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On some classes of retro Banach frames in Banach spaces

MAYUR PURI GOSWAMI

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Abstract. In this paper, we study certain classes of retro Banach frames, namely, retro bi-Banach frame, retro Bessel sequences. In the sequel, we characterize retro Bessel sequences with the help of examples and establish some relations between retro Bessel sequence and retro Banach frame. Finally, results on the existence of retro bi-Banach frames have been obtained.

1. Introduction

In 1952, frames for Hilbert spaces were introduced by Duffin and Schaeffer [5], in the context of non-harmonic Fourier series. Nowadays frames play a crucial role in various branches of pure and applied mathematics and engineering sciences. In 1988, Hilbert frames were generalized to Banach spaces by Feichtinger and Gröchenig [6], called atomic decomposition. For more on Banach frame literature one may refer to [2, 11, 3].

In 2004, Jain et al. [10] extended Banach frames to the conjugate Banach spaces and introduced retro Banach frames. Recently, Shalu Sharma [15], introduced a notion of the bi-Banach frame as a pair of Banach frame and retro Banach frame. Motivated by the paper [15], Goswami et al. [7] defined the retro bi-Banach frame in conjugate Banach spaces and established some relations of retro bi-Banach frame with some existing notions in Banach frame theory. Bessel sequences, another important notion in the frame theory were studied in [12, 13]. In this paper, we defined retro Bessel sequences and obtained some results regarding construction of retro Bessel sequences and retro bi-Banach frames. Our results extend some previous results proved for G-Banach Bessel sequences [13] and strong retro Banach frames [9].

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2. Preliminaries

Throughout the paper, a Banach space and its first dual will be denoted by E and E^*, respectively, over the scalar field $\mathbb{K}(\mathbb{R}$ or $\mathbb{C})$. E_d and $(E^*)_d$ denotes the Banach spaces of scalar valued sequences associated with E and E^*, respectively. Also, $[x_n]$ denotes the closed linear subspace of E, spanned by $\{x_n\}$. The set of positive integers will be denoted by $\mathbb{N}$.

A sequence $\{x_n\}$ in E is said to be complete if $[x_n] = E$ and is said to be total if $\{f \in E^* : f(x_n) = 0, \forall n \in \mathbb{N}\} = \{0\}$. Also, if $M \subseteq E^*$, then we define $\gamma(M) = \inf\limits_{\substack{x \in E \\ x \neq 0}} \sup\limits_{\substack{f \in M \\ \|f\| \leq 1}} \left| f\left(\frac{x}{\|x\|}\right)\right|$.

Definition 2.1 ([8]). Let E be a Banach space and E_d be an associated Banach space of scalar valued sequences, indexed by $\mathbb{N}$. Let $\{f_n\} \subset E^*$ and $S : E_d \to E$ be given. The pair $(\{f_n\}, S)$ is called a *Banach frame* for E with respect to E_d if

(i) $\{f_n(x)\} \in E_d$, for each $x \in E$

(ii) there exist constants A_1 and A_2 with $0 < A_1 \leq A_2 < \infty$ such that

$$A_1\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq A_2\|x\|_E, \qquad x \in E \tag{2.1}$$

(iii) S is a bounded linear operator such that

$$S(\{f_n(x)\}) = x, \qquad x \in E.$$

The positive constants A_1 and A_2 are called *lower* and *upper frame bounds* of the Banach frame $(\{f_n\}, S)$, respectively. The operator $S : E_d \to E$ is called the *reconstruction operator*. The inequality (2.1) is called the *Banach frame inequality*.

Definition 2.2 ([10]). Let E be a Banach space and E^* be its conjugate space. Let $(E^*)_d$ be a Banach space of scalar valued sequences indexed by $\mathbb{N}$ associated with E^*. Let $\{x_n\}$ be a sequence in E and $T : (E^*)_d \to E^*$ be given. The pair $(\{x_n\}, T)$ is called a *retro Banach frame*(RBF) for E^* with respect to $(E^*)_d$ if

(i) $\{f(x_n)\} \in (E^*)_d$, for each $f \in E^*$

(ii) there exist constants A_1 and A_2 with $0 < A_1 \leq A_2 < \infty$ such that

$$A_1||f||_{E^*} \leq ||\{f(x_n)\}||_{(E^*)_d} \leq A_2||f||_{E^*}, \qquad \text{for all } f \in E^* \tag{2.2}$$

(iii) T is a bounded linear operator such that

$$T(\{f(x_n)\}) = f, \qquad \text{for all } f \in E^*.$$

The positive constants A_1 and A_2 are called, *lower* and *upper frame bounds* of the retro Banach frame $(\{x_n\}, T)$, respectively. The operator $T : (E^*)_d \to E^*$ is called the *reconstruction operator* for the retro Banach frame $(\{x_n\}, T)$. The inequality (2.2) is called the *retro frame inequality*.

A retro Banach frame $(\{x_n\}, T)$ is called *tight* if $A_1 = A_2$ and is called *normalized tight* if $A_1 = A_2 = 1$. If removal of one x_n renders the collection $\{x_n\} \subset E$ no longer a retro Banach frame for E^*, then $(\{x_n\}, T)$ is called an *exact* retro Banach frame.

Definition 2.3 ([15]). Let E be a Banach space. A pair $(\{x_n\}, \{f_n\})$ (where $\{f_n\} \subset E^*$ and $\{x_n\} \subset E$) is called a *bi-Banach frame* for E if there exist associated Banach spaces E_d and $(E^*)_d$ and bounded linear operators $S : E_d \to E$, $T : (E^*)_d \to E^*$ such that $(\{f_n\}, S)$ is a Banach frame for E and $(\{x_n\}, T)$ is a retro Banach frame for E^*.

Definition 2.4. [7] Let E be a Banach space. Let $\{x_n\}$ and $\{f_n\}$ be sequences in E and E^*, respectively. Then the system $(\{x_n\}, \{f_n\})$ is called a *retro bi-Banach frame* for E^* if there exist associated Banach spaces $(E^*)_d$ and $(E^{**})_d$ and bounded linear operators $S : (E^*)_d \to E^*$, $T : (E^{**})_d \to E^{**}$ such that $(\{x_n\}, S)$ is a retro Banach frame for E^* with respect to $(E^*)_d$ and $(\{f_n\}, T)$ is a retro Banach frame for E^{**} with respect to $(E^{**})_d$.

Definition 2.5. [9] Let $\{x_n\} \subset E$ be an exact retro Banach frame for E^* with admissible sequence $\{f_n\} \subset E^*$. Let $E_n = [x_1, x_2, ..., x_n]$, $n \in \mathbb{N}$. If there exists a sequence $\{v_n\}$, where each $v_n : E_n \to E_n$ is a continuous linear mapping, such that $x = \lim_{n\to\infty} v_n \sum_{i=1}^{n} f_i(x)x_i$, $x \in E$, then $(\{x_n\}, \{f_n\}, \{v_n\})$ is called a strong retro Banach frame (SRBF) for E^*.

We finish this section with some key results that will be used in the subsequent part of the paper.

Lemma 2.6 ([14])**.** *If E is a Banach space and $\{f_n\} \subset E^*$ is total over E, then E is linearly isometric to the BK-space $E_d = \{\{f_n(x)\} : x \in E\}$, where the norm is given by $||\{f_n(x)\}||_{E_d} = ||x||_E$, $x \in E$.*

Lemma 2.7 ([10])**.** *Let E be a Banach space. Then E^* has a retro Banach frame if and only if E is separable.*

3. Retro Bessel sequence

In the last decade, Banach frames in conjugate Banach spaces have been studied with various perspectives. One of them is due to Vashisht [18], where they studied retro Banach frames consisting of eigenfunctions associated with a given boundary value problem. Also, Chugh et al. [4] presented some duality properties of retro Banach frames including appropriate applications. Besides, Rathore and Mittal [13], defined g-Banach Bessel sequence in the context of the g-Banach frame, which was introduced by Abdollahpour et al. [1] as a generalization of g-frames in Banach spaces. More precisely, they discussed some characterizations of g-Banach Bessel sequences and gave some methods regarding the construction of g-Banach Bessel sequences.

Motivated by Rathore and Mittal [13], in this section, we define and study retro Bessel sequences in conjugate Banach spaces under some weaker conditions rather than retro Banach frames. More precisely, in the definition of retro Banach frames, we focus to the reconstruction of vectors of the underlying space by a pre-frame operator, on the other hand, in the retro Bessel sequence, we do not reconstruct the elements of the space, however, we just need to obtain an upper bound satisfying the inequality (3.1). Now, it could be interesting to establish relations between retro Bessel sequence and retro Banach frame. This thought leads us to the various characterizations of retro Bessel sequences which shows the essence of the notion introduced as we discussed subsequently.

Let us define the notion of Retro Bessel sequence as follows.

Definition 3.1. Let E be a Banach space and E^* the dual of E. Let $(E^*)_d$ be a Banach space of scalar valued sequences associated with E^* indexed by $\mathbb{N}$. A sequence $\{x_n\} \subset E$ is said to be retro Bessel sequence for E^* with respect to $(E^*)_d$ if

(a) $\{f(x_n)\} \in (E^*)_d$, $f \in E^*$.

(b) there exists a positive constant M such that

$$\|\{f(x_n)\}\|_{(E^*)_d} \leq M\|f\|_{E^*}, \qquad f \in E^*. \tag{3.1}$$

The positive constant M is called retro Bessel bound for the retro Bessel sequence $\{x_n\}$.

Notice that the following observations shows the essence of retro Bessel sequences in the study of frames in conjugate Banach spaces.

(O1) If $(\{x_n\}, T)$ (where $\{x_n\} \subset E$, $T : (E^*)_d \to E^*$) is a retro Banach frame for E^*, then $\{x_n\}$ is a retro Bessel sequence for E^*.

(O2) A sequence $\{x_n\} \subset E$ is a retro Bessel sequence for E^* with respect to $(E^*)_d$ if and only if the coefficient mapping $T : E^* \to (E^*)_d$ given by $T(f) = \{f(x_n)\}$, $f \in E^*$ is bounded.

(O3) For each $i \in \{1, 2, ..., k\}$, if $\{x_{i,n}\} \subset E$ is a retro Bessel sequence for E^* with respect to $(E^*)_d$ and with bound M_i, then $\{\sum_{i=1}^{k} f(\alpha_i x_{i,n})\}$ is also a retro Bessel sequence for E^* with respect to $(E^*)_d$ with bound $\sum_{i=1}^{k} |\alpha_i| M_i$, where $\alpha_i \in \mathbb{R}$, $i = 1, 2, ..., k$.

(O4) Let $\{x_n\}$ be a retro Bessel sequence for E^* with respect to $(E^*)_d$ with bound B. Let $\{y_n\} \subset E$ be a sequence in E such that $\{x_n + y_n\} \subset E$ is a retro Bessel sequence for E^* with respect to $(E^*)_d$ with bound K. Then $\{y_n\}$ is a retro Bessel sequence for E^* with respect to $(E^*)_d$ with bound $K + B$.

Remark 3.2. The converse of the observation (O1) need not be true. In this regard, we give the following example:

Example 3.3. Let $E = l^\infty(X) = \{\{x_n\} : x_n \in X, \sup \|x_n\|_X < \infty\}$ equipped with the norm

$$\|\{x_n\}\|_E = \sup_{1 \leq n < \infty} \|x_n\|_X, \qquad \{x_n\} \subset E,$$

where $(X, \|.\|)$ is a Banach space. Let $\{x_n\} \subset E$ be a sequence in E and $f = (f_1, f_2, f_3, ...) \in E^*$ such that

$$f(x_1) = f_2, \qquad f(x_n) = f_{n+1}, \ n \geq 2.$$

Then $\{f(x_n)\} \in (E^*)_d$, $f \in E^*$, and $\|\{f(x_n)\}\|_{(E^*)_d} \leq \|f\|_{E^*}$. Which shows that, $\{x_n\}$ is a retro Bessel sequence for E^*. However, by Lemma 2.7, there exists no reconstruction operator $T : (E^*)_d \to E^*$ such that $(\{x_n\}, T)$ is a retro Banach frame for E^* with respect to $(E^*)_d$.

Remark 3.4. In view of observation (O4), there exists, in general, no associated Banach space $(E^*)_{d_0} = \{\{f(y_n)\} : f \in E^*\}$ and reconstruction operator $T_0 : (E^*)_{d_0} \to E^*$ such that $(\{y_n\}, T_0)$ is a retro Banach frame for E^* with respect to $(E^*)_{d_0}$.(see Example 3.5)

Example 3.5. Let $E = l^1$ and $\{x_n\} \subset E$ be the sequence of unit vectors in E. Then there is a bounded linear operator $T : (E^*)_d = \{\{f(x_n)\} : f \in E^*\} \to E^*$ such that $(\{x_n\}, T)$ is a retro Banach frame for E^* with respect to $(E^*)_d$ with the norm given by $\|\{f(x_n)\}\|_{(E^*)_d} = \|f\|_{E^*}$, $f \in E^*$.

Let $\{y_n\} \subset E$ be defined by $y_1 = x_2$ and $y_n = x_n$, $n \geq 2$, $n \in \mathbb{N}$. Then for each $f \in E^*$, $\{f(x_n)\} \in (E^*)_d$ and

$$\begin{aligned}\|\{f(x_n + y_n)\}\| \leq & \|\{f(x_n)\}\| + \|\{f(y_n)\}\| \\ \leq & \|\{f(x_n)\}\| + \|\{f(x_n)\}\| \\ = & 2\|\{f(x_n)\}\| \\ = & 2\|f\|_{E^*}\end{aligned}$$

Therefore, $\{x_n + y_n\} \subset E$, is a retro Bessel sequence for E^* with bound $M = 2$. Thus by observation (O4), $\{y_n\}$ is a retro Bessel sequence for E^* with bound 3. But there exists no associated Banach space $(E^*)_{d_0} = \{\{f(y_n)\} : f \in E^*\}$ and reconstruction operator $T_0 : (E^*)_{d_0} \to E^*$ such that $(\{y_n\}, T_0)$ is a retro Banach frame for E^* with respect to $(E^*)_{d_0}$.

In the following result, we give a condition that if the Bessel bound of retro Bessel sequence $\{x_n + y_n\}$ is $M < \|T\|^{-1}$, then there exists an associated Banach space $(E^*)_{d_0}$ and a reconstruction operator $T_0 : (E^*)_{d_0} \to E^*$ such that $(\{y_n\}, T_0)$ is a retro Banach frame for E^* with respect to $(E^*)_{d_0}$.

Theorem 3.6. *Let $(\{x_n\}, T)$ (where $\{x_n\} \subset E$, $T : (E^*)_d \to E^*$) be a retro Banach frame for E^* with respect to $(E^*)_d$. Let $\{y_n\}$ be a sequence in E such that $\{x_n + y_n\}$ is a retro Bessel sequence for E^* with respect to $(E^*)_d$ and with bound $M < \|T\|^{-1}$. Then there exists a reconstruction operator $T_0 : (E^*)_{d_0} \to E^*$ such that $(\{y_n\}, T_0)$ is a normalized tight retro Banach frame for E^* with respect to $(E^*)_{d_0}$.*

Proof. Let A and B be the retro frame bounds for the retro Banach frame $(\{x_n\}, T)$. Then the retro frame inequality is given by

$$A\|f\|_{E^*} \leq \|\{f(x_n)\}\|_{(E^*)_d} \leq B\|f\|_{E^*}, \qquad f \in E^*.$$

Since $\{x_n + y_n\}$ is a retro Bessel sequence for E^* with respect to $(E^*)_d$ with bound M, we have

$$\|\{f(x_n + y_n)\}\|_{(E^*)_d} \leq M\|f\|_{E^*}, \qquad f \in E^*.$$

Since the operator $T : (E^*)_d \to E^*$ is given by $T(\{f(x_n)\}) = f$, $f \in E^*$. We obtain $\|\{f(x_n)\}\| \geq \|T\|^{-1}\|f\|$. Hence, we compute,

$$\begin{aligned}(\|T\|^{-1} - M)\|f\|_{E^*} \leq & \|\{f(x_n)\}\|_{(E^*)_d} - \|\{f(x_n + y_n)\}\|_{(E^*)_d} \\ \leq & \|\{f(y_n)\}\|_{(E^*)_d}.\end{aligned}$$

Thus $\{y_n\}$ is total over E. Therefore by Lemma 2.6, there exists an associated Banach space $(E^*)_{d_0} = \{\{f(y_n)\} : f \in E^*\}$ with norm given by $\|\{f(y_n)\}\|_{(E^*)_{d_0}} = \|f\|_{E^*}$, $f \in E^*$.

Define $T_0 : (E^*)_{d_0} \to E^*$ by $T_0(\{f(y_n)\}) = f$, $f \in E^*$. Then T_0 is bounded linear operator such that $(\{y_n\}, T_0)$ is a retro Banach frame for E^* with respect to $(E^*)_{d_0}$. □

Regarding the converse of Theorem 3.6, we prove the next result.

Theorem 3.7. *Let $(\{x_n\}, T)$ and $(\{y_n\}, S)$ be retro Banach frames for E^* with respect to $(E^*)_d$. Let $\mathcal{W} : (E^*)_d \to (E^*)_d$ be a linear homeomorphism such that*

$$\mathcal{W}(\{f(x_n)\}) = \{f(y_n)\}, \qquad f \in E^*.$$

Then $\{x_n + y_n\}$ *is a retro Bessel sequence for* E^* *with respect to* $(E^*)_d$ *and with bound*

$$M = \min\{\|U\|\|\mathcal{I} + \mathcal{W}\|, \|V\|\|\mathcal{I} + \mathcal{W}^{-1}\|\},$$

where $U, V : E^* \to (E^*)_d$ *are the coefficient mappings given by* $U(f) = \{f(x_n)\}$, $f \in E^*$ *and* $V(f) = \{f(y_n)\}$, $f \in E^*$ *and* $\mathcal{I}$ *denote the identity mapping on* $(E^*)_d$.

Proof. For each $f \in E^*$, we have

$$\begin{aligned}\|\{f(x_n + y_n)\}\|_{(E^*)_d} &= \|\{f(x_n)\} + \{f(y_n)\}\| \\ &= \|\mathcal{I}(\{f(x_n)\}) + \mathcal{W}(\{f(x_n)\})\| \\ &= \|(\mathcal{I} + \mathcal{W})(\{f(x_n)\})\| \\ &= \|(\mathcal{I} + \mathcal{W})U(f)\| \\ &\le \|(\mathcal{I} + \mathcal{W})\|\|U\|\|f\|\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|\{f(x_n + y_n)\}\|_{(E^*)_d} &= \|\{f(x_n)\} + \{f(y_n)\}\| \\ &= \|\mathcal{W}^{-1}(\{f(y_n)\}) + \mathcal{I}(\{f(y_n)\})\| \\ &= \|(\mathcal{I} + \mathcal{W}^{-1})(\{f(y_n)\})\| \\ &= \|(\mathcal{I} + \mathcal{W}^{-1})V(f)\| \\ &\le \|(\mathcal{I} + \mathcal{W}^{-1})\|\|V\|\|f\|\end{aligned}$$

Hence, $\{x_n + y_n\}$ is a retro Bessel sequence for E^* with respect $(E^*)_d$ and with bound

$$M = \min\{\|U\|\|\mathcal{I} + \mathcal{W}\|, \|V\|\|\mathcal{I} + \mathcal{W}^{-1}\|\}.$$

□

Remark 3.8. One may observe that, if $(\{x_n\}, T)$ and $(\{y_n\}, S)$ are two retro Banach frames for E^* with respect to $(E^*)_d$, then $\{x_n + y_n\}$ is a retro Bessel sequence for E^* with respect to $(E^*)_d$ with bound $\|U\| + \|V\|$, where $U, V : E^* \to (E^*)_d$ are the coefficient mappings given by $U(f) = \{f(x_n)\}$ and $V(f) = \{f(y_n)\}$, $f \in E^*$.

In the following result we construct a retro Bessel sequence by using the action of a bounded linear operator on $(E^*)_d$. This result extend Theorem 4.1 of [13] to the case of retro Bessel sequence.

Theorem 3.9. *Let* $\{x_n\} \subset E$ *be a retro Bessel sequence for* E^* *with respect to* $(E^*)_d$. *Let* $\{y_n\} \subset E$ *be such that* $\{f(y_n)\} \in (E^*)_d$, $f \in E^*$ *and let* $\mathcal{W} : (E^*)_d \to (E^*)_d$ *be a bounded linear operator such that*

$$\mathcal{W}(\{f(x_n)\}) = \{f(y_n)\}, \qquad f \in E^*.$$

Then $\{y_n\}$ *is a retro Bessel sequence for* E^* *with respect to* $(E^*)_d$.

Proof. Let $\{x_n\}$ be a retro Bessel sequence for E^* with Bessel bound M. Then for each $f \in E^*$, we have

$$\begin{aligned}\|\{f(y_n)\}\| &= \|\mathcal{W}(\{f(x_n)\})\| \\ &\le \|\mathcal{W}\|\|\{f(x_n)\}\| \\ &\le M\|\mathcal{W}\|\|f\|.\end{aligned}$$

□

In view of Theorem 3.9, we observe that if $(\{x_n\}, T)$ is a retro Banach frame for E^* with respect to $(E^*)_d$ and $\mathcal{W} : (E^*)_d \to (E^*)_d$ is a bounded linear operator such that $\mathcal{W}(\{f(x_n)\}) = \{f(y_n)\}$, $f \in E^*$. Then there exists, in general no reconstruction operator $T_0 : (E^*)_d \to E^*$ such that $(\{y_n\}, T_0)$ is a retro Banach frame for E^* with respect to $(E^*)_d$. (see Example 3.10)

Example 3.10. Let $E = l^1$. Let $\{x_n\}$ be a sequence of standard unit vectors in E. Then there exists an associated Banach space $(E^*)_d = \{\{f(x_n)\} : f \in E^*\}$ and a reconstruction operator $T : (E^*)_d \to E^*$ such that $(\{x_n\}, T)$ is a retro Banach frame for E^* with respect to $(E^*)_d$ with the norm given by $\|\{f(x_n)\}\|_{(E^*)_d} = \|f\|_{E^*}$, $f \in E^*$.

Now, define $\mathcal{W} : (E^*)_d \to (E^*)_d$ as $\mathcal{W}(\{f(x_n)\}) = \{f(x_n)\}_{n\neq 1}$, $f \in E^*$. Then $\mathcal{W}$ is a bounded linear operator on $(E^*)_d$. But there exists no reconstruction operator $T_0 : (E^*)_d \to (E^*)_d$ such that $(\{x_n\}_{n\neq 1}, T_0)$ is a retro Banach frame for E^* with respect to $(E^*)_d$. Indeed, if possible, suppose that $(\{x_n\}_{n\neq 1}, T_0)$ is a retro Banach frame for E^* with respect to $(E^*)_d$ having retro frame bounds A_0, B_0. Then we have,

$$A_0\|f\|_{E^*} \leq \|\{f(x_n)\}\|_{(E^*)_d} \leq \|f\|_{E^*}, \qquad f \in E^*. \tag{3.2}$$

Let $f = (1, 0, 0, ...)$ be a nonzero functional in E^* such that $f(x_n) = 0$, $n > 1$, $n \in \mathbb{N}$. Then by frame inequality (3.2) we obtain $f = 0$, a contradiction.

In view of Theorem 3.9 and Example 3.10, we give a necessary and sufficient condition for a retro Bessel sequence to be a retro Banach frame.

Theorem 3.11. *Let $(\{x_n\}, T)$ be a retro Banach frame for E^* with respect to $(E^*)_d$. Let $\{y_n\} \subset E$ be such that $\{f(y_n)\} \in (E^*)_d$, $f \in E^*$. Let $\mathcal{W} : (E^*)_d \to (E^*)_d$ be a bounded linear operator such that*

$$\mathcal{W}(\{f(x_n)\}) = \{f(y_n)\}, \qquad f \in E^*.$$

Then there exists a reconstruction operator $U : (E^*)_d \to E^*$ such that $(\{y_n\}, U)$ is a retro Banach frame for E^* with respect to $(E^*)_d$ if and only if

$$\|\{f(y_n)\}\|_{(E^*)_d} \geq M\|\mathcal{L}(\{f(y_n)\})\|_{(E^*)_d}, \qquad f \in E^*,$$

where $M < \infty$ is a positive constant and $\mathcal{L} : (E^*)_d \to (E^*)_d$ is a bounded linear operator such that

$$\mathcal{L}(\{f(y_n)\}) = \{f(x_n)\}, \qquad f \in E^*.$$

Proof. Let A_1 and B_1 be the retro frame bounds for the retro Banach frame $(\{x_n\}, T)$. Then for all $f \in E^*$, we have

$$\begin{aligned} MA_1\|f\|_{E^*} &\leq M\|\{f(x_n)\}\|_{(E^*)_d} \\ &= M\|\mathcal{L}(\{f(y_n)\})\|_{(E^*)_d} \\ &\leq \|\{f(y_n)\}\|_{(E^*)_d} \\ &= \|\mathcal{W}(\{f(x_n)\})\|_{(E^*)_d} \\ &\leq \|\mathcal{W}\|\|\{f(x_n)\}\|_{(E^*)_d} \\ &\leq \|\mathcal{W}\|B_1\|f\|_{E^*} \end{aligned}$$

Therefore,

$$(MA_1)\|f\|_{E^*} \leq \|\{f(y_n)\}\|_{(E^*)_d} \leq (\|\mathcal{W}\|B_1)\|f\|_{E^*}, \qquad f \in E^*.$$

Put $U = TL$. Then $U : (E^*)_d \to E^*$ is a bounded linear operator such that $U(\{f(y_n)\}) = f$, $f \in E^*$. Hence, $(\{y_n\}, U)$ is a retro Banach frame for E^* with respect to $(E^*)_d$.

For the reverse part, assume that $(\{y_n\}, U)$ is a retro Banach frame for E^* with respect to $(E^*)_d$ and having frame bounds A_0 and B_0. Let $S : E^* \to (E^*)_d$ be the coefficient mapping given by $S(f) = \{f(x_n)\}$, $f \in E^*$. Then $\mathcal{L} = SU : (E^*)_d \to (E^*)_d$ is a bounded linear operator such that $\mathcal{L}(\{f(y_n)\}) = \{f(x_n)\}$, $f \in E^*$. Thus we write, $M = \frac{A_0}{B_1}$, where B_1 is an upper bound for retro Banach frame $(\{x_n\}, T)$, we have

$$\begin{aligned} \|\{f(y_n)\}\|_{(E^*)_d} &\geq A_0\|f\|_{E^*} \\ &= MB_1\|f\|_{E^*} \\ &\geq M\|\{f(x_n)\}\|_{(E^*)_d} \\ &= M\|\mathcal{L}(\{f(y_n)\})\|_{(E^*)_d}. \end{aligned}$$

□

In 2015, C. Shekhar [16] constructed a normalized tight operator Banach frame in terms of operator Bessel sequence. Using this method, in the next result, we construct a retro Banach frame, in terms of retro Bessel sequence.

Theorem 3.12. *Let $(\{x_n\}, T)$ be a retro Banach frame for E^* with respect to $(E^*)_d$. Let $\{y_n\} \subset E$ be a sequence in E such that $\{x_n + y_n\}$ is a retro Bessel sequence for E^* with respect to $(E^*)_d$ and with bound $K < \|T\|^{-1}$. Then there exists a reconstruction operator $U : (E^*)_d \to E^*$ such that $(\{y_n\}, U)$ is a retro Banach frame for E^* with respect to $(E^*)_d$ if and only if there exists a bounded linear operator $\mathcal{L} : (E^*)_d \to E^*$ such that $\mathcal{L}(\{f(y_n)\}) = \{f(x_n)\}$, $f \in E^*$.*

Proof. Let $(\{y_n\}, U)$ is a retro Banach frame for E^* with respect to $(E^*)_d$. Let $S : E^* \to (E^*)_d$ be the coefficient mapping given by $S(f) = \{f(x_n)\}$, $f \in E^*$. Then $\mathcal{L} = SU : (E^*)_d \to (E^*)_d$ is a bounded linear operator such that $\mathcal{L}(\{f(y_n)\}) = \{f(x_n)\}$, $f \in E^*$.

Conversely, by hypothesis $\{f(y_n)\} \in (E^*)_d$, $f \in E^*$. Also, for each $f \in E^*$, we have

$$\begin{aligned} (\|T\|^{-1} - K)\|f\|_{E^*} &= \|T\|^{-1}\|f\|_{E^*} - K\|f\|_{E^*} \\ &\leq \|\{f(x_n)\}\|_{(E^*)_d} - \|\{f(x_n + y_n)\}\|_{(E^*)_d} \\ &\leq \|\{f(y_n)\}\|_{(E^*)_d} \\ &\leq \|\{f(x_n + y_n)\}\|_{(E^*)_d} + \|\{f(x_n)\}\|_{(E^*)_d} \\ &\leq \|S\|\|f\|_{E^*} + K\|f\|_{E^*} \\ &= (\|S\| + K)\|f\|_{E^*}, \end{aligned}$$

where S is a coefficient mapping. Now put $U = T\mathcal{L}$. Then $U : (E^*)_d \to E^*$ is a bounded linear operator such that $U(\{f(y_n)\}) = f$, $f \in E^*$. Hence, $(\{y_n\}, U)$ is a retro Banach frame for E^* with respect to $(E^*)_d$ and frame bounds $(\|T\|^{-1} - K)$ and $(\|S\| + K)$. □

Next we give a necessary and sufficient condition for a retro Bessel sequence to be a retro Banach frame. This result extends Theorem 2.12 of [16] to the case of retro Bessel sequence.

Theorem 3.13. *Let $(\{x_n\}, T)$ be a retro Banach frame for E^* with respect to $(E^*)_d$. Let $\{y_n\} \subset E$ be a retro Bessel sequence for E^* with respect to $(E^*)_d$. Let $V : (E^*)_d \to (E^*)_d$*

be a bounded linear operator such that $V(\{f(y_n)\}) = \{f(x_n)\}$, $f \in E^*$. Then there exists a reconstruction operator $U : (E^*)_d \to E^*$ such that $(\{y_n\}, U)$ is a retro Banach frame for E^* with respect to $(E^*)_d$ if and only if there exist a constant $M > 0$ such that

$$\|\{f(x_n - y_n)\}\|_{(E^*)_d} \leq M\|\{f(y_n)\}\|_{(E^*)_d}, \qquad f \in E^*.$$

Proof. Let $K > 0$ be a constant such that $\|\{f(y_n)\}\|_{(E^*)_d} \leq K\|f\|_{E^*}$, $f \in E^*$. Let A_1 be a lower bound for the retro Banach frame $(\{x_n\}, T)$. Then

$$\begin{aligned} A_1\|f\|_{E^*} \leq& \|\{f(x_n)\}\|_{(E^*)_d} \\ \leq& \|\{f(y_n)\}\|_{(E^*)_d} + \|\{f(x_n - y_n)\}\|_{(E^*)_d} \\ \leq& (1 + M)\|\{f(y_n)\}\|_{(E^*)_d}. \end{aligned}$$

Put $U = TV$, then $U : (E^*)_d \to E^*$ is a bounded linear operator such that $U(\{f(y_n)\}) = f$, $f \in E^*$. Hence $(\{y_n\}, U)$ is a retro Banach frame for E^* with respect to $(E^*)_d$.

Conversely, let A_0, B_0; A_1, B_1 be bounds for the retro Banach frames $(\{x_n\}, T)$ and $(\{y_n\}, S)$, respectively. Then,

$$A_0\|f\|_{E^*} \leq \|\{f(x_n)\}\|_{(E^*)_d} \leq B_0\|f\|_{E^*}, \qquad f \in E^*$$

and

$$A_1\|f\|_{E^*} \leq \|\{f(y_n)\}\|_{(E^*)_d} \leq B_1\|f\|_{E^*}, \qquad f \in E^*.$$

Therefore,

$$\begin{aligned} \|\{f(x_n - y_n)\}\|_{(E^*)_d} \leq& \|\{f(x_n)\}\|_{(E^*)_d} + \|\{f(y_n)\}\|_{(E^*)_d} \\ \leq& (B_1 + B_0)\|f\|_{E^*} \\ \leq& M\|\{f(y_n)\}\|_{(E^*)_d}. \end{aligned}$$

where, $M = \frac{B_1+B_0}{A_1}$ □

In the following result, we show that the retro Bessel sequences are stable under small perturbation on given retro Bessel sequence. This result is inspired by Theorem 5 of [17].

Theorem 3.14. *Let $\{x_n\}$ be a retro Bessel sequence for E^* with respect to $(E^*)_d$ and let $\{y_n\} \subset E$ be such that $\{f(y_n)\} \in (E^*)_d$, $f \in E^*$. Then $\{y_n\}$ is a retro Bessel sequence for E^* with respect to $(E^*)_d$ if there exist nonnegative constants λ, μ ($\mu < 1$) and δ such that*

$$\|\{f(x_n - y_n)\}\|_{(E^*)_d} \leq \lambda\|\{f(x_n)\}\|_{(E^*)_d} + \mu\|\{f(y_n)\}\|_{(E^*)_d} + \delta\|f\|_{E^*}, f \in E^*.$$

Proof. Let M be the Bessel bound for the retro Bessel sequence $\{x_n\}$. Then

$$\begin{aligned} \|\{f(y_n)\}\|_{(E^*)_d} \leq& \|\{f(x_n)\}\|_{(E^*)_d} + \|\{f(x_n - y_n)\}\|_{(E^*)_d} \\ =& (1 + \lambda)M\|f\|_{E^*} + \mu\|\{f(y_n)\}\|_{E^*} + \delta\|f\|_{E^*}, f \in E^* \\ \leq& (\frac{(1 + \lambda)M + \delta}{1 - \mu})\|f\|_{E^*}, f \in E^*. \end{aligned}$$

Hence, $\{y_n\}$ is a retro Bessel sequence for E^* with respect to $(E^*)_d$. □

4. Retro bi-Banach frame

Very recently, Shah Jahan et al.[9] defined the notion of *Strong Retro Banach Frame* (*SRBF*) in order to correlate this notion with the existing notions like approximative Schauder frames and bi-Banach frames. In this section, we correlate *SRBF* with the notion of *Retro bi-Banach Frames*. In fact, our results generalizes some results of Shah Jahan et al.[9] for retro bi-Banach frames.

First result of this section shows that the existence of *SRBF* in the conjugate Banach space guarantees the existence of retro bi-Banach frame in the conjugate Banach space with some geometric properties.

Theorem 4.1. *Let X be a Banach space and $(\{x_n\},\{f_n\},\{v_n\})$ be a* SRBF *for X^* with admissible sequence $\{f_n\}\subset X^*$. Then $(\{x_n\},\{f_n\})$ is a retro bi-Banach frame for X^* and there exists a sequence $\{\Phi_n\}\subset X^{**}$ such that $\gamma([\Phi_n])>0$.*

Proof. By Theorem 2.4 in [9], there exists a sequence of continuous linear mappings $\{\sigma_n\}$, where $\sigma_n : V_n \to V_n$, $V_n=[f_1,f_2,...f_n]$, $n\in\mathbb{N}$ such that

$$f(x)=\lim_{n\to\infty}\left(\sigma_n\sum_{i=1}^{n} f(x_i)f_i\right)(x), \qquad f\in X^*.$$

Then, it follows that $\{\Phi\in X^{**} : \Phi(f_n)=0, \text{ for all } n\in\mathbb{N}\}=\{0\}$. Therefore, $\{f_n\}$ is total over X^*. Hence by Lemma 2.6, there exists an associated Banach space $(X^{**})_d=\{\{\Phi(f_n)\} : \Phi\in X^{**}\}$ with norm given by $\|\{\Phi(f_n)\}\|_{(X^{**})_d}=\|\Phi\|_{(X^{**})}$, $\Phi\in X^{**}$, and a reconstruction operator $S:(X^{**})_d\to E^{**}$, such that $(\{f_n\},S)$ is an RBF for X^{**} with respect to $(X^{**})_d$. Hence, $(\{x_n\},\{f_n\})$ is a retro bi-Banach frame for X^*.

Now, write

$$\sigma_n(f_j)=\sum_{i=1}^{n} a_{ji}^{(n)} f_i, \qquad 1\le j\le n, n\in\mathbb{N},$$

where $a_{ji}^{(n)}=\Phi_i(\sigma_n(f_j))$, for all $i,j=1,2,3,...,n$, $n\in\mathbb{N}$. Thus

$$\begin{aligned}\sigma_n\left(\sum_{i=1}^{n}\Phi_i(f)f_i\right)&=\sum_{i=1}^{n}\Phi_i(f)\sigma_n(f_i)\\&=\sum_{i=1}^{n}\Phi_i(f)\left(\sum_{j=1}^{n}a_{ij}^{(n)}f_j\right)\\&=\sum_{j=1}^{n}\left(\sum_{i=1}^{n}a_{ij}^{(n)}\Phi_i(f)\right)f_j, \qquad f\in X^*, n\in\mathbb{N}.\end{aligned}$$

Define

$$h_{n,i}=\sum_{i=1}^{n}a_{ij}^{(n)}\Phi_i, \qquad 1\le j\le n, n\in\mathbb{N}.$$

Then $h_{n,i}\in[\Phi_i]_{i=1}^{n}, 1\le j\le n, n\in\mathbb{N}$. Hence, we conclude that $\gamma([\Phi_n])>0$.

□

Finally, we give a necessary and sufficient condition for a conjugate Banach space having a retro Banach frame also has a retro bi-Banach frame. Although, Theorem 4.2 is a particular case of Theorem 2.10 of [4].

Theorem 4.2. *Let* $(\{x_n\}, T)$ $(\{x_n\} \in E, T : (E^*)_d \to E^*)$ *be a retro Banach frame for* E^* *and* $\{f_n\}$ *a sequence in* E^*. *Then there exists a reconstruction operator* S *such that* $(\{f_n\}, S)$ *is a retro Banach frame for* E^{**} *(and hence* $(\{x_n\}, \{f_n\})$ *is a retro bi-Banach frame for* E^**) if and only if* $dist(f, M_n) \to 0$ *as* $n \to \infty$, *for all* $f \in E^*$, *where* $M_n = [f_1, f_2, ..., f_n]$, *for all* $n \in \mathbb{N}$.

Proof. First suppose that $(\{x_n\}, \{f_n\})$ is a retro bi-Banach frame for E^*. Then there exists a reconstruction operator S such that $(\{f_n\}, S)$ is a retro Banach frame for E^{**}. So, there are positive constants A_0 and B_0 such that

$$A_0\|\Phi\|_{E^{**}} \leq \|\{\Phi(f_n)\}\|_{(E^{**})_d} \leq B_0\|\Phi\|_{E^{**}}, \qquad \Phi \in E^{**}. \tag{4.1}$$

Suppose that the condition $dist(f, M_n) \to 0$ as $n \to \infty$, for all $f \in E^*$, is false. Then there exists a nonzero functional $f_0 \in E^*$ such that $\lim_{n\to\infty} dist(f_0, M_n) \neq 0$.

Note that $dist(f_0, M_n) \geq dist(f_0, M_{n+1})$, for all $n \in \mathbb{N}$, since $\{M_n\}$ is a nested system of subspaces. Now, $\{dist(f_0, M_n)\}$ is bounded below monotone decreasing sequence. So, $\{dist(f_0, M_n)\}$ is convergent and its limit is a positive real number, since otherwise $dist(f_0, M_n) \to 0$ as $n \to \infty$ (which is not possible).

Let $\lim_{n\to\infty} dist(f_0, M_n) = \alpha > 0$ and $D = \cup_n M_n$. Then we obtain $dist(f_0, D) \geq \alpha > 0$, since $dist(f_0, D) = \inf dist(f_0, M_n)$.

Now we show that $f_0 \notin \overline{D}$. Let if possible, $f_0 \in \overline{D}$. Then we can find a sequence $\{\xi_n\} \subset D$ such that $dist(\xi_n, f_0) \to 0$ as $n \to \infty$. Since, we have $dist(f_0, D) \geq \alpha$. Therefore, $dist(\xi_n, f_0) \geq \alpha > 0$. This contradict the fact that $dist(\xi_n, f_0) \to 0$ as $n \to \infty$. Hence, $f_0 \notin \overline{D}$. Thus, by using Hahn Banach theorem, there exists a nonzero functional $\Phi_0 \in E^{**}$ such that $\Phi_0(f_n) = 0$, for all $n \in \mathbb{N}$. Therefore, by retro frame inequality (4.1) we obtain, $\Phi_0 = 0$, a contradiction. Hence, $dist(f, M_n) \to 0$ as $n \to \infty$, for all $f \in E^*$.

Conversely, suppose that $dist(f, M_n) \to 0$ as $n \to \infty$, for all $f \in E^*$. Then, in particular, for each $\epsilon > 0$ and for each $f \in E^*$, we can find a f_j from some M_k such that $\|f - f_k\| < \epsilon$.

Therefore by using Lemma 2.6, $(E^{**})_d = \{\{\Phi(f_n)\} : \Phi \in E^{**}\}$ is a Banach space of sequences of scalars with norm given by $\|\{\Phi(f_n)\}\|_{(E^{**})_d} = \|\Phi\|_{E^{**}}$, $\Phi \in E^{**}$. Define $S : (E^{**})_d \to E^{**}$ by $S(\{\Phi(f_n)\}) = \Phi$, $\Phi \in E^{**}$. Then S is a bounded linear operator such that $(\{f_n\}, S)$ is a retro Banach frame for E^{**} with respect to $(E^{**})_d$. Hence, $(\{x_n\}, \{f_n\})$ is a retro bi-Banach frame for E^*. □

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(Mayur Puri Goswami) School of Studies in Mathematics, Pt. Ravishankar Shukla University, Raipur (C.G.) 492010, India.

E-mail address: `mayurpuri89@gmail.com`