2)}[g] > 0$ instead of $0 < \lambda_{(\alpha_2, \beta_2)}[g] \leq \rho_{(\alpha_2, \beta_2)}[g] < +\infty$, the theorem remains true with “limit replaced by limit inferior”.

Theorem 3.13. Let f and g be any two entire functions such that $0 < \lambda_{(\alpha_1, \beta_1)}[f] \leq \rho_{(\alpha_1, \beta_1)}[f] < +\infty$ and $\rho_{(\alpha_2, \beta_2)}[g] < +\infty$. Also let γ be a positive continuous function on

$[0, +\infty)$ increasing to $+\infty$ and $A \geq 0$ be any number.

(i) If $\beta_1(\alpha_2^{-1}(\log r)) \leq r$ and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = +\infty$, then

$$\lim_{r \rightarrow +\infty} \frac{\{\alpha_1(\mu_{f \circ g}(\beta_2^{-1}(\log r)))\}^{1+A}}{\alpha_1(\mu_f(\beta_1^{-1}(\gamma(r))))} = 0 \text{ and}$$

(ii) if either $\beta_1(r) = B\alpha_2(r)$ where B is any positive constant and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = +\infty$ or $\beta_1(\alpha_2^{-1}(r)) \in L^0$ and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))} = +\infty$, then

$$\lim_{r \rightarrow +\infty} \frac{\{\exp(\alpha_1(\mu_{f \circ g}(\beta_2^{-1}(\log r))))\}^{1+A}}{\alpha_1(\mu_f(\beta_1^{-1}(\gamma(r))))} = 0.$$

Theorem 3.14. Let f and g be any two entire functions such that $0 < \lambda_{(\alpha_2, \beta_2)}[g] \leq \rho_{(\alpha_2, \beta_2)}[g] < +\infty$ and $\rho_{(\alpha_1, \beta_1)}[f] < +\infty$. Also let γ be a positive continuous function on $[0, +\infty)$ increasing to $+\infty$ and $A \geq 0$ be any number.

(i) If $\beta_1(\alpha_2^{-1}(\log r)) \leq r$ and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = +\infty$, then

$$\lim_{r \rightarrow +\infty} \frac{\{\alpha_1(\mu_{f \circ g}(\beta_2^{-1}(\log r)))\}^{1+A}}{\alpha_2(\mu_g(\beta_2^{-1}(\gamma(r))))} = 0 \text{ and}$$

(ii) if either $\beta_1(r) = B\alpha_2(r)$ where B is any positive constant and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = +\infty$ or $\beta_1(\alpha_2^{-1}(r)) \in L^0$ and $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))} = +\infty$, then

$$\lim_{r \rightarrow +\infty} \frac{\{\exp(\alpha_1(\mu_{f \circ g}(\beta_2^{-1}(\log r))))\}^{1+A}}{\alpha_2(\mu_g(\beta_2^{-1}(\gamma(r))))} = 0.$$

A similar arguments in the proofs of Theorem 3.9 and Theorem 3.11 respectively will establish the results in Theorem 3.13 and Theorem 3.14 by the help of Lemma 2.2. Therefore we omit the details.

Remark 3.15. In Theorem 3.13 if we take the condition $\rho_{(\alpha_1, \beta_1)}[f] > 0$ instead of $0 < \lambda_{(\alpha_1, \beta_1)}[f] \leq \rho_{(\alpha_1, \beta_1)}[f] < +\infty$, the theorem remains true with “limit inferior” in place of “limit”.

Remark 3.16. In Theorem 3.14, if we take the condition $\rho_{(\alpha_2, \beta_2)}[g] > 0$ instead of $0 < \lambda_{(\alpha_2, \beta_2)}[g] \leq \rho_{(\alpha_2, \beta_2)}[g] < +\infty$, the theorem remains true with “limit replaced by limit inferior”.

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