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On weighted $G-$Banach frames in Banach spaces

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Abstract. In the present paper, we introduce and study weighted $g-$Banach frames in Banach spaces. Examples and counter examples to distinguish various types of weighted $g-$Banach frames in Banach spaces have been given. A necessary and sufficient condition for a weighted $g-$Banach frame to be an exact has been given. Also, the construction of weighted $g-$Banach frames from bounded linear operator has been discussed. Finally, we study the finite sum of weighted $g-$Banach frames and give a sufficient condition for the finite sum of weighted $g-$Banach frames to be a weighted $g-$Banach frame.

1. Introduction

Frames are main tools for use in signal processing, image processing, data compression and sampling theory etc. Today even more uses are being found for the theory such as optics, filter banks, signal detection as well as study of Besov spaces, Banach space theory etc. Frames for Hilbert spaces were introduced by Duffin and Schaeffer[14]. Later, in 1986, Daubechies, Grossmann and Meyer [13] reintroduced frames and found a new application to wavelet and Gabor transforms. For a nice introduction to frames, one may refer [6].

Coifman and Weiss [12] introduced the notion of atomic decomposition for function spaces. Feichtinger and Gröchening [16] extended the notion of atomic decomposition to Banach spaces. Gröchening [18] introduced a more general concept for Banach spaces called Banach frame. He gave the following definition of a Banach frame:

Definition 1.1. ([18]) Let $\mathcal{X}$ be a Banach space and $\mathcal{X}_{d_1}$ an associated Banach space of scalar-valued sequences indexed by $\mathbb{N}$. Let $\{f_n\} \subset \mathcal{X}^*$ and $S : \mathcal{X}_{d_1} \to \mathcal{X}$ be given. Then the pair $(\{f_n\}, S)$ is called a Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_{d_1}$, if

(1) $\{f_n(x)\} \in \mathcal{X}_{d_1}$, for each $x \in \mathcal{X}$.

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(2) there exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_{\mathcal{X}} \leq \|\{f_n(x)\}\|_{\mathcal{X}_{d_1}} \leq B\|x\|_{\mathcal{X}}, \quad x \in \mathcal{X}. \tag{1.1}$$

(3) S is a bounded linear operator such that $S(\{f_n(x)\}) = x, \quad x \in \mathcal{X}$.

The positive constants A and B, respectively, are called lower and upper frame bounds of the Banach frame $(\{f_n\}, S)$. The operator $S : \mathcal{X}_{d_1} \to \mathcal{X}$ is called the reconstruction operator (or the pre-frame operator). The inequality (1.1) is called the frame inequality.

Banach frames and Banach Bessel sequence in Banach spaces were further studied in [5, 19, 20, 21].

Over the last decade, various other generalizations of frames for Hilbert spaces have been introduced and studied. Some of them are bounded quasi-projectors by Fornasier [17]; Pseudo frames by Li and Ogawa [22]; Oblique frames by Eldar [15]; Christensen and Eldar [7]; $(p;Y)$-Operators frames by Cao et.al [4]. W. Sun [25] introduced and defined $g-$frames in Hilbert spaces and observed that bounded quasi-projectors, frames of subspaces, Pseudo frames and Oblique frames are particular cases of $g-$frames in Hilbert spaces. $G-$frames are further studied in [8, 9, 10, 11, 23, 26].

In practice, the ratio of frame bounds for a given non-tight frame can be reduced by multiplying the elements by suitable weights. In 2005, Bogdanova et.al [3] introduced and defined weighted frames to get a numerically more efficient approximation algorithm for spherical wavelets. Thus, by weighting the elements, we can improve the numerical efficiency of iterative algorithm like "frame algorithm" [6] for the inversion of the frame operator. Weighted frames are further studied in [2, 27].

In the present paper, we introduce and study weighted $g-$Banach frames in Banach spaces. Examples and counter examples to distinguish various types of weighted $g-$Banach frames in Banach spaces have been given. A necessary and sufficient condition for a weighted $g-$Banach frame to be an exact has been given. Also, the construction of weighted $g-$Banach frames from bounded linear operator has been discussed. Finally, we study the finite sum of weighted $g-$Banach frames and obtain a sufficient condition for the finite sum of weighted $g-$Banach frames to be a weighted $g-$Banach frame.

2. Preliminaries

Throughout this paper, $\mathcal{X}$ will denote a Banach space over the scalar field $\mathbb{K}$($\mathbb{R}$ or $\mathbb{C}$), $\mathcal{X}_d$ is an associated Banach space of vector valued sequences indexed by $\mathbb{N}$. $\mathcal{H}$ is a separable Hilbert space, $B(\mathcal{X}, \mathcal{H})$ is the collection of all bounded linear operators from $\mathcal{X}$ into $\mathcal{H}$ and $\overline{span}\{\Lambda_n\}$ is closed linear span of $\{\Lambda_n\}$ in the strong operator topology in $B(\mathcal{X}, \mathcal{H})$. A weight $\omega = \{\omega_n\}$ is a sequence of positive real numbers.

A Banach space of vector valued sequences (or BV-space) is a linear space of sequences with a norm which makes it a Banach space. Let $\mathcal{X}$ be a Banach space and $1 < p < \infty$, then

$$Y = \left\{\{x_n\} : x_n \in \mathcal{X}, \ \left(\sum_{n\in\mathbb{N}} \|x_n\|^p\right)^{1/p} < \infty\right\}$$

and

$$\ell^\infty = \{\{x_n\} : \sup_{n\in\mathbb{N}} \|x_n\| < \infty, \ x_n \in \mathcal{X}\}$$

are BV spaces of $\mathcal{X}$.

Abdollahpour et.al [1] generalized the concepts of frames for Banach Spaces and defined $g-$Banach frames in Banach spaces. They gave the following definition:

Definition 2.1. ([1]) Let $\mathcal{X}$ be a Banach space and $\mathcal{H}$ be a separable Hilbert space. Let $\mathcal{X}_d$ be an associated Banach space of vector-valued sequences indexed by $\mathbb{N}$. Let $\{\Lambda_n\}_{n\in\mathbb{N}} \subset B(\mathcal{X},\mathcal{H})$ and $S : \mathcal{X}_d \to \mathcal{X}$ be given. Then the pair $(\{\Lambda_n\}, S)$ is called a $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{H}$ and $\mathcal{X}_d$, if

(1) $\{\Lambda_n(x)\} \in \mathcal{X}_d$, for each $x \in \mathcal{X}$.
(2) there exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_{\mathcal{X}} \leq \|\{\Lambda_n(x)\}\|_{\mathcal{X}_d} \leq B\|x\|_{\mathcal{X}}, \quad x \in \mathcal{X}. \tag{2.1}$$

(3) S is a bounded linear operator such that $S(\{\Lambda_n(x)\}) = x, \quad x \in \mathcal{X}$.

The positive constants A and B, respectively, are called the lower and upper frame bounds of the $g-$Banach frame $(\{\Lambda_n\}, S)$. The operator $S : \mathcal{X}_d \to \mathcal{X}$ is called the reconstruction operator and the inequality (2.1) is called the frame inequality for $g-$Banach frame $(\{\Lambda_n\}, S)$.

The following lemma, proved in [24], is used in the sequel:

Lemma 2.2. *Let $\mathcal{X}$ be a Banach space and $\mathcal{H}$ be a separable Hilbert space. Let $\{\Lambda_n\} \subset B(\mathcal{X},\mathcal{H})$ be a sequence of non-zero operators. If $\{\Lambda_n\}$ is total over $\mathcal{X}$, i.e., $\{x \in \mathcal{X} : \Lambda_n(x) = 0, n \in \mathbb{N}\} = \{0\}$, then $\mathcal{A} = \{\{\Lambda_n(x)\} : x \in \mathcal{X}\}$ is a Banach space with the norm given by $\|\{\Lambda_n(x)\}\|_{\mathcal{A}} = \|x\|_{\mathcal{X}}, x \in \mathcal{X}$.*

3. $G-$ Weighted Banach Frames

We begin this section with the following definition of weighted $g-$Banach frames in Banach spaces.

Definition 3.1. Let $\omega = \{\omega_n\}_{n\in\mathbb{N}}$ be a weight. A pair $(\{\omega_n\Lambda_n\}, \mathfrak{U})$ ($\{\Lambda_n\} \subset B(\mathcal{X},\mathcal{H})$), $\mathfrak{U} : \mathcal{X}_d \to \mathcal{X}$) is called a *weighted $g-$Banach frame* (or $\omega g-$Banach frame) for $\mathcal{X}$ with respect to $\mathcal{H}$ and $\mathcal{X}_d$, if

(1) $\{(\omega_n\Lambda_n)(x)\} \in \mathcal{X}_d$, for each $x \in \mathcal{X}$.
(2) there exist constants C and D ($0 < C \leq D < \infty$) such that

$$C\|x\|_{\mathcal{X}} \leq \|\{(\omega_n\Lambda_n)(x)\}\|_{\mathcal{X}_d} \leq D\|x\|_{\mathcal{X}}, \quad x \in \mathcal{X}. \tag{3.1}$$

(3) $\mathfrak{U}$ is a bounded linear operator such that $\mathfrak{U}(\{(\omega_n\Lambda_n)(x)\}) = x, \ x \in \mathcal{X}$.

The positive constants C and D, respectively, are called the lower and upper frame bounds of the weighted $g-$Banach frame $(\{\omega_n\Lambda_n\}, \mathfrak{U})$. The operator $\mathfrak{U} : \mathcal{X}_d \to \mathcal{X}$ is called the reconstruction operator and the inequality (3.1) is called frame inequality for weighted $g-$Banach frame $(\{\omega_n\Lambda_n\}, \mathfrak{U})$.

Definition 3.2. A weighted $g-$Banach frame $(\{\omega_n\Lambda_n\}, \mathfrak{U})$ for $\mathcal{X}$ with respect to $\mathcal{H}$ and $\mathcal{X}_d$ with frame bounds C and D is called

- *Tight*, if it is possible to choose $C = D$ satisfying (3.1).
- *Normalized tight*, if it is possible to choose $C = D = 1$ satisfying (3.1).
- *Exact*, if atleast one of the conditions in Definition 3.1 is not satisfied whenever $\omega_m\Lambda_m, m \in \mathbb{N}$ is removed from $\{\omega_n\Lambda_n\}$.

Definition 3.3. Let $\omega = \{\omega_n\}_{n\in\mathbb{N}}$ be a weight. A sequence of operators $\{\omega_n\Lambda_n\} \subset B(\mathcal{X}, \mathcal{H})$ is said to be a *weighted $g-$Banach Bessel sequence* for $\mathcal{X}$ with respect to $\mathcal{X}_d$, if

(1) $\{(\omega_n\Lambda_n)(x)\} \in \mathcal{X}_d$, for each $x \in \mathcal{X}$.

(2) there exist a positive constant M such that

$$\|\{(\omega_n\Lambda_n)(x)\}\|_{\mathcal{X}_d} \leq M\|x\|_{\mathcal{X}}, \quad x \in \mathcal{X}. \tag{3.2}$$

The positive constant M is called a *Bessel bound* (or simply, a bound) for the weighted $g-$Banach Bessel sequence $\{\omega_n\Lambda_n\}$.

Regarding the existence of various type of weighted $g-$Banach frames, we have the following examples:-

Example 3.4. Let $\mathcal{H}$ be a separable Hilbert space. Let $E = \ell^\infty(\mathcal{H}) = \{\{x_n\} : x_n \in \mathcal{H};\ \sup_{1\leq n<\infty} \|x_n\|_{\mathcal{H}} < \infty\}$ be a Banach space with the norm given by

$$\|\{x_n\}\|_E = \sup_{1\leq n<\infty} \|x_n\|_{\mathcal{H}},\ \{x_n\} \in E$$

and $E_d = \ell^\infty(\mathcal{H})$.
Now, for each $n \in \mathbb{N}$, define $\Lambda_n : E \to \mathcal{H}$ by

$$\Lambda_n(x) = x_n,\ x = \{x_n\} \in E.$$

Then, for each $x \in E$, $\{\Lambda_n(x)\} \in E_d$ such that

$$\|\{\Lambda_n(x)\}\|_{E_d} = \|x\|_E,\ x \in E.$$

Now, define $S : E_d \to E$ by $S(\{\Lambda_n(x)\}) = x,\ x \in E$. Then, $(\{\Lambda_n\}, S)$ is a $g-$Banach frame for E with respect to E_d.

(1) **Weighted $G-$Banach Frame**. Let $\omega = \left\{\omega_n = \dfrac{1}{n}\right\}_{n\in\mathbb{N}}$ be a family of weights. Let $x \in E$ be an element such that $(\omega_n\Lambda_n)(x) = 0$, for all $n \in \mathbb{N}$. This gives $x = 0$. Thus $\{\omega_n\Lambda_n\}$ is total over E. Then, by Lemma 2.2, there exists an associated Banach space $\mathcal{A} = \{\{(\omega_n\Lambda_n)(x)\} : x \in E\}$ with the norm given by $\|\{(\omega_n\Lambda_n)(x)\}\|_{\mathcal{A}} = \|x\|_E,\ x \in E$. Define $\mathfrak{U} : \mathcal{A} \to E$ given by $\mathfrak{U}(\{(\omega_n\Lambda_n)(x)\}) = x,\ x \in E$. Then $(\{\omega_n\Lambda_n\}, \mathfrak{U})$ is a weighted $g-$Banach frame for E with respect to $\mathcal{A}$.

(2) **Tight and Exact Weighted $G-$Banach Frames**. A pair $(\{\omega_n\Lambda_n\}, \mathfrak{U})$ in (1) is a weighted $g-$Banach frame for E with respect to $\mathcal{A}$ with bounds $C = D = 1$. Also, there exists no reconstruction operator $\mathfrak{U}_0$ such that $(\{\omega_n\Lambda_n\}_{n\neq m}, \mathfrak{U}_0),\ (m \in \mathbb{N})$ is a weighted $g-$Banach frame for E. Indeed, if $x = (0, \ldots, 0, \underbrace{1}_{m^{th}\text{place}}, 0, \ldots)$ be a non zero element in E, then $(\omega_n\Lambda_n)(x) = 0$, for $n \neq m,\ m \in \mathbb{N}$.

(3) **Non-Exact and Non-Tight Weighted $G-$Banach Frames**. Let $\omega = \{\omega_n\}_{n\in\mathbb{N}}$ be weights defined as

$$\omega_3 = \frac{1}{2} \text{ and } \omega_n = 1,\ n \neq 3,\ n \in \mathbb{N}.$$

Now, define a sequence of operators $\{\mathfrak{T}_n\} \subset B(E, \mathcal{H})$ as

$$\begin{cases} \mathfrak{T}_1(x) = \Lambda_1(x) \\ \mathfrak{T}_n(x) = \Lambda_{n-1}(x),\ n \geq 2,\ n \in \mathbb{N},\ x = \{x_n\} \in E. \end{cases}$$

Then, for each $x \in E$, $\{(\omega_n \mathfrak{T}_n)(x)\} \in E_d$ such that

$$\frac{1}{2}\|x\|_E \leq \|\{(\omega_n \mathfrak{T}_n)(x)\}\|_{E_d} \leq \|x\|_E,\ x \in E.$$

Also, there exists a bounded linear operator $\mathfrak{U}_1 : E_d \to E$ such that $(\{\omega_n \mathfrak{T}_n\}, \mathfrak{U}_1)$ is non-tight weighted $g-$Banach frame for E with respect to E_d. Further, by Lemma 2.2, there exists an associated Banach space $\mathcal{A}_1 = \{\{(\omega_n \mathfrak{T}_n)(x)\}_{n \neq 1},\ x \in E\}$ and an operator $\tilde{\mathfrak{U}} : \mathcal{A}_1 \to E$ defined by $\tilde{\mathfrak{U}}(\{(\omega_n \mathfrak{T}_n)(x)\}_{n \neq 1}) = x,\ x \in E$ such that $(\{\omega_n \mathfrak{T}_n\}_{n \neq 1}, \tilde{\mathfrak{U}})$ is a weighted $g-$Banach frame for E with respect to $\mathcal{A}_1$. Hence $(\{\omega_n \mathfrak{T}_n\}, \mathfrak{U}_1)$ is non-tight and non-exact weighted $g-$Banach frame for E with respect to E_d.

(4) **Not a Weighted $G-$Banach Frame**. Let $\omega^* = \{\omega^*_n = n\}_{n \in \mathbb{N}}$ be a weight. Then, there exists no reconstruction operator $\tilde{\mathfrak{U}}_1 : E_d \to E$ such that $(\{\omega^*_n \Lambda_n\}, \tilde{\mathfrak{U}}_1)$ is a weighted $g-$Banach frame for E with respect to E_d.

Remark 3.5. However, there are some other associated Banach space of vector valued sequences with respect to which $(\{\omega^*_n \Lambda_n\}, \tilde{\mathfrak{U}}_1)$ forms a weighted $g-$Banach frame for E. Thus, various types of Banach space of vector valued sequences involved in the definition of weighted $g-$Banach frame. This leads us to study the weighted $g-$Banach frame for a Banach space $\mathcal{X}$ with respect to various types of Banach space of vector valued sequences $\mathcal{X}_d$.

Now, we give the necessary and sufficient condition for exactness of a weighted $g-$Banach frame.

Theorem 3.6. *Let $\omega = \{\omega_n\}$ be a weight and let $(\{\omega_n \Lambda_n\}, \mathfrak{U})$ be a weighted $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$. Then $(\{\omega_n \Lambda_n\}, \mathfrak{U})$ is an exact if and only if $\omega_i \Lambda_i \notin \overline{span}\{\omega_n \Lambda_n\}_{n \neq i}$, for all $i \in \mathbb{N}$.*

Proof. Let $(\{\omega_n \Lambda_n\}, \mathfrak{U})$ be an exact weighted $g-$Banach frame for $\mathcal{X}$ and let $\omega_i \Lambda_i \in \overline{span}\{\omega_n \Lambda_n\}_{n \neq i}$, for some $i \in \mathbb{N}$.
Then $(\omega_i \Lambda_i)(x) = \lim_{k \to \infty} \sum_{\substack{n=1 \\ n \neq i}}^{k} \alpha^{(k)}_n ((\omega_n \Lambda_n)(x)),\ x \in \mathcal{X}$.
Thus, by frame inequality of weighted $g-$Banach frame $(\{\omega_n \Lambda_n\}, \mathfrak{U})$, $\{\omega_n \Lambda_n\}_{n \neq i}$ is total over $\mathcal{X}$. So, by Lemma 2.2, there exists an associated Banach space $\mathcal{A} = \{\{(\omega_n \Lambda_n)(x)\}_{n \neq i} : x \in \mathcal{X}\}$ with the norm given by $\|\{(\omega_n \Lambda_n)(x)\}_{n \neq i}\|_{\mathcal{A}} = \|x\|_{\mathcal{X}},\ x \in \mathcal{X}$. Let $\mathfrak{U}_0 : \mathcal{A} \to \mathcal{X}$ be given by $\mathfrak{U}_0(\{(\omega_n \Lambda_n)(x)\}_{n \neq i}) = x,\ x \in \mathcal{X}$. Then $(\{\omega_n \Lambda_n\}_{n \neq i}, \mathfrak{U}_0)$ is a weighted $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{A}$, which is a contradiction to given hypothesis.

Conversely, let $\omega_i \Lambda_i \notin \overline{span}\{\omega_n \Lambda_n\}_{n \neq i}$, for all $i \in \mathbb{N}$. Let $(\{\omega_n \Lambda_n\}, \mathfrak{U})$ be not an exact weighted $g-$Banach frame for $\mathcal{X}$. Then, there exists a positive integer m and a reconstruction operator $\mathfrak{U}_1$ such that $(\{\omega_n \Lambda_n\}_{n \neq m}, \mathfrak{U}_1)$ is a weighted $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{A}_1$, where $\mathcal{A}_1$ is some associated Banach space of vector valued sequences indexed by $\mathbb{N}$. Let C_1 and D_1 be choice of bounds for $(\{\omega_n \Lambda_n\}_{n \neq m}, \mathfrak{U}_1)$. Then

$$C_1\|x\|_{\mathcal{X}} \leq \|\{(\omega_n \Lambda_n)(x)\}_{n \neq m}\|_{\mathcal{A}_1} \leq D_1\|x\|_{\mathcal{X}},\ x \in \mathcal{X}. \tag{3.3}$$

This gives $\{\omega_n \Lambda_n\}_{n \neq m}$ is total over $\mathcal{X}$, since otherwise there exists a non-zero element $x \in \mathcal{X}$ such that $(\omega_n \Lambda_n)(x) = 0,\ n \in \mathbb{N},\ n \neq m$. Then, by frame inequality (3.3), $x = 0$.

Which is a contradiction.
Therefore, $\overline{span}\{\omega_n\Lambda_n\}_{n\neq m} = B(\mathcal{X},\mathcal{H})$. This gives $\omega_m\Lambda_m \in \overline{span}\{\omega_n\Lambda_n\}_{n\neq m}$. Which is again a contradiction to the hypothesis. Hence $(\{\omega_n\Lambda_n\},\mathfrak{U})$ is an exact weighted $g-$Banach frame for $\mathcal{X}$. □

Now, the next result provides the condition under which the Banach space possesses a weighted $g-$Banach frame, when it is related in some sense to another Banach space which is having a weighted $g-$Banach frame.

Theorem 3.7. *Let $\omega = \{\omega_n\}$ be a weight and $(\{\omega_n\Lambda_n\},\mathfrak{U})$ $(\{\Lambda_n\} \subset B(\mathcal{X},\mathcal{H}), \mathfrak{U} : \mathcal{X}_d \to \mathcal{X})$ be a weighted $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$. Let $\mathcal{Y}$ be another Banach space and $\{\mathfrak{T}_n\} \subset B(\mathcal{Y},\mathcal{H})$ be a sequence of operators. Let T be a continuous linear operator from $\mathcal{X}$ onto $\mathcal{Y}$ such that $(\omega_n\mathfrak{T}_n)oT = \omega_n\Lambda_n,\ n \in \mathbb{N}$. Then $\mathcal{Y}$ has a weighted $g-$Banach frame with respect to some associated Banach space $\mathcal{A}$.*

Proof. For each $y \in \mathcal{Y}$, there exists $x \in \mathcal{X}$ such that $T(x) = y$. Let $(\omega_n\mathfrak{T}_n)(y) = 0$, for all $n \in \mathbb{N}$. Then $(\omega_n\Lambda_n)(x) = 0$, for all $n \in \mathbb{N}$. Then, by frame inequality of weighted $g-$Banach frame $(\{\omega_n\Lambda_n\},\mathfrak{U})$, $x = 0$. Therefore $\{\omega_n\mathfrak{T}_n\}$ is total over $\mathcal{Y}$. Hence, by Lemma (2.2), there exists an associated Banach space $\mathcal{A} = \{\{(\omega_n\mathfrak{T}_n)(y)\} :\ y \in \mathcal{Y}\}$ with the norm given by $\|\{(\omega_n\mathfrak{T}_n)(y)\}\|_{\mathcal{A}} = \|y\|_{\mathcal{Y}}$. Define $\tilde{\mathfrak{U}} : \mathcal{A} \to \mathcal{Y}$ by $\tilde{\mathfrak{U}}(\{(\omega_n\mathfrak{T}_n)(y)\}) = y,\ y \in \mathcal{Y}$. Then $\tilde{\mathfrak{U}}$ is a bounded linear operator such that $(\{\omega_n\mathfrak{T}_n\},\tilde{\mathfrak{U}})$ is a weighted $g-$Banach frame for $\mathcal{Y}$ with respect to $\mathcal{A}$. □

Next, we give a result which shows that the weighted $g-$Banach frames are invariant under linear homeomorphism.

Theorem 3.8. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces and let $(\{\omega_n\Lambda_n\},\mathfrak{U})$ be a weighted $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$ and with best bounds $C_\Lambda,\ D_\Lambda$. Let $U : \mathcal{X} \to \mathcal{Y}$ be a linear homeomorphism. Then $\left(\{(\omega_n\Lambda_n)U^{-1}\}, U\mathfrak{U}\right)$ is a weighted $g-$Banach frame for $\mathcal{Y}$ with respect to $\mathcal{X}_d$ and its best bounds $C,\ D$ satisfy the inequalities*

$$C_\Lambda\|U\|^{-1} \leq C \leq C_\Lambda\|U^{-1}\| \text{ and } D_\Lambda\|U\|^{-1} \leq D \leq D_\Lambda\|U^{-1}\|.$$

Proof. By hypothesis

$$C_\Lambda\|x\|_{\mathcal{X}} \leq \|\{(\omega_n\Lambda_n)(x)\}\|_{\mathcal{X}_d} \leq \|x\|_{\mathcal{X}},\ x \in \mathcal{X}. \tag{3.4}$$

Let $y \in \mathcal{Y}$ be any arbitrary. Then

$$\begin{aligned} \|y\|_{\mathcal{Y}} &= \|UU^{-1}(y)\|_{\mathcal{Y}} \\ &\leq \|U\|\|U^{-1}(y)\|_{\mathcal{X}}. \end{aligned}$$

Then, by using (3.4), we have

$$\begin{aligned} C_\Lambda\|U\|^{-1}\|y\|_{\mathcal{Y}} &\leq C_\Lambda\|U^{-1}(y)\|_{\mathcal{X}} \\ &\leq \|\{(\omega_n\Lambda_n)(U^{-1}(y))\}\|_{\mathcal{X}_d} \\ &\leq D_\Lambda\|U^{-1}(y)\|_{\mathcal{X}} \\ &\leq D_\Lambda\|U^{-1}\|\|y\|_{\mathcal{Y}}. \end{aligned}$$

Now, $\{(\omega_n\Lambda_n)(U^{-1}(y))\} \in \mathcal{X}_d$, for all $y \in \mathcal{Y}$ and $U(\mathfrak{U}(\{(\omega_n\Lambda_n)(U^{-1}(y))\})) = y$, for all $y \in \mathcal{Y}$. Hence $(\{(\omega_n\Lambda_n)(U^{-1})\}, U\mathfrak{U})$ is a weighted $g-$Banach frame for $\mathcal{Y}$ with respect

to $\mathcal{X}_d$ and with bounds $C_\Lambda\|U\|^{-1},\ D_\Lambda\|U^{-1}\|$.
Let C and D be the best bounds for $(\{(\omega_n\Lambda_n)(U^{-1})\}, U\mathfrak{U})$, then

$$C_\Lambda\|U\|^{-1} \leq C \text{ and } D \leq D_\Lambda\|U^{-1}\|. \tag{3.5}$$

Now, for all $x \in \mathcal{X}$, we have

$$\|x\|_{\mathcal{X}} = \|U^{-1}U(x)\|_{\mathcal{X}} \leq \|U^{-1}\|\|U(x)\|_{\mathcal{Y}}.$$

Therefore

$$\begin{aligned} C\|U^{-1}\|^{-1}\|x\|_{\mathcal{X}} &\leq C\|U(x)\|_{\mathcal{Y}} \\ &\leq \|\{(\omega_n\Lambda_n)U^{-1}(Ux)\}\|_{\mathcal{X}_d}\Big(= \|\{(\omega_n\Lambda_n)(x)\}\|_{\mathcal{X}_d}\Big) \\ &\leq D\|U(x)\|_{\mathcal{Y}} \\ &\leq D\|U\|\|x\|_{\mathcal{X}}. \end{aligned}$$

By using the fact that C_Λ and D_Λ are the best bounds for $(\{\omega_n\Lambda_n\}, \mathfrak{U})$, we get

$$C\|U^{-1}\|^{-1} \leq C_\Lambda \text{ and } D_\Lambda \leq D\|U\|. \tag{3.6}$$

Hence, by combining (3.5) and (3.6), we get
$C_\Lambda\|U\|^{-1} \leq C \leq C_\Lambda\|U^{-1}\|$ and $D_\Lambda\|U\|^{-1} \leq D \leq D_\Lambda\|U^{-1}\|$. □

4. Construction of Weighted $G-$Banach Frame for a Banach Space $\mathcal{X}$ from Bounded Linear Operators on $\mathcal{X}_d$

Let $(\{\omega_n\Lambda_n\}, \mathfrak{U})$ be a weighted $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$ and let $\{\mathfrak{T}_n\} \subset B(\mathcal{X}, \mathcal{H})$. Let $W : \mathcal{X}_d \to \mathcal{X}_d$ be a bounded linear operator such that $W(\{(\omega_n\Lambda_n)(x)\}) = \{(\omega_n\mathfrak{T}_n)(x)\},\ x \in \mathcal{X}$. Then, in general, there exists no reconstruction operator $\tilde{\mathfrak{U}} : \mathcal{X}_d \to \mathcal{X}$ such that $(\{\omega_n\mathfrak{T}_n\}, \tilde{\mathfrak{U}})$ is a weighted $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$.
In this regard, we give the following example:

Example 4.1. A pair $(\{\omega_n\Lambda_n\}, \mathfrak{U})$ is a weighted $g-$Banach frame for E with respect to E_d (Example 3.4(1)). Define $W : E_d \to E_d$ as $W(\{(\omega_n\Lambda_n)(x)\}) = \{(\omega_n\Lambda_n)(x)\}_{n\neq i}$, for some $i \in \mathbb{N}$. Then W is a bounded linear operator on E_d. But, there exists no reconstruction operator $\tilde{\mathfrak{U}} : E_d \to E$ such that $(\{\omega_n\Lambda_n\}_{n\neq i}, \tilde{\mathfrak{U}})(i \in \mathbb{N})$ is a weighted $g-$Banach frame for E with respect to E_d.
Indeed, if possible $(\{\omega_n\Lambda_n\}_{n\neq i}, \tilde{\mathfrak{U}})(i \in \mathbb{N})$ is a weighted $g-$Banach frame for E with respect to E_d. Let C_1 and D_1 be the choice of bounds for weighted $g-$Banach frame $(\{\omega_n\Lambda_n\}_{n\neq i}, \tilde{\mathfrak{U}})(i \in \mathbb{N})$. Then

$$C_1\|x\|_E \leq \|\{(\omega_n\Lambda_n)(x)\}_{n\neq i}\|_{E_d} \leq D_1\|x\|_E,\ x \in \mathcal{X},\ \text{for some } i \in \mathbb{N}. \tag{4.1}$$

Let $x = (0, \ldots, 0, \underbrace{1}_{i^{th}\text{place}}, 0, \ldots)$ be any non-zero element in E. Then $(\omega_n\Lambda_n)(x) = 0$ for $n \neq i,\ i \in \mathbb{N}$. Thus, by frame inequality (4.1), we get $x = 0$. Which is a contradiction.

In view of the Example 4.1, we give the following result:

Theorem 4.2. *Let $(\{\omega_n\Lambda_n\}, \mathfrak{U})$ be a weighted $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$. Let $\{\mathfrak{T}_n\} \subset B(\mathcal{X}, \mathcal{H})$ be such that $\{(\omega_n\mathfrak{T}_n)(x)\} \in \mathcal{X}_d,\ x \in \mathcal{X}$. Let $W : \mathcal{X}_d \to \mathcal{X}_d$ be a bounded linear operator such that $W(\{(\omega_n\Lambda_n)(x)\}) = \{(\omega_n\mathfrak{T}_n)(x)\},\ x \in \mathcal{X}$. Then,* *there exists a reconstruction operator* $\tilde{\mathfrak{U}} : \mathcal{X}_d \to \mathcal{X}$ *such that* $(\{\omega_n \mathfrak{T}_n\}, \tilde{\mathfrak{U}})$ *is a weighted* $g-$*Banach frame for* $\mathcal{X}$ *with respect to* $\mathcal{X}_d$ *if and only if*

$$\|\{(\omega_n \mathfrak{T}_n)(x)\}\|_{\mathcal{X}_d} \geq M \|L(\{(\omega_n \mathfrak{T}_n)(x)\})\|_{\mathcal{X}_d},\ x \in \mathcal{X},$$

where, M *is a positive constant and* $L : \mathcal{X}_d \to \mathcal{X}_d$ *is a bounded linear operator such that* $L(\{(\omega_n \mathfrak{T}_n)(x)\}) = \{(\omega_n \Lambda_n)(x)\},\ x \in \mathcal{X}$.

Proof. Let C_1 and D_1 be the bounds for the weighted $g-$Banach frame $(\{\omega_n \Lambda_n\}, \mathfrak{U})$. Then, for all $x \in \mathcal{X}$, we have

$$\begin{aligned} MC_1 \|x\|_{\mathcal{X}} &\leq M \|\{(\omega_n \Lambda_n)(x)\}\|_{\mathcal{X}_d} \\ &= M \|L(\{(\omega_n \mathfrak{T}_n)(x)\})\|_{\mathcal{X}_d} \\ &\leq \|\{(\omega_n \mathfrak{T}_n)(x)\}\|_{\mathcal{X}_d} \\ &= \|W(\{(\omega_n \Lambda_n)(x)\})\|_{\mathcal{X}_d} \\ &\leq \|W\| \|\{(\omega_n \Lambda_n)(x)\}\|_{\mathcal{X}_d} \\ &\leq \|W\| D_1 \|x\|_{\mathcal{X}}. \end{aligned}$$

Therefore,

$$(MC_1)\|x\|_{\mathcal{X}} \leq \|\{(\omega_n \mathfrak{T}_n)(x)\}\|_{\mathcal{X}_d} \leq (\|W\| D_1)\|x\|_{\mathcal{X}},\ x \in \mathcal{X}.$$

Put $\tilde{\mathfrak{U}} = \mathfrak{U}L$. Then $\tilde{\mathfrak{U}} : \mathcal{X}_d \to \mathcal{X}$ is a bounded linear operator such that $\tilde{\mathfrak{U}}(\{(\omega_n \mathfrak{T}_n)(x)\}) = x,\ x \in \mathcal{X}$. Hence $(\{\omega_n \mathfrak{T}_n\}, \tilde{\mathfrak{U}})$ is a weighted $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$.

Conversely, let $(\{\omega_n \mathfrak{T}_n\}, \tilde{\mathfrak{U}})$ be a weighted $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$ and with frame bounds C_2, D_2. Let $T : \mathcal{X} \to \mathcal{X}_d$ be the coefficient mapping given by $T(x) = \{(\omega_n \Lambda_n)(x)\}$, $x \in \mathcal{X}$.Then $L = T\tilde{\mathfrak{U}} : \mathcal{X}_d \to \mathcal{X}_d$ is a bounded linear operator such that $L(\{(\omega_n \mathfrak{T}_n)(x)\}) = \{(\omega_n \Lambda_n)(x)\}$, $x \in \mathcal{X}$. Thus, on writing $M = \dfrac{C_2}{D_1}$, where D_1 is an upper bound for weighted $g-$Banach frame $(\{\omega_n \Lambda_n\}, \mathfrak{U})$, we have

$$\begin{aligned} \|\{(\omega_n \mathfrak{T}_n)(x)\}\|_{\mathcal{X}_d} &\geq C_2 \|x\|_{\mathcal{X}} \\ &= MD_1 \|x\|_{\mathcal{X}} \\ &\geq M \|\{(\omega_n \Lambda_n)(x)\}\|_{\mathcal{X}_d} \\ &= M \|L(\{(\omega_n \mathfrak{T}_n)(x)\})\|_{\mathcal{X}_d}. \end{aligned}$$

□

Next, we give the result which provide the better Bessel bound for the sum of two weighted $g-$Banach frames.

Theorem 4.3. *Let* $(\{\omega_n \Lambda_n\}, \mathfrak{U})$ *and* $(\{\omega_n \mathfrak{T}_n\}, \tilde{\mathfrak{U}})$ *be two weighted* $g-$*Banach frames for* $\mathcal{X}$ *with respect to* $\mathcal{X}_d$. *Let* $L : \mathcal{X}_d \to \mathcal{X}_d$ *be a linear homeomorphism such that* $L(\{(\omega_n \Lambda_n)(x)\}) = \{(\omega_n \mathfrak{T}_n)(x)\}$, $x \in \mathcal{X}$. *Then* $\{\omega_n \Lambda_n + \omega_n \mathfrak{T}_n\}$ *is a weighted* $g-$*Bessel sequence for* $\mathcal{X}$ *with respect to* $\mathcal{X}_d$ *and with bound* $K = min\{\|U\| \|I + L\|, \|V\| \|I + L^{-1}\|\}$ *where* $U, V : \mathcal{X} \to \mathcal{X}_d$ *are the coefficient mappings given by* $U(x) = \{(\omega_n \Lambda_n)(x)\}$ *and* $V(x) = \{(\omega_n \mathfrak{T}_n)(x)\}$, $x \in \mathcal{X}$ *and* I *is an identity mapping on* $\mathcal{X}_d$.

Proof. For each $x \in \mathcal{X}$, we have

$$\begin{aligned} \|\{(\omega_n\Lambda_n + \omega_n\mathfrak{T}_n)(x)\}\|_{\mathcal{X}_d} &= \|\{(\omega_n\Lambda_n)(x)\} + \{(\omega_n\mathfrak{T}_n)(x)\}\|_{\mathcal{X}_d} \\ &= \|I(\{(\omega_n\Lambda_n)(x)\}) + L(\{(\omega_n\Lambda_n)(x)\})\|_{\mathcal{X}_d} \\ &\leq \|I + L\|\|\{(\omega_n\Lambda_n)(x)\}\|_{\mathcal{X}_d} \\ &= \|I + L\|\|U(x)\|_{\mathcal{X}_d} \\ &\leq (\|I + L\|\|U\|)\|x\|_{\mathcal{X}}. \end{aligned}$$

Therefore

$$\|\{(\omega_n\Lambda_n + \omega_n\mathfrak{T}_n)(x)\}\|_{\mathcal{X}_d} \leq (\|U\|\|I + L\|)\|x\|_{\mathcal{X}}, x \in \mathcal{X}. \tag{4.2}$$

Similarly, we have

$$\|\{(\omega_n\Lambda_n + \omega_n\mathfrak{T}_n)(x)\}\|_{\mathcal{X}_d} \leq (\|V\|\|I + L^{-1}\|)\|x\|_{\mathcal{X}}, x \in \mathcal{X}. \tag{4.3}$$

Using (4.2) and (4.3), it follows that $\{\omega_n\Lambda_n + \omega_n\mathfrak{T}_n\}$ is a weighted $g-$Banach Bessel sequence for $\mathcal{X}$ with respect to $\mathcal{X}_d$ and with bound $K = min\{\|U\|\|I + L\|, \|V\|\|I + L^{-1}\|\}$. □

Remark 4.4. One may observe that, if $(\{\omega_n\Lambda_n\}, \mathfrak{U})$ and $(\{\omega_n\mathfrak{T}_n\}, \tilde{\mathfrak{U}})$ are two weighted $g-$Banach frames for $\mathcal{X}$ with respect to $\mathcal{X}_d$, then $\{\omega_n\Lambda_n+\omega_n\mathfrak{T}_n\}$ is a weighted $g-$Banach Bessel sequence for $\mathcal{X}$ with respect to $\mathcal{X}_d$ with bound $\|U\| + \|V\|$, where $U, V : \mathcal{X} \to \mathcal{X}_d$ are the coefficient mappings given by $U(x) = \{(\omega_n\Lambda_n)(x)\}$ and $V(x) = \{(\omega_n\mathfrak{T}_n)(x)\}$, $x \in \mathcal{X}$.

Theorem 4.5. *Let $(\{\omega_n\Lambda_n\}, \mathfrak{U})$ be a weighted $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$ and with bounds C, D. Let $\{\omega_n\mathfrak{T}_n\}$ be a weighted $g-$Banach Bessel sequence for $\mathcal{X}$ with respect to $\mathcal{X}_d$ with bound M. Then, there exists a reconstruction operator $\tilde{\mathfrak{U}}$ such that $\left(\{\omega_n(\Lambda_n \pm \mathfrak{T}_n)\}, \tilde{\mathfrak{U}}\right)$ is a weighted $g-$Banach frame for $\mathcal{X}$, provided $M < C$.*

Proof. Let U and V be the coefficient mappings defined as $U(x) = \{(\omega_n\Lambda_n)(x)\}$ and $V(x) = \{(\omega_n\mathfrak{T}_n)(x)\}$, $x \in \mathcal{X}$. For each $x \in \mathcal{X}$, we have

$$\begin{aligned} \|\{(\omega_n(\Lambda_n \pm \mathfrak{T}_n))(x)\}\|_{\mathcal{X}_d} &= \|U(x) \pm V(x)\|_{\mathcal{X}_d} \\ &\geq \|U(x)\|_{\mathcal{X}_d} - \|V(x)\|_{\mathcal{X}_d} \\ &\geq (C - M)\|x\|_{\mathcal{X}}. \end{aligned}$$

Similarly, we have

$$\|\{(\omega_n(\Lambda_n \pm \mathfrak{T}_n))(x)\}\|_{\mathcal{X}_d} \leq (D + M)\|x\|_{\mathcal{X}},\ x \in \mathcal{X}.$$

Thus, we have

$$(C - M)\|x\|_{\mathcal{X}} \leq \|\{(\omega_n(\Lambda_n \pm \mathfrak{T}_n))(x)\}\|_{\mathcal{X}_d} \leq (D + M)\|x\|_{\mathcal{X}},\ x \in \mathcal{X}. \tag{4.4}$$

Let $x \in \mathcal{X}$ be any non-zero element such that $(\omega_n(\Lambda_n \pm \mathfrak{T}_n))(x) = 0$. Then, by inequality (4.4), $x = 0$. which is a contradiction. Therefore $\{\omega_n(\Lambda_n \pm \mathfrak{T}_n)\}$ is total over $\mathcal{X}$. Thus, by Lemma 2.2, there exists an associated Banach space $\mathcal{A} = \left\{\{(\omega_n(\Lambda_n \pm \mathfrak{T}_n))(x)\} : x \in \mathcal{X}\right\}$ with the norm given by $\left\|\{(\omega_n(\Lambda_n \pm \mathfrak{T}_n))(x)\}\right\|_{\mathcal{A}} = \|x\|_{\mathcal{X}},\ x \in \mathcal{X}$. Define an operator

$\tilde{\mathfrak{U}} : \mathcal{A} \to \mathcal{X}$ by $\tilde{\mathfrak{U}}(\{(\omega_n(\Lambda_n \pm \mathfrak{T}_n))(x)\}) = x,\ x \in \mathcal{X}$. Hence $\left(\{\omega_n(\Lambda_n \pm \mathfrak{T}_n)\}, \tilde{\mathfrak{U}}\right)$ is a weighted $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{A}$. □

Next, we give a necessary condition for a weighted $g-$Banach frame, in terms of weighted $g-$Banach Bessel sequence.

Theorem 4.6. *Let $(\{\omega_n\Lambda_n\}, \mathfrak{U})$ be a weighted $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$. Let $\{\mathfrak{T}_n\} \subset B(\mathcal{X}, H)$ be a sequence of operators such that $\{\omega_n\Lambda_n + \omega_n\mathfrak{T}_n\}$ is a weighted $g-$Banach Bessel sequence for $\mathcal{X}$ with respect to $\mathcal{X}_d$ and with bound $K < \|\mathfrak{U}\|^{-1}$. Then there exists a reconstruction operator $\tilde{\mathfrak{U}} : \mathcal{A} \to \mathcal{X}$ such that $(\{\omega_n\mathfrak{T}_n\}, \tilde{\mathfrak{U}})$ is a weighted $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{A} = \{\{(\omega_n\mathfrak{T}_n)(x)\} : x \in \mathcal{X}\}$.*

Proof. Since $\{\omega_n\Lambda_n + \omega_n\mathfrak{T}_n\}$ is a weighted $g-$Banach Bessel sequence for $\mathcal{X}$ with respect to $\mathcal{X}_d$ and with bound K, then

$$\begin{aligned}\left(\|\mathfrak{U}\|^{-1} - K\right)\|x\|_{\mathcal{X}} &\leq \|\{(\omega_n\Lambda_n)(x)\}\|_{\mathcal{X}_d} - \|\{(\omega_n(\Lambda_n + \mathfrak{T}_n))(x)\}\|_{\mathcal{X}_d} \\ &\leq \|\{(\omega_n\mathfrak{T}_n)(x)\}\|_{\mathcal{X}_d}, \text{ for all } x \in \mathcal{X}.\end{aligned}$$

Thus, $\{\omega_n\mathfrak{T}_n\}$ is total over $\mathcal{X}$. Therefore, by Lemma 2.2, there exists an associated Banach space $\mathcal{A} = \{\{(\omega_n\mathfrak{T}_n)(x)\} :\ x \in \mathcal{X}\}$ with the norm given by $\|\{(\omega_n\mathfrak{T}_n)(x)\}\|_{\mathcal{A}} = \|x\|_{\mathcal{X}},\ x \in \mathcal{X}$. Define $\tilde{\mathfrak{U}} : \mathcal{A} \to \mathcal{X}$ by $\tilde{\mathfrak{U}}(\{(\omega_n\mathfrak{T}_n)(x)\}) = x,\ x \in \mathcal{X}$. Then $\tilde{\mathfrak{U}}$ is a bounded linear operator such that $(\{\omega_n\mathfrak{T}_n\}, \tilde{\mathfrak{U}})$ is a weighted $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{A}$. □

5. Finite Sum of Weighted $G-$Banach Frames

Let $(\{\omega_n\Lambda_{i,n}\}, \mathfrak{U}_i),\ i = 1, 2, \ldots, k$ be weighted $g-$Banach frames for $\mathcal{X}$ with respect to $\mathcal{X}_d$ and $\mathcal{H}$. Then, there exists, in general no reconstruction operator $\tilde{\mathfrak{U}}$ such that $\left(\{\sum_{i=1}^{k} \omega_n\Lambda_{i,n}\}, \tilde{\mathfrak{U}}\right)$ is a weighted $g-$Banach frame for $\mathcal{X}$.

Regarding this, we have the following example:

Example 5.1. Let $\mathcal{H}$ be a separable Hilbert space and let $E = \ell^\infty(\mathcal{H}) = \{\{x_n\} : x_n \in \mathcal{H}; \sup_{1 \leq n < \infty} \|x_n\|_{\mathcal{H}} < \infty\}$ be a Banach space with the norm given by

$$\|\{x_n\}\|_E = \sup_{1 \leq n < \infty} \|x_n\|_{\mathcal{H}},\ \{x_n\} \in E.$$

Let $\omega = \{\omega_n = 2^{1-n}\}$ be weight and for each $n \in \mathbb{N}$, define $\Lambda_{1,n} : E \to \ell^2(\mathcal{H})$ by

$$\Lambda_{1,n}(x) = \delta_n^{x_n} - 2^{n-1}\delta_{n+1}^{x_n},\ x = \{x_n\} \in E,$$

where $\delta_n^x = (0, \ldots, 0, \underbrace{x}_{n^{th}\text{place}}, 0, \ldots)$ for all $n \in \mathbb{N}$ and $x \in \mathcal{H}$.

Then $\{\omega_n\Lambda_{1,n}\}$ is total over E. Thus, by Lemma (2.2), there exists an associated Banach space $\mathcal{A}_1$ and an operator $\mathfrak{U}_1 : \mathcal{A}_1 \to E$ such that $(\{\omega_n\Lambda_{1,n}\}, \mathfrak{U}_1)$ is a weighted $g-$Banach frame for E with respect to $\mathcal{A}_1$.

Now, define a sequence of operators $\Lambda_{2,n} \subset B(E, \ell^2(\mathcal{H}))$ by

$$\begin{cases}\Lambda_{2,1}(x) = \delta_2^{x_1} - \delta_1^{x_1} \\ \Lambda_{2,n}(x) = \delta_n^{x_n} - 2^{n-1}\delta_{n+1}^{x_n},\ n \geq 2,\ n \in \mathbb{N},\ x = \{x_n\} \in E.\end{cases}$$

Then, by Lemma (2.2), there exists an associated Banach space $\mathcal{A}_2$ and an operator $\mathfrak{U}_2 : \mathcal{A}_2 \to E$ such that $(\{\omega_n\Lambda_{2,n}\}, \mathfrak{U}_2)$ is a weighted $g-$Banach frame for E with respect to $\mathcal{A}_2$. But there exists no reconstruction operator $\tilde{\mathfrak{U}}$ such that $\bigg(\{\sum_{i=1}^{2}\omega_n\Lambda_{i,n}\}, \tilde{\mathfrak{U}}\bigg)$ is a weighted $g-$Banach frame for E with respect to some associated Banach space $\mathcal{A}$. Indeed, let C_0 and D_0 be the choice of bounds for $\bigg(\{\sum_{i=1}^{2}\omega_n\Lambda_{i,n}\}, \tilde{\mathfrak{U}}\bigg)$.
Then

$$C_0\|x\|_E \leq \left\|\left\{\bigg(\sum_{i=1}^{2}\omega_n\Lambda_{i,n}\bigg)(x)\right\}\right\|_A \leq D_0\|x\|_E,\ x \in E. \tag{5.1}$$

Let $x = (1, 0, 0, \ldots)$ be a non-zero element in E. Then $(\sum_{i=1}^{2}\omega_n\Lambda_{i,n})(x) = 0$. Thus, by frame inequality (5.1), we get $x = 0$. Which is a contradiction. Hence $\bigg(\{\sum_{i=1}^{2}\omega_n\Lambda_{i,n}\}, \tilde{\mathfrak{U}}\bigg)$ is not a weighted $g-$Banach frame for E with respect to some associated Banach space $\mathcal{A}$.

In view of example 5.1, we give a condition under which the finite sum of weighted $g-$Banach frame for a Banach space $\mathcal{X}$ is a weighted $g-$Banach frame for $\mathcal{X}$.

Theorem 5.2. *Let $(\{\omega_n\Lambda_{i,n}\}, \mathfrak{U}_i)$, $i = 1, 2, ..., k$ be a weighted $g-$Banach frame $\mathcal{X}$ with respect to $\mathcal{X}_d$. Then, there exists a reconstruction operator $\tilde{\mathfrak{U}}$ such that $\bigg(\{\sum_{i=1}^{k}\omega_n\Lambda_{i,n}\}, \tilde{\mathfrak{U}}\bigg)$ is a weighted $g-$Banach frame for $\mathcal{X}$, provided*

$$\|\{(\omega_n\Lambda_{p,n})(x)\}\|_{\mathcal{X}_d} \leq \left\|\left\{\bigg(\sum_{i=1}^{k}\omega_n\Lambda_{i,n}\bigg)(x)\right\}\right\|_{\mathcal{X}_d}, x \in \mathcal{X},$$

for some $p \in \{1, 2, ..., k\}$.

Proof. Since for each $i = 1, 2, ..., k$, $(\{\omega_n\Lambda_{i,n}\}, \mathfrak{U}_i)$ is a weighted $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$. Then

$$\begin{aligned} \|x\|_{\mathcal{X}} &= \|\mathfrak{U}_p(\{(\omega_n\Lambda_{p,n})(x)\})\|_{\mathcal{X}} \\ &\leq \|\mathfrak{U}_p\|\left\|\left\{\bigg(\sum_{i=1}^{k}\omega_n\Lambda_{i,n}\bigg)(x)\right\}\right\|_{\mathcal{X}_d},\ x \in \mathcal{X}. \end{aligned}$$

Then

$$\|\mathfrak{U}_p\|^{-1}\|x\|_{\mathcal{X}} \leq \left\|\left\{\bigg(\sum_{i=1}^{k}\omega_n\Lambda_{i,n}\bigg)(x)\right\}\right\|_{\mathcal{X}_d},\ x \in \mathcal{X}.$$

Therefore, by Lemma (2.2), there exists an associated Banach space

$$\mathcal{A} = \left\{\left\{\bigg(\sum_{i=1}^{k}\omega_n\Lambda_{i,n}\bigg)(x)\right\} :\ x \in \mathcal{X}\right\}$$

with the norm given by $\left\|\left\{\left(\sum_{i=1}^{k}\omega_n\Lambda_{i,n}\right)(x)\right\}\right\|_{\mathcal{A}} = \|x\|_{\mathcal{X}},\ x \in \mathcal{X}$. Define $\tilde{\mathfrak{U}} : \mathcal{A} \to \mathcal{X}$ by $\tilde{\mathfrak{U}}\left(\left\{\left(\sum_{i=1}^{k}\omega_n\Lambda_{i,n}\right)(x)\right\}\right) = x,\ x \in \mathcal{X}$. Thus $\tilde{\mathfrak{U}}$ is a bounded linear operator such that $\left(\left\{\sum_{i=1}^{k}\omega_n\Lambda_{i,n}\right\}, \tilde{\mathfrak{U}}\right)$ is a weighted $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{A}$. □

Remark 5.3. Towards the converse of the Theorem 5.2, we observe that $\left(\left\{\sum_{i=1}^{k}\omega_n\Lambda_{i,n}\right\}, \tilde{\mathfrak{U}}\right)$ is a weighted $g-$Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$. Then, there exists, in general, no reconstruction operators $\mathfrak{U}_i$, for $i = 1, 2 \dots, k$ such that $\left(\left\{\omega_n\Lambda_{i,n}\right\}, \mathfrak{U}_i\right)$, $i = 1, 2, \dots, k$ is a weighted $g-$Banach frame for $\mathcal{X}$ with respect to some associated Banach space $\mathcal{A}_0$.

Regarding this, we give the following example:

Example 5.4. Let $\mathcal{H}$ be a separable Hilbert space and $E = \ell^\infty(\mathcal{H}) = \{\{x_n\} : x_n \in \mathcal{H}; \sup_{1\le n<\infty}\|x_n\|_{\mathcal{H}} < \infty\}$ be a Banach space with the norm given by

$$\|\{x_n\}\|_E = \sup_{1\le n<\infty}\|x_n\|_{\mathcal{H}},\ \{x_n\} \in E.$$

Let $\omega = \begin{cases} \omega_1 = \dfrac{1}{2} \\ \omega_n = 1, n \ge 2, n \in \mathbb{N} \end{cases}$ be weight and and for each $n \in \mathbb{N}$, define $\{\Lambda_{1,n}\}$, $\{\Lambda_{2,n}\} \subset B(E, \ell^2(\mathcal{H}))$ as

$$\begin{cases} \Lambda_{1,1}(x) = 0,\ \Lambda_{1,n}(x) = \delta_n^{x_n},\ n > 1 \\ \Lambda_{2,1}(x) = 2\delta_1^{x_1},\ \Lambda_{2,n}(x) = 0, n > 1,\ x = \{x_n\} \in E. \end{cases}$$

Then $\left\{\sum_{i=1}^{2}\omega_n\Lambda_{i,n}\right\}$ is total over E. Therefore, by Lemma 2.2, there exists an associated Banach space $\mathcal{A} = \left\{\left\{\left(\sum_{i=1}^{2}\omega_n\Lambda_{i,n}\right)(x)\right\} : x \in E\right\}$ with the norm given by $\left\|\left\{\left(\sum_{i=1}^{2}\omega_n\Lambda_{i,n}\right)(x)\right\}\right\|_{\mathcal{A}} = \|x\|_E,\ x \in E$. Define $\tilde{\mathfrak{U}} : \mathcal{A} \to E$ by $\tilde{\mathfrak{U}}\left(\left\{\left(\sum_{i=1}^{2}\omega_n\Lambda_{i,n}\right)(x)\right\}\right) = x,\ x \in E$. Then $\tilde{\mathfrak{U}}$ is a bounded linear operator such that $\left(\left\{\sum_{i=1}^{2}\omega_n\Lambda_{i,n}\right\}, \tilde{\mathfrak{U}}\right)$ is a weighted $g-$Banach frame for E with respect $\mathcal{A}$. But there exist no reconstruction operators $\mathfrak{U}_1$ and $\mathfrak{U}_2$ such that $\left(\left\{\omega_n\Lambda_{1,n}\right\}, \mathfrak{U}_1\right)$ and $\left(\left\{\omega_n\Lambda_{2,n}\right\}, \mathfrak{U}_2\right)$ are weighted $g-$Banach frame for E.

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