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J-fusion frame and its application in Krein spaces

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Abstract. In this article we introduce the notion of J-fusion frame for a Krein space $\mathbb{K}$. We prove some basic results regarding J-fusion frame. We also relate this concept with fusion frame for Hilbert spaces and also with J-frame for Krein spaces. Further, we prove that J-fusion frame is actually a generalization of J-frames and we also explore some basic properties of J-fusion frame operator.

1. Introduction

The frame theory was introduced by Duffin and Schaeffer [1] in the year 1952. But after the work of Daubechies et al. [2], frame theory developed rapidly. Now a days frame theory has major applications in signal processing, image processing, data compression, sampling theory and many more. One of the most important applications of frame theory is to calculate the effect of losses in packet-based communication system and in data transmission. In order to tackle this problem fusion frame theory was introduced. The idea behind fusion frame is to construct local frames and add them together to get the global frame. Fornasier [8] used this idea to quasi-orthogonal subspaces. Casazza et al. [5] formulated a general method to introduce fusion frame in Hilbert spaces.

Since fusion frame in Hilbert space has such huge applications, so it is a natural demand to extend these ideas in Banach space frame theory and also in Krein space frame theory. Some works had already been done in this direction [9, 10]. In this article we are interested to extend the idea of fusion frame in Krein spaces. Krein space has some interesting applications in modern analysis. The theory of frame for Krein spaces can be found in [11, 12, 13, 15, 16]. Acosta-Humánez et al. [14] first defined fusion

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frame in Krein spaces. They found a correspondence between fusion frame in Hilbert spaces and fusion frame in Krein spaces. But their definition involves fundamental symmetry in Krein spaces which is not unique.

In this article we define fusion frame for Krein spaces in a more geometric setting motivated by the work of Giribet et al. [12]. We also relate this concept with fusion frame for Hilbert spaces and also with J-frame for Krein spaces. Further, we prove that J-fusion frame is a generalization of J-frame and we also explore some properties of J-fusion frame operator.

2. Preliminaries and some basic definitions

In this section we briefly recall some basic notations, definitions and some important properties useful to our further study. For more detailed information we refer [3, 4, 5, 7, 11, 12, 15].

2.1. Hilbert space frame theory. A family of vectors $\{f_n\}_{n\in\mathbb{N}}$ is said to be a *frame* for a Hilbert space $(\mathbb{H}, \langle\cdot,\cdot\rangle)$, if there exist positive real numbers A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \tag{2.1}$$

for all $f \in \mathbb{H}$, where A, B are respectively known as the *lower and upper frame bounds*. If $A = B$, then the frame is known as A-*tight frame* while for $A = B = 1$, the frame is known as *Parseval frame*.

Let $\{f_n\}_{n\in\mathbb{N}}$ be a frame for the Hilbert space H and $\{e_n\}_{n\in\mathbb{N}}$ be the natural orthonormal basis of $\ell_2(\mathbb{N})$. An operator $T : \mathbb{H} \to \ell_2(\mathbb{N})$ defined by $T(f) = \sum_{n\in\mathbb{N}}\langle f, f_n\rangle e_n$, for all $f \in \mathbb{H}$ is known as *analysis operator* and its adjoint operator defined by $T^*(e_n) = f_n$ is known as *synthesis operator* for the frame $\{f_n\}_{n\in\mathbb{N}}$. The operator $S\ (= TT^*) : \mathbb{H} \to \mathbb{H}$ given by $S(f) = \sum_{n\in\mathbb{N}}\langle f, f_n\rangle f_n$, for all $f \in \mathbb{H}$ is called the *frame operator*. A mapping $G : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$ defined as $G = T^*T$ is known as the *Gramian operator*. It is clear that S is a selfadjoint, positive and invertible operator and $A.I \leq S \leq B.I$.

Definition 2.1. Let $\mathbb{H}$ be a Hilbert space. A sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be a *frame sequence* in $\mathbb{H}$ if it is a frame for the Hilbert space $\overline{span}\{f_n : n \in \mathbb{N}\}$.

2.2. On Krein spaces. An abstract space $\mathbb{K}$ that satisfies the following is called a Krein space.

(i) $\mathbb{K}$ is a linear space over the field F, where F is either $\mathbb{R}$ or $\mathbb{C}$.
(ii) there exists a bilinear form $[\cdot,\cdot] \in F$ on $\mathbb{K}$ such that

$$[y, x] = \overline{[x, y]}$$
$$[ax + by, z] = a[x, z] + b[y, z]$$

for any $x, y, z \in \mathbb{K}$, $a, b \in F$, where $\overline{[\cdot,\cdot]}$ denotes the complex conjugation.

(iii) The vector space $\mathbb{K}$ admits a canonical decomposition $\mathbb{K} = \mathbb{K}^+[\dot{+}]\mathbb{K}^-$ such that $(\mathbb{K}^+, [\cdot,\cdot])$ and $(\mathbb{K}^-, -[\cdot,\cdot])$ are respectively the Hilbert spaces relative to the norms $\|x\| = [x,x]^{\frac{1}{2}}(x \in \mathbb{K}^+)$ and $\|x\| = (-[x,x])^{\frac{1}{2}}(x \in \mathbb{K}^-)$.

Now, every canonical decomposition of $\mathbb{K}$ generates two mutually complementary projectors P_+ and P_- ($P_+ + P_- = I$, the identity operator on $\mathbb{K}$) mapping $\mathbb{K}$ onto $\mathbb{K}^+$ and $\mathbb{K}^-$ respectively. Thus, for any $x \in \mathbb{K}$, we have $P_\pm = x^\pm$, where $x^+ \in \mathbb{K}^+$ and $x^- \in \mathbb{K}^-$. The projectors P_+ and P_- are called canonical projectors. The linear operator $J : \mathbb{K} \to \mathbb{K}$ defined by the formula $J = P_+ - P_-$ is called the canonical symmetry of the Krein space $\mathbb{K}$. The J-metric is defined by the formula $[x,y]_J = [x, Jy]$, where $x, y \in \mathbb{K}$. The vector space $\mathbb{K}$ associated with the J-metric is a Hilbert space, called the *associated Hilbert space* of the Krein space $\mathbb{K}$.

Definition 2.2. Let M be a subspace of a Krein space $\mathbb{K}$, then M is said to be *projectively complete* if $\mathbb{K} = M + M^{[\perp]}$.

The *J-adjoint* of an operator T in Krein spaces, denoted by $T^\#$, satisfies $[T(x), y] = [x, T^\#(y)]$.

Definition 2.3. A linear operator on a Krein space $\mathbb{K}$ is said to be *J-selfadjoint* if $T = T^\#$.

A J-selfadjoint projection is called a *J-projection*.

Definition 2.4. [18] A subspace M is said to be *regular* if it is the range of a J-projection.

Every regular subspace is closed. If M is regular with J-projection Q, then its J-orthocomplement $M^{[\perp]}$ with J-projection $(I - Q)$ is also regular.

Let $M(\neq \{0\})$ be a subspace of $\mathbb{K}$. Then the operator G_M defined by $G_M = \pi_M J|_M$ is the Gram operator of M, here π_M is the orthogonal projection from $\mathbb{K}$ onto M (in Hilbert space sense).

Theorem 2.5. [15] *For a subspace $M(\neq \{0\}) \subset \mathbb{K}$ with a Gram operator G_M the following are equivalent:*
(i) M is projectively complete.
(ii) M is regular.

Definition 2.6. Let $(\mathbb{K}, [\cdot,\cdot], J)$ be a Krein space. Then a collection of vectors $\mathbb{F} = \{f_n : n \in \mathbb{N}\}$ is said to be a *Bessel sequence of vectors* in $\mathbb{K}$ if it is a Bessel sequence in the associated Hilbert space $(\mathbb{K}, [\cdot,\cdot]_J)$ *i.e.* $\sum_{n\in\mathbb{N}} |[f, f_n]_J|^2 < \infty$, for all $f \in \mathbb{K}$.

2.3. J-frames in Krein Spaces. Let $(\mathbb{K}, [\cdot,\cdot], J)$ be a Krein space. Suppose $\mathbb{F} = \{f_n : n \in \mathbb{N}\}$ is a Bessel sequence in $\mathbb{K}$ and $T \in L(\ell_2(\mathbb{N}), \mathbb{K})$ $\big(\ell_2(\mathbb{N}) := \{(c_n) : \sum_{n\in\mathbb{N}} |c_n|^2 < \infty\}\big)$ is the synthesis operator for the Bessel sequence $\mathbb{F}$.

Let $I_+ = \{i \in \mathbb{N} : [f_i, f_i] \geq 0\}$ and $I_- = \{i \in \mathbb{N} : [f_i, f_i] < 0\}$, then $\ell_2(\mathbb{N}) = \ell_2(I_+) \oplus \ell_2(I_-)$. Also let $P_\pm$ denote the orthogonal projection of $\ell_2(\mathbb{N})$ onto $\ell_2(I_\pm)$. Let $T_\pm = TP_\pm$, $M_\pm = \overline{span}\{f_i : i \in I_\pm\}$ then we have $R(T) = R(T_+) + R(T_-)$, where $R(T)$ represents the range of the operator T.

Definition 2.7. [12] A Bessel sequence $\mathbb{F}$ is said to be a *J-frame* for $\mathbb{K}$ if $R(T_+)$ is a maximal uniformly J-positive subspace of $\mathbb{K}$ and $R(T_-)$ is a maximal uniformly J-negative subspace of $\mathbb{K}$.

2.4. On Hilbert space fusion frame. Here we briefly mention some definitions and results from Hilbert space fusion frame theory. Let π_M be the orthogonal projection from the Hilbert space $\mathbb{H}$ onto the subspace M of $\mathbb{H}$. Then $R(\pi_M) = M$ and the null space *i.e.* $N(\pi_M) = M^\perp$.

Definition 2.8. Let I be some index set and $\{W_i : i \in I\}$ be a family of closed subspaces in $\mathbb{H}$. Also let $\{v_i : i \in I\}$ be a family of weights *i.e.* $v_i > 0,\ \forall i \in I$. Then $\{(W_i, v_i) : i \in I\}$ is a *fusion frame* if there exist constants C, D with $0 < C \leq D < \infty$ such that

$$C\|f\|^2 \leq \sum_{i\in I} v_i^2 |[\pi_{W_i} f, f]| \leq D\|f\|^2, \quad \text{for every } f \in \mathbb{H} \tag{2.2}$$

where C and D are known as the lower and upper bounds respectively for the fusion frame. If $C = D$ then the fusion frame is known as *C-tight fusion frame* and if $C = D = 1$ then the fusion frame is known as *Parseval fusion frame*. Moreover, a fusion frame is called *v-uniform*, if $v := v_i = v_j$ for all $i,\ j \in I$. The family of subspaces $\{W_i : i \in I\}$ is an *orthonormal basis of subspaces* if $\mathbb{H} = \dot{\oplus}_{i\in I} W_i$.

Definition 2.9. A family of subspaces $\{W_i : i \in I\}$ of $\mathbb{H}$ is called complete, if $\overline{\sum_{i\in I} W_i} = \mathbb{H}$.

Theorem 2.10. *Let $\{(W_i, v_i) : i \in I\}$ is a fusion frame for $\mathbb{H}$, then it is complete* [3]*. The converse of the above theorem holds if $\mathbb{H}$ is finite dimensional.*

The following theorem provides a nice interaction between frames in Hilbert spaces and fusion frames in Hilbert spaces.

Theorem 2.11. [3] *For each $i \in I$, let $v_i > 0$ and let $\{f_{ij}\}_{j\in J_i}$ be a frame sequence in $\mathbb{H}$ with frame bounds A_i and B_i. Define $W_i = \overline{span}_{j\in J_i}\{f_{ij}\}$ for all $i \in I$ and choose an orthonormal basis $\{e_{ij}\}_{j\in J_i}$ for each subspace W_i. Suppose that $0 < A = inf_{i\in I} A_i \leq B = sup_{i\in I} B_i < \infty$. The following conditions are equivalent:*

(i) $\{v_i f_{ij}\}_{i\in I, j\in J_i}$ *is a frame for* $\mathbb{H}$.

(ii) $\{v_i e_{ij}\}_{i\in I, j\in J_i}$ *is a frame for* $\mathbb{H}$.

(iii) $\{(W_i, v_i) : i \in I\}$ *is a fusion frame for* $\mathbb{H}$.

For a family of subspaces $\{W_i\}_{i\in I}$ of $\mathbb{H}$, we define the space $(\sum_{i\in I} \oplus W_i)_{\ell_2}$ in the following way

$$\Big(\sum_{i\in I} \oplus W_i\Big)_{\ell_2} = \Big\{\{f_i\}_{i\in I} : f_i \in W_i \text{ and } \sum_{i\in i} \|f_i\|^2 < \infty\Big\}$$

with inner product given by $\langle \{f_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle = \sum_{i\in I}\langle f_i, g_i\rangle$.

Let $\{(W_i, v_i) : i \in I\}$ be a fusion frame for $\mathbb{H}$. Then $T_{W,v} : \big(\sum_{i\in I} \oplus W_i\big)_{\ell_2} \to \mathbb{H}$ defined by $T_{W,v}(f) = \sum_{i\in I} v_i f_i$, for all $f = \{f_i\}_{i\in I} \in \big(\sum_{i\in I} \oplus W_i\big)_{\ell_2}$ is the synthesis operator for $\{(W_i, v_i) : i \in I\}$. The adjoint of this operator $T^*_{W,v} : \mathbb{H} \to \big(\sum_{i\in I} \oplus W_i\big)_{\ell_2}$ defined by $T^*_{W,v}(f) = \{v_i \pi_{W_i}(f)\}_{i\in I}$ is the analysis operator for $\{(W_i, v_i) : i \in I\}$. The fusion frame operator $S_{W,v}$ for $\{(W_i, v_i) : i \in I\}$ is defined by $S_{W,v}(f) = T_{W,v}T^*_{W,v}(f) = T_{W,v}(\{v_i \pi_{W_i}(f)\}_{i\in I}) = \sum_{i\in I} v_i^2 \pi_{W_i}(f)$. It is a positive, selfadjoint and invertible linear operator on $\mathbb{H}$.

3. Main results

3.1. On orthogonal and J-orthogonal projection on Krein spaces.

Let M be a closed subspace of a Krein space $\mathbb{K}$ and π_M be an orthogonal projection from $\mathbb{K}$ onto M. Then it is an orthogonal projection in the associated Hilbert space $(\mathbb{K}, [\cdot,\cdot]_J)$. Thus, we have $\pi_M^2 = \pi_M$ and $\pi_M^* = \pi_M$. Here range of the projection *i.e.* $R(\pi_M) = M$ and Null space of π_M *i.e.* $N(\pi_M) = M^{\perp}$. Now if M is a projectively complete subspace of $\mathbb{K}$ then the J-orthogonal projection from $\mathbb{K}$ onto M exists. Let Q_M be the J-orthogonal projection from $\mathbb{K}$ onto M. Here range of the J-projection Q_M is $R(Q_M) = M$ and Null space *i.e.* $N(Q_M) = M^{[\perp]}$.

Let $\pi_M^{\#}$ be the J-adjoint of π_M. Then we have $\pi_M^{\#} = J\pi_M J$ and since $\pi_{JM} = J\pi_M J$, thus we have $\pi_{JM} = \pi_M^{\#}$.

Let W be a closed subspace of M, where M is a uniformly J-definite subspace of $\mathbb{K}$. Then W is a regular subspace of $\mathbb{K}$. Hence, the J-orthogonal projection from $\mathbb{K}$ onto W exists, let it be Q_W. We have the following important result.

Lemma 3.1. [17] *Let M be a uniformly J-definite subspace for the Krein space $\mathbb{K}$. If W is a closed subspace of M then $Q_W|_M = \pi_W|_M$.*

Since $(M, [\cdot,\cdot])$ is itself a Hilbert space, so if P_W be the orthogonal projection from M onto W, then the above lemma states that $P_W = Q_W|_M = \pi_W|_M$.

Let W be a subspace of a Krein space $\mathbb{K}$. Also let us assume that $\mathbb{P}^{++}$ denotes the set of all J-positive subspaces of $\mathbb{K}$ while $\mathbb{P}^{+}$ denotes the set of all J-non-negative subspaces of $\mathbb{K}$. Similarly, let $\mathbb{P}^{--}$ and $\mathbb{P}^{-}$ respectively denote the set of all J-negative and J-non-positive subspaces of $\mathbb{K}$. Also let $\tilde{\mathbb{P}}$ be the set of all J-indefinite subspaces of $\mathbb{K}$. We note that the set of all neutral subspaces sometimes referred as $\mathbb{P}^0$ satisfies

$\mathbb{P}^0 \subset \mathbb{P}^+$ or $\mathbb{P}^0 \subset \mathbb{P}^-$. Then $W \in \mathbb{P}^+ \cup \mathbb{P}^- \cup \tilde{\mathbb{P}}$. Throughout in our work we consider either $W \in \mathbb{P}^+ \cup \mathbb{P}^{--}$ or $W \in \mathbb{P}^{++} \cup \mathbb{P}^-$. Without any loss of generality, we assume $W \in \mathbb{P}^+ \cup \mathbb{P}^{--}$ to establish our results.

Let $\{W_i : i \in I\}$ be a collection of subspaces of the Krein space $\mathbb{K}$ such that $W_i \in \mathbb{P}^+ \cup \mathbb{P}^{--}$, $\forall i \in I$. We consider the space $\big(\sum_{i\in I} \oplus W_i\big)$. If $f \in \big(\sum_{i\in I} \oplus W_i\big)$ then $f = \{f_i\}_{i\in I}$, where $f_i \in W_i$ for each $i \in I$. Let $I_+ = \{i \in I : [f_i, f_i] \geq 0 \text{ for all } f_i \in W_i\}$ and $I_- = \{i \in I : [f_i, f_i] < 0 \text{ for all } f_i \in W_i\}$. We define $[f, g] = \sum_{i\in I}[f_i, g_i]$, where $f, g \in \big(\sum_{i\in I} \oplus W_i\big)$. If the series is unconditionally convergent then $[\cdot,\cdot]$ defines an inner product on $\big(\sum_{i\in I} \oplus W_i\big)$.

Definition 3.2. Let I be some index set and let $\{v_i : i \in I\}$ be a family of weights *i.e.* $v_i > 0$, $\forall\ i \in I$. Let $\{W_i : i \in I\}$ be a family of subspaces of a Krein space $\mathbb{K}$. Then the collection $\mathbb{F} = \{(W_i, v_i) : i \in I\}$ is said to be a *Bessel family of subspaces* in $\mathbb{K}$ if $\sum_{i\in I} v_i^2[\pi_{W_i}(f), f]_J \leq C[f, f]_J$, for all $f \in \mathbb{K}$, where $C > 0$.

3.2. Definition of J-fusion frames for Krein spaces. Let $\mathbb{F} = \big\{(W_i, v_i) : i \in I\big\}$ be a Bessel family of closed subspaces of a Krein space $\mathbb{K}$ with synthesis operator $T_{W,v} \in L\Big(\big(\sum_{i\in I} \oplus W_i\big)_{\ell_2}, \mathbb{K}\Big)$ such that $W_i \in \mathbb{P}^+ \cup \mathbb{P}^{--}$, $\forall i \in I$. Let $I_+ = \{i \in I : [f_i, f_i] \geq 0 \text{ for all } f_i \in W_i\}$ and $I_- = \{i \in I : [f_i, f_i] < 0 \text{ for all } f_i \in W_i\}$. Now consider the orthogonal decomposition of $\big(\sum_{i\in I} \oplus W_i\big)_{\ell_2}$ given by

$$\big(\sum_{i\in I} \oplus W_i\big)_{\ell_2} = \big(\sum_{i\in I_+} \oplus W_i\big)_{\ell_2} \bigoplus \big(\sum_{i\in I_-} \oplus W_i\big)_{\ell_2}$$

and denote by $P_\pm$ the orthogonal projection onto $(\sum_{i\in I_\pm} \oplus W_i)_{\ell_2}$. Also, let $T_{W,v_\pm} = T_{W,v}P_\pm$. If $M_\pm = \overline{\sum_{i\in I_\pm} W_i}$, notice that $\sum_{i\in I_\pm} W_i \subseteq R(T_{W,v_\pm}) \subseteq M_\pm$ and $R(T_{W,v}) = R(T_{W,v_+}) + R(T_{W,v_-})$.

Definition 3.3. The Bessel family of closed subspaces $\mathbb{F} = \{(W_i, v_i) : i \in I\}$ is a *J-fusion frame* for $\mathbb{K}$ if $R(T_{W,v_+})$ is a maximal uniformly J-positive subspace of $\mathbb{K}$ and $R(T_{W,v_-})$ is a maximal uniformly J-negative subspace of $\mathbb{K}$.

Let $\{(W_i, v_i) : i \in I\}$ be a J-fusion frame for $\mathbb{K}$ then $\big(\sum_{i\in I} \oplus W_i, [\cdot,\cdot]\big)$ is a Krein space. The fundamental symmetry, denoted by J_2, is defined as $J_2(f) = \{f_i : i \in I_+\} \cup \{-f_i : i \in I_-\}$ for all f. Also $[f, g]_{J_2} = \sum_{i\in I_+}[f_i, g_i] - \sum_{i\in I_-}[f_i, g_i]$. Now we consider the following space which is useful in our work

$$\big(\sum_{i\in I} \oplus W_i\big)_{\ell_2} = \Big\{f \in \big(\sum_{i\in I} \oplus W_i\big) : \sum_{i\in I} \|f_i\|_J^2 < \infty\Big\}.$$

Let $\mathbb{H}$ be a finite dimensional Hilbert space. Then from Hilbert space fusion frame theory we know that any complete family of subspaces is a fusion frame for $\mathbb{H}$ with respect to any arbitrary weights $\{v_i\}$. But similar to the J-frame theory of Krein spaces, any complete family of closed non-neutral subspaces in a finite dimensional Krein space may not be a J-fusion frame for that Krein space.

Example 3.4. Consider the vector space $\mathbb{R}^3(\mathbb{R})$. Let $\{e_1, e_2, e_3\}$ be the standard orthonormal basis for $\mathbb{R}^3$. Define $[e_1, e_1] = 1$, $[e_2, e_2] = 1$ and $[e_3, e_3] = -1$. Also $[e_i, e_j] = 0$ for $i \neq j,\ i, j \in \{1, 2, 3\}$. Then $(\mathbb{R}^3, [\cdot,\cdot])$ is a Krein space.

Now let $W = \{W_1, W_2, W_3\}$ be a family of closed uniformly J-definite subspace of $\mathbb{R}^3$, where $W_1 = span\{e_1 + \frac{1}{\sqrt{2}}e_3\}$, $W_2 = span\{e_2 + \frac{1}{\sqrt{2}}e_3\}$ and $W_3 = span\{e_3\}$. Then for any family of weights $\{v_i : i \in \{1, 2, 3\}\}$, the collection W is not a J-fusion frame of $\mathbb{R}^3$. But it is a fusion frame for $(\mathbb{R}^3, [\cdot,\cdot]_J)$, considered as a Hilbert space.

If $\mathbb{F} = \{(W_i, v_i) : i \in I\}$ is a J-fusion frame for $\mathbb{K}$ then $R(T_{W,v_+})$ is a maximal uniformly J-positive subspace of $\mathbb{K}$ and $R(T_{W,v_-})$ is a maximal uniformly J-negative subspace of $\mathbb{K}$. So $R(T_{W,v_\pm})$ is closed and we have $R(T_{W,v_\pm}) = M_\pm$. Thus, we have $\mathbb{K} = M_+ \oplus M_-$ and $\mathbb{F} = \{(W_i, v_i) : i \in I\}$ is a fusion frame for the associated Hilbert space $(\mathbb{K}, [\cdot,\cdot]_J)$. So, in terms of the frame inequality we have

$$A\|f\|_J^2 \leq \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|_J^2 \leq B\|f\|_J^2, \text{ for all } \ f \in \mathbb{K}$$

where $0 < A \leq B < \infty$. Let $T^*_{W,v}$ be the analysis operator for the above fusion frame, then $T^*_{W,v}(f) = \{v_i \pi_{W_i}(f)\}_{i\in I}$, for all $f \in \mathbb{K}$.

The following theorem is a generalization of a theorem in [12] in J-fusion frame setting.

Theorem 3.5. *Let $\mathbb{F} = \{(W_i, v_i)\}_{i\in I}$ be a J-fusion frame for $\mathbb{K}$. Then $\mathbb{F}_\pm = \{(W_i, v_i)\}_{i\in I_\pm}$ is fusion frame for the Hilbert space $(M_\pm, \pm[\cdot,\cdot])$, i.e. there exist constants, $-\infty < B_- \leq A_- < 0 < A_+ \leq B_+ < \infty$, such that*

$$A_\pm[f, f] \leq \sum_{i\in I_\pm} v_i^2 |[\pi_{W_i}|_{M_\pm}(f), f]| \leq B_\pm[f, f], \text{ for every } f \in M_\pm. \quad (3.1)$$

Proof. It is clear that $\mathbb{F}_\pm = \{(W_i, v_i)\}_{i\in I_\pm}$ is a fusion frame for the Hilbert space $(M_\pm, \pm[\cdot,\cdot])$. Without any loss of generality we only consider the positive part. So, we have $A_\pm[f, f] \leq \sum_{i\in I_+} |[v_i P_{W_i}(f), v_i P_{W_i}(f)]| \leq B_\pm[f, f]$, for all $f \in M_\pm$, where P_{W_i} is the orthogonal projection from the Hilbert space $(M_\pm, \pm[\cdot,\cdot])$ onto W_i. Now from lemma (3.1) we have $[P_{W_i}(f), f] = [\pi_{W_i}|_{M_\pm}(f), f]$. So, we are done. $\square$

Remark 3.6. Let $\{f_i : i \in I\}$ be a J-frame for the Krein space $\mathbb{K}$. We assume that M is a uniformly J-definite subspace of $\mathbb{K}$. Then we know that $\{\pi_M(f_i) : i \in I\}$ is also a J-frame for the subspace M. But since M is uniformly J-definite hence $(M, [\cdot,\cdot])$ is a Hilbert space. So $\{\pi_M(f_i) : i \in I\}$ is also a frame for the subspace M in Hilbert space sense.

Definition 3.7. A sequence $\{f_i : i \in I\}$ in $\mathbb{K}$ is said to be a *J-frame sequence* in $\mathbb{K}$ if $\overline{span}\{f_i : i \in I_+\}$ and $\overline{span}\{f_i : i \in I_-\}$ are uniformly J-positive and uniformly J-negative subspace of $\mathbb{K}$ respectively.

Let $\mathbb{F} = \{f_i : i \in I\}$ be a J-frame for $\mathbb{K}$, then there exist constants B_-, A_-, A_+ and B_+ such that, $-\infty < B_- \leq A_- < 0 < A_+ \leq B_+ < \infty$. These constants are the J-frame bounds for the J-frame $\mathbb{F}$. Now for every J-frame sequence such constants exist. But if the J-frame sequence consists of only positive or negative elements respectively then the positive or negative parts of the constants redundant accordingly.

Theorem 3.8. *For each $i \in I$, let $v_i > 0$ and $\{f_{ij}\}_{j\in J_i}$ be a J-frame sequence in $\mathbb{K}$ with J-frame bounds B_{i-}, A_{i-}, A_{i+} and B_{i+} for each $i \in I$, such that, $-\infty < sup\ _iB_{i-} \leq inf\ _iA_{i-} < 0 < inf\ _iA_{i+} \leq sup\ _iB_{i+} < \infty$. Let $W_i = \overline{span}_{j\in J_i}\{f_{ij}\}$ for each $i \in I$. If W_i is J-definite for each $i \in I$, then the following conditions are equivalent:*
(i) $\{v_if_{ij}\}_{i\in I, j\in J_i}$ is a J-frame for $\mathbb{K}$.

(ii) $\{(W_i, v_i) : i \in I\}$ is a J-fusion frame for $\mathbb{K}$.

Proof. Let $\{v_if_{ij}\}_{i\in I_+, j\in J_i}$ be a J-frame for $\mathbb{K}$. So, let $I_+ = \{(i,j) : [f_{ij}, f_{ij}] > 0\}$ and $I_- = \{(i,j) : [f_{ij}, f_{ij}] < 0\}$. Also let $M_+ = \overline{span}\{f_{ij} : (i,j) \in I_+\}$ and $M_- = \overline{span}\{f_{ij} : (i,j) \in I_-\}$. Also we have

$$A_\pm[f,f] \leq \sum_{(i,j)\in I_\pm} |[f, v_if_{ij}]|^2 \leq B_\pm[f,f], \text{ for all } \ f \in M_\pm.$$

Now, $\{f_{ij}\}_{j\in J_i}$ is a J-frame sequence in $\mathbb{K}$ and $W_i = \overline{span}\{f_{ij}\}_{j\in J_i}$. But W_i is a J-definite subspace of $\mathbb{K}$, so either $W_i \subset M_+$ or $W_i \subset M_-$ for each i. Hence, $\{f_{ij}\}_{j\in J_i}$ is a J-frame for $\mathbb{K}$. Here these J-frames either consist of only positive elements or only negative elements. Without any loss of generality, we choose the pair (i,j), such that, $(i,j) \in I_+$, then $W_i \subset M_+$. So we have

$$A_{i+}[\pi_{W_i}|_{M_+}(f), f] \leq \sum_{j\in J_i} |[\pi_{W_i}f, f_{ij}]|^2 \leq B_{i+}[\pi_{W_i}|_{M_+}(f), f], \ \forall\ f \in M_+.$$

Now, let us assume $0 < \ A_1 = inf_i A_{i+} \leq B_1 = sup_i B_{i+} < \ \infty$, then

$$\begin{aligned} A_1 \sum_i v_i^2 [\pi_{W_i}|_{M_+}(f), f] &\leq \sum_i A_{i+} v_i^2 [\pi_{W_i}|_{M_+}(f), f] \\ &\leq \sum_i \sum_{j\in J_i} |[\pi_{W_i}|_{M_+}(f), v_i f_{ij}]|^2 \\ &\leq \sum_i B_{i+} [\pi_{W_i}|_{M_+}(f), f] \\ &\leq B_1 \sum_i v_i^2 [\pi_{W_i}|_{M_+}(f), f] \end{aligned}$$

We observe that $\sum_i \sum_{j\in J_i} |[\pi_{W_i}|_{M_+}(f), v_i f_{ij}]|^2 = \sum_i \sum_{j\in J_i} |[f, v_i f_{ij}]|^2$. Since $[\pi_{W_i}|_{M_+}(f), f] > 0$ for all $f \in M_+$, then

$$\begin{aligned} A_1 \sum_i v_i^2 [\pi_{W_i}|_{M_+}(f), f] &\leq \sum_i \sum_{j\in J_i} |[f, v_i f_{ij}]|^2 \\ &\leq B_+ [f, f] \end{aligned}$$

So, we have $\sum_i v_i^2 |[\pi_{W_i}|_{M_+}(f), f]| \leq \frac{B_+}{A_1}[f, f]$.

Similarly, we can show that $\sum_i v_i^2 |[\pi_{W_i}|_{M_+}(f), f, J(f)]| \leq \frac{A_+}{B_1}[f, f]$. When $W_i \subset M_-$ for some i, then we can proceed as above. This proves that $\{(W_i, v_i) : i \in I\}$ is a J-fusion frame of subspaces for $\mathbb{K}$. A careful investigation of the above implications reveal that they are vice versa. So, we prove that $(i) \Leftrightarrow (ii)$. □

3.3. The J-fusion frame operator. Let $\{(W_i, v_i)\}_{i\in I}$ be a J-fusion frame for the Krein space $\mathbb{K}$. Then $I_+ = \{i \in I : [f_i, f_i] \geq 0$ for all $f_i \in W_i\}$ and $I_- = \{i \in I : [f_i, f_i] < 0$ for all $f_i \in W_i\}$.

Definition 3.9. The linear operator $S_{W,v} : \mathbb{K} \to \mathbb{K}$ defined by $S_{W,v}(f) = \sum_{i\in I} \sigma_i v_i^2 \pi_{J(W_i)}(f)$ is said to be the *J-fusion frame operator* for the J-fusion frame $\{(W_i, v_i)\}_{i\in I}$. Here $\sigma_i = 1$ if $i \in I_+$ and $\sigma_i = -1$ if $i \in I_-$.

From the definition it readily follows that $S_{W,v} = T_{W,v} T^{\#}_{W,v}$. Also $S_{W,v}$ is the sum of two J-positive operators. The statement also easily follows from the definition. Let $S_{W,v_+} : \mathbb{K} \to \mathbb{K}$ is defined by $S_{W,v_+}(f) = \sum_{i\in I_+} v_i^2 \pi_{J(W_i)}(f)$. Then $[S_{W,v_+}(f), f] = [\sum_{i\in I_+} v_i^2 \pi_{J(W_i)}(f), f] = \sum_{i\in I_+} v_i^2 [\pi_{W_i}(f), f]_J$. So, S_{W,v_+} is a J-positive operator and also $S_{W,v_+} = T_{W,v_+} T^{\#}_{W,v_+}$. Similarly, let $S_{W,v_-} = -T_{W,v_-} T^{\#}_{W,v_-}$. Then S_{W,v_-} is also a J-positive operator. We have $S_{W,v} = S_{W,v_+} - S_{W,v_-}$.

Theorem 3.10. *If $\{(W_i, v_i)\}_{i\in I}$ is a J-fusion frame for the Krein space $\mathbb{K}$ with synthesis operator $T_{W,v} \in L\Big(\big(\sum_{i\in I} \oplus W_i\big)_{\ell_2}, \mathbb{K}\Big)$ then the J-fusion frame operator $S_{W,v}$ is bijective and J-selfadjoint.*

Proof. Let $S_{W,v}(f) = 0$ then $S_{W,v_+}(f) = S_{W,v_-}(f)$. Again $R(S_{W,v_+}) = M_+$ and $R(S_{W,v_-}) = M_-$. But $M_+ \cap M_- = \{0\}$. These together implies that $f = 0$. Hence, $S_{W,v}$ is injective. To prove the surjectivity of $S_{W,v}$ we consider the oblique decomposition of $\mathbb{K}$. We can write $\mathbb{K} = M_+^{[\perp]} \oplus M_-^{[\perp]}$. Now, $N(S_{W,v_\pm}) = M_\pm^{[\perp]}$. So, $S_{W,v}(M_\pm^{[\perp]}) = S_{W,v_\mp}(M_\pm^{[\perp]})$ and $R(S_{W,v}) = S_{W,v_-}(M_+^{[\perp]}) + S_{W,v_+}(M_-^{[\perp]}) = R(S_{W,v_+}) + R(S_{W,v_-}) = M_+ + M_+ = \mathbb{K}$. So, $S_{W,v}$ is bijective. Since, $S_{W,v} = T_{W,v}T_{W,v}^{\#}$ therefore $S_{W,v}$ is J-selfadjoint. $\square$

Theorem 3.11. *If $\{(W_i, v_i)\}_{i\in I}$ is a J-fusion frame for the Krein space $\mathbb{K}$ with J-fusion frame operator $S_{W,v}$, then $\{(S_{W,v}^{-1}(W_i), v_i)\}_{i\in I}$ is a J-fusion frame for $\mathbb{K}$ with J-fusion frame operator $S_{W,v}^{-1}$.*

Proof. We observe that the operator $S_{W,v}^{-1}T_{W,v} \in L\Big(\big(\sum_{i\in I} \oplus W_i\big)_{\ell_2}, \mathbb{K}\Big)$ is the synthesis operator of $\{(S_{W,v}^{-1}(W_i), v_i)\}_{i\in I}$. Furthermore we have $S_{W,v}^{-1}(M_\pm) = M_\mp^{[\perp]}$. So, $\{S_{W,v}^{-1}(W_i)\}_{i\in I_+}$ is a collection of uniformly J-definite subspace of $M_-^{[\perp]}$. Similarly, $\{S_{W,v}^{-1}(W_i)\}_{i\in I_-}$ is a collection of uniformly J-definite subspace of $M_+^{[\perp]}$. Also, $R(S_{W,v}^{-1}T_{W,v_+})$ and $R(S_{W,v}^{-1}T_{W,v_-})$ is maximal uniformly J-positive and J-negative subspace respectively. So, from the definition of J-fusion frame we have that $\{(S_{W,v}^{-1}(W_i), v_i)\}_{i\in I}$ is a J-fusion frame for $\mathbb{K}$. $\{(S_{W,v}^{-1}(W_i), v_i)\}_{i\in I_\pm}$ is fusion frame for the Hilbert space $(M_\mp^{\perp}, \pm[\cdot,\cdot])$. Since $(S_{W,v}^{-1}T_{W,v})(T_{W,v}^{\#}S_{W,v}^{-1}) = S_{W,v}^{-1}$ therefore $S_{W,v}^{-1}$ is the required J-fusion frame operator. $\{(S_{W,v}^{-1}(W_i), v_i)\}_{i\in I}$ is called the *canonical J-dual fusion frame* for $\{(W_i, v_i)\}_{i\in I}$ in $\mathbb{K}$. $\square$

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