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Some results and problems in dyadic analysis: an overview

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1. Notations

Let us introduce some notations. For the number $x \in R_+ = [0, +\infty)$, we will use dyadic expansion $x = \sum\limits_{n=-\infty}^{+\infty} 2^{-n-1} x_n$ of x, where x_n be equal to 0 or 1. It is evident, that x_{-n} for $n \geq n(x)$. If x is dyadic rational, then we use finite expansion, i.e. $x_n = 0$ for $n \geq n(x)$.
Let us introduce the generalized Walsh functions

$$\psi(x,y) \equiv \psi_y(x) = (-1)^{t(x,y)},$$

where

$$t(x,y) = \sum_{n=-\infty}^{\infty} x_n\; y_{-n-1} \text{ for } (x,y) \in R_+ \times R_+.$$

It is evident that

$$\psi_y(x) = \psi_x(y),\;\; \psi_y(x) = \pm 1 \text{ for } x, y \in R_+.$$

Let us note that

$$w_n(x) \equiv \psi_n(x) \equiv \psi(x,n),\;\; n \in \mathbb{Z}_+,\;\; x \in R_+$$

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are Walsh-Paley functions (see [14], p. 2). The system $\{w_n\,(x)\}_{n=0}^{\infty}$ is orthonormal on the interval [0,1).
N. J. Fine [10] introduced Walsh-Fourier transform of the function $f \in L(R_+)$ by the formula

$$F[f](x) \equiv \tilde{f}(x) = \int_{R_+} \psi_x(y)\ f(y)\ dy.$$

Let us introduce the dyadic sum $x \oplus y = z$ of the numbers $x, y \in R_+$, where $z_n \equiv x_n + y_n (mod\ 2),\ n \in \mathbb{Z}$.

2. The known concepts of dyadic derivatives and integrals

I. The pointwise dyadic derivative for functions defined on the interval $[0,1)$ was introduced by P. L. Butzer and H. J. Wagner [6] and has its roots in the work of J. E. Gibbs and M. S. Millard [13]. P. L. Butzer and H. J. Wagner [6] introduced the following

Definition 2.1. If for a function $f : [0,1) \to R$ there exists finite limit

$$f^{[1]}(x) = \lim_{n\to\infty} \sum_{m=0}^{n} 2^{m-1}[f(x) - f(x \oplus 2^{-m-1})] \tag{2.1}$$

at a given point $x \in [0,1)$, then $f^{[1]}(x)$ is called dyadic derivative of the function f at the point x.

P. L. Butzer and H. J. Wagner proved that each Walsh-Paley function $w_n(x)$ has dyadic derivative at every point $x \in [0,1)$. More precisely,

$$w_n^{[1]}(x) = nw_n(x),\ n \in \mathbb{Z}_+,\ x \in [0,1).$$

The concept of strong dyadic derivative also is known.

Definition 2.2 (P.L. Butzer and H.J. Wagner [6])**.** If for the function $f \in L^p[0,1)$, $1 \leq p \leq +\infty$, there exists

$$D(f)_{L^p} = \lim_{n\to\infty} \sum_{m=0}^{n} 2^{m-1}[f(\circ) - f(\circ \oplus 2^{-m-1})] \tag{2.2}$$

where the limit is taken by the norm of the space $L^p[0,1)$, then $D(f)_{L^p}$ is called strong $L^p[0,1)$ -derivative of the function f.

P. L. Butzer and H. J. Wagner proved that if a function $f \in L^p[0,1),\ 1 \leq p \leq \infty$ has $L^p[0,1)$-derivative $D(f)_{L^p} = g$, then

$$\hat{g}(n) = n\ \hat{f}(n),\ n \in \mathbb{Z}_+,$$

where $\hat{f}(n)$ are Walsh-Fourier coefficients of the function f. The Walsh functions have $L^p[0,1)$-derivative for all $1 \leq p \leq \infty$ and $D(w_n)_{L^p} = n\ w_n,\ n \in \mathbb{Z}_+$.
C. W. Onneweer [37] generalized the concept of strong derivative for functions defined on Vilenkin groups. After that J. Pal and P. Simon [40] introduced corresponding integral and proved the fundamental theorem of integral calculus for bounded Vilenkin groups.
For functions f defined on R_+ the pointwise derivative is introduced as follows

$$df(x) = \lim_{n\to+\infty} \sum_{m=-n}^{n} 2^{m-1}[f(x) - f(x \oplus 2^{-m-1})] \tag{2.3}$$

(see, for example, [45]).

It is known (see F. Pichler [41]) that $d\psi_y(x) = y\psi_y(x),\ x \in R_+$.

For the functions $f \in L^p(R_+),\ 1 \le p \le \infty$, the strong dyadic derivative is defined as follows:

$$D(f)_{L^p(R_+)} = \lim_{n\to\infty} \sum_{m=-n}^{n} 2^{m-1}[f(\circ) - f(\circ \oplus 2^{-m-1})] \tag{2.4}$$

where the limit is taken by the norm of the space $L^p(R_+)$.

It is known that if $f \in L^p(R_+),\ 1 \le p \le 2$, and $D(f)_{L^p(R_+)}$ exists, then

$$D(f)^{\sim}_{L^p(R_+)}(x) = x\tilde{f}(x).$$

For $p = 1$ this was proved by P. L. Butzer and H. J. Wagner [6] in 1973 and for $p = 2$ by J. Pal [39] in 1977. For $f \in L^p\,(R_+)\,, 1 < p < 2$, the statement mentioned above may be proved by representation $f = f_1 + f_2,\ f_1 \in L^1\,(R_+),\ f_2 \in L^2\,(R_+)$.
C. W. Onneweer [38] introduced modified pointwise and strong dyadic derivatives for functions defined on dyadic group D or dyadic field K. (The characters of the group D are Walsh functions $w_n,\ n \in \mathbb{Z}_+$). He proved that the characters of dyadic group D or dyadic field K are differentiable in his sense and they are eigenfunctions of modified differential operator d.

For example, he proved the equalities

$$d(w_0) = 0,\ d(w_n) = 2^k\ w_n,\ 2^k \le n < 2^{k+1},\ k \in \mathbb{Z}_+.$$

In another article C.W. Onneweer [36] introduced fractional differentiation and integration on the group of integers of a p-adic or p-series field and proved fundamental theorem of dyadic calculus.

II. The dyadic integral for functions defined on the interval $[0, 1)$ was introduced by P. L. Butzer and H. J. Wagner [6] in 1973 as follows. Let us set

$$W(x) = 1 + \sum_{n=1}^{+\infty} \frac{w_n(x)}{n}.$$

It is evident that $W \in L[0, 1]$. If $f \in L^p(R_+),\ 1 \le p < \infty$, then there exists dyadic convolution

$$I(f)(x) = \int_0^1 f(y)\ W(x \oplus y)\ dy \tag{2.5}$$

and $I(f) \in L^p[0, 1)$. The function $I(f)$ is called *dyadic integral* of the function f in the space $L^p[0, 1)$.

From (2.5) it follows that for the function $f \in L[0, 1)]$ its dyadic integral $I(f)$ has Walsh-Paley series of the form

$$\hat{f}\,(0) + \sum_{n=1}^{\infty} \frac{\hat{f}(n) w_n\,(x)}{n},$$

where $\hat{f}\,(n)$ are the Fourier coefficients of the function f with respect to Walsh-Paley system.

P. L. Butzer and H. J Wagner proved that if for a function $f \in L^p[0, 1),\ 1 \le p < \infty$, there exists $L^p[0, 1)$-derivative $D(f)$ and $\hat{f}(0) = 0$, then

$$I(D(f)) = f \text{ and } D(I(f)) = f.$$

F. Schipp [44] proved that for the function $f \in L[0,1)$ its dyadic integral $I(f)$ has pointwise dyadic derivative almost everywhere on $[0,1)$, which is equal to f almost everywhere on $[0,1)$.

For the functions $f \in L^p(R_+),\ 1 \le p < \infty$ the strong dyadic integral was defined by J. Pal in the paper [38] as follows. For $n \in \mathbb{Z}_+$, we set

$$W_n(x) = \lim_{k \to +\infty} \int_{2^{-n}}^{2^k} \frac{1}{t}\ \psi_x(t)\ dt,\ x \in R_+.$$

This limit exists a.e. on R_+ and also in $L(R_+)$-metric. Therefore, there exists dyadic convolution

$$(f * W_n)(x) = \int_{R_+} f(t)\ W_n(x \oplus t)\ dt \tag{2.6}$$

and $f * W_n \in L^p(R_+)$, if $f \in L^p(R_+),\ 1 \le p < \infty$.

Definition 2.3. If for a function $f \in L^p(R_+),\ 1 \le p < \infty$, the sequence (2.6) converges to a function $g \in L^p(R_+)$ as $n \to +\infty$ in the space $L^p(R_+)$, then $g \equiv J(f)$ is called strong dyadic integral in the space $L^p(R_+)$.

The following result is known (see [38]):

(a) *If a function $f \in L(R_+)$ dyadic differentiable in the space $L(R_+)$ and $\tilde{f}(0) = 0$, then*

$$J(D(f)) = D(J(f)) = f;$$

(b) *The equality $g = J(f)$ holds iff $\tilde{g}(0) = 0$ and $\tilde{g}(x) = \tilde{f}(x)/x, x > 0$.*

3. Modified strong dyadic integral and derivative of fractional order on R_+

For $x > 0$ we set $h(x) = 2^{-n}, 2^n \le x < 2^{n+1}, n \in Z$. Then $x^{-1} \le h(x) < 2x^{-1}$.

Lemma 3.1. *If $\alpha > 0$, then for each $x > 0$ and $n \in Z$ there exists finite limit*

$$W_n^\alpha(x) = \lim_{m \to +\infty} \int_{2^{-n}}^{2^m} (h(y))^\alpha\, \psi_x(y)\, dy \quad \textit{and } W_n^\alpha \in L(R_+).$$

More precisely $W_n^\alpha(x) = 0$ for $x \ge 2^n$ and

(1) if $0 < \alpha < 1$, then $W_n^\alpha(x) \approx x^{\alpha-1}, x \to +0$;

(2) $W_n^1(x) \approx \log_2(x^{-1}), x \to +0$;

(3) if $\alpha > 1$, then $W_n^\alpha(x)$ is bounded on R_+.

Here we write $f(x) \approx g(x), x \to a$, if

$$f(x) = O(g(x)), x \to a \text{ and } g(x) = O(f(x)), x \to a.$$

For $f, g \in L(R_+)$ we set

$$(f * g)(x) = \int_{R_+} f(x \oplus y)\, g(y)\, dy, x \in R_+,$$

i.e. $f * g$ is dyadic convolution of the functions f and g.

Definition 3.2. If $\alpha > 0$, $f, g \in L(R_+)$ and

$$\lim_{n\to+\infty} \|f * W_n^\alpha - g\|_{L(R_+)} = 0,$$

then the function $g = J_\alpha(f)$ is called modified strong dyadic integral (MSDI) of order α of the function f.

Theorem 3.3. *If $f, g \in L(R_+)$ and $\alpha > 0$, then $g = J_\alpha(f)$ iff $\tilde{g}(0) = 0$ and $\tilde{g}(x) = \tilde{f}(x)(h(x))^\alpha$ for $x > 0$.*

Let us set

$$\Lambda_n^\alpha(x) = \sum_{k=-n}^{n} \frac{\psi\left(x, 2^{-k}\right)}{2^{(1+\alpha)k}} \chi_{[0,2^k)}(x), n \in Z_+,$$

where χ_E is characteristic function of the set E.

Definition 3.4. If $\alpha > 0$, $f, \varphi \in L(R_+)$ and

$$\lim_{n\to+\infty} \|f * \Lambda_n^\alpha - \varphi\|_{L(R_+)} = 0,$$

then the function $\varphi = D^\alpha(f)$ is called modified strong dyadic derivative (MSDD) of order α of the function f.

Theorem 3.5. *If $\alpha > 0$ and $f, g \in L(R_+)$, then $g = D^\alpha(f)$ iff $\tilde{g}(0) = 0$ and $\tilde{g}(x) = \tilde{f}(x)(h(x))^{-\alpha}$ for $x > 0$.*

Several generalizations of Theorems 3.3 and 3.5 were proved by S.S. Volosivets [50] in terms of usual multiplicative Fourier transform of the functions $f, g \in L^p(R_+), 1 < p \leq 2$, and in terms of distributional one of the functions $f, g \in L^p(R_+), p > 2$.

Theorem 3.6. *If $\alpha > 0$ and the function $f \in L(R_+)$ has MSDD $D^\alpha(f)$ and $\tilde{f}(0) = 0$, then $J_\alpha(D^\alpha(f)) = f$.*

Theorem 3.7. *If $\alpha > 0$ and the function $f \in L(R_+)$ has MSDI $J_\alpha(f)$ and $\tilde{f}(0) = 0$, then $D^\alpha(J_\alpha(f)) = f$.*

Theorem 3.7 was generalized by S.S. Volosivets (see [48], Theorems 7 and 13) to the general P-nary case and the spaces $L^p(R_+)$.

Theorem 3.8. *Let $\alpha > 0, \beta > 0$ and $f \in L(R_+)$. Then $D^\alpha\left(D^\beta(f)\right) = D^{\alpha+\beta}(f)$. (correspondingly $J^\alpha\left(J^\beta(f)\right) = J^{\alpha+\beta}(f)$) if the left side or right side exists.*

This theorem was generalized to modified multiplicative integral and derivative of arbitrary orders on the positive semyaxis in the paper [48] (see there Corollaries 3 and 6).

Theorem 3.9. *The functions $a_{m,n}(x) = \psi(x, m2^{-n})\chi_{[0,2^n)}(x)$, where $m \in N, n \in Z$, for each $\alpha > 0$ are eigenfunctions of the operators J_α and D^a with eigenvalues $2^{-r\alpha}$ and $2^{r\alpha}$ respectively. Here $r = r(m,n) \in Z$ is uniquely determined by the imbedding $[m2^{-n}, (m+1)2^{-n}) \subset [2^r, 2^{r+1})$.*

Theorems 3.3-3.9 are R_+-analogues of the results of C. W. Onneweer [36]. He considered fractional differentiation and integration on the group of integers of a p-adic or p-series field.

Let us denote by $L_{J_\alpha}(R_+)$ (or$L_{D^\alpha}(R_+)$) the natural domain of the operator J_α (D^α respectively), i.e. the set of all functions $f \in L(R_+)$ for which $J_\alpha(f)$or $D^\alpha(f)$ respectively exists. It is evident that $L_{J_\alpha}(R_+)$ and $L_{D^\alpha}(R_+)$ are linear subspaces of the space $L(R_+)$.

It follows from the theorem 3.9 that

$$J_\alpha(a_{1,n}) = 2^{n\alpha}a_{1,n}, D^\alpha(a_{1,n}) = 2^{-n\alpha}a_{1,n}, n \in Z, \alpha > 0.$$

Therefore, we have

Corollary 3.10. The linear operators $J_\alpha : L_{J_\alpha}(R_+) \to L(R_+)$ and $D^\alpha : L_{D^\alpha}(R_+) \to L(R_+)$ are unbounded for each $\alpha > 0$.

Let us define the pointwise derivative of fractional order.

Definition 3.11. Let $\alpha > 0$, $x \in R_+$ and $f \in L^\infty(R_+)$. If there exists finite limit

$$d^\alpha(f)(x) \equiv \lim_{n\to+\infty}(\Lambda_n^\alpha * f)(x),$$

then we will say that the function f has the modified dyadic derivative (MDD) $d^\alpha(f)(x)$ of order α at the point x.

Theorem 3.12. *For each $\alpha > 0$ and a fixed $y \in R_+$ the Walsh generalized function $\psi_y(\circ)$ has MDD of order α at each point $x \in R_+$. More precisely,*

$$d^\alpha(\psi_0)(x) \equiv 0 \text{ on } R_+ \text{ and } d^\alpha(\psi_y)(x) = (h(y))^{-\alpha}\psi_y(x) \text{ for } x \in R_+, y > 0.$$

Theorem 3.13. *If $\alpha > 0$ and the function $f \in L(R_+)$ is such that $(h(x))^{-\alpha}\tilde{f}(x) \in L(R_+)$, then at each point $x \in R_+$ it has MDD of order α equal to $\int_0^{+\infty}(h(y))^{-\alpha}\tilde{f}(y)\psi(x,y)\,dy$.*

For the case $\alpha = 1$ Theorems 3.1-3.13 were published in our paper [21] and for $\alpha > 0$ in the papers [24], [25].

For $\alpha = 1$ Theorem 3.13 is an analogue of the following theorem:

Theorem 3.14. *Under the assumption $\sum\limits_{n=0}^{\infty} n|a_n| < +\infty$ the series $\sum\limits_{n=0}^{\infty} a_n w_n(x)$ converges absolutely and uniformly on $[0,1)$ to a function f, which has dyadic derivative $f^{[1]}(x)$ for all $x \in [0,1)$ and*

$$f^{[1]}(x) = \sum_{n=0}^{\infty} na_n w_n(x).$$

This theorem was proved by P. L. Butzer and H. J. Wagner [7].

Definition 3.15. *If $\alpha > 0$, $x \in R_+$ and for a function $f \in L^\infty(R_+)$ there exists finite limit*

$$j_\alpha(f)(x) \equiv \lim_{n\to+\infty}(f * W_n^\alpha)(x),$$

then we say that the function f has modified dyadic integral $j_\alpha(f)(x)$ (MDI) of order α at the point x.

Theorem 3.16. *For each $\alpha > 0$ and fixed $y \in R_+$ the Walsh generalized function $\psi_y(\circ)$ has MDI of order α at each point $x \in R_+$. More precisely, $j_\alpha(\psi_0)(x) \equiv 0$ on R_+ and $j_\alpha(\psi_y)(x) = (h(y))^\alpha \psi_y(x)$ for $x \in R_+, y > 0$.*

Problem. *Let $\alpha > 0$ and the function $f \in L(R_+)$ has MSDI of order α. Let $J_\alpha(f)(x)$ has pointwise MDD of order α almost everywhere on R_+. Is the equality $d^\alpha(J_\alpha(f))(x) = f(x)$ holds almost everywhere on R_+?*

Probably this is true only for $\alpha \geq 1$.

4. Uniform integral in dyadic Hardy spaces

Let us introduce the following notations:

$\Delta \equiv \Delta_n^k = [k2^{-n}, (k+1)2^{-n})$, $k \in Z_+, n \in Z$ - dyadic interval;

$M_d(f)(x) = \sup\limits_{x \in \Delta \subset R_+} \frac{1}{|\Delta|} \int_\Delta |f(t)|\, dt$, $x \in R_+$ - dyadic maximal function on R_+;

$H_d(R_+) = \{f \in L(R_+) : M_d(f) \in L(R_+)\}$– dyadic Hardy space on R_+;

$\|f\|_{H(R_+)} = \|M_d(f)\|_{L(R_+)} < \infty$ - the norm of the function $f \in H_d(R_+)$;

$M^d_{[0,1)}(f)(x) = \sup\limits_{x \in \Delta \subset [0,1)} \frac{1}{|\Delta|} \int_\Delta |f(t)|\, dt$, $x \in [0,1)$ - dyadic maximal function on $[0,1)$;

$H_d([0,1)) = \left\{f \in L[0,1) : M^d_{[0,1)}(f) \in L[0,1)\right\}$– dyadic Hardy space on $[0,1)$;

$C_W(R_+)$ - the space of uniformly $W-$continuous functions on R_+, i.e. $f \in C_W(R_+)$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\sup\limits_{x \in R_+} |f(x \oplus y) - f(x)| < \varepsilon$ for $0 < y < \delta$.

The following theorem of Hardy and Littlewood [27] is known:

Theorem A. *If the function $f(z) = \sum\limits_{n=0}^{+\infty} a_n z^n$ belongs to the Hardy space $H(|z| < 1)$ on the unit disc $|z| < 1$ of the complex plane $\boldsymbol{C}$ and $f(e^{it})$ is its boundary function on the unit circle $|z| = 1$, then*

$$\sum_{n=0}^{+\infty} \frac{|a_n|}{n+1} \leq \frac{1}{2} \int_0^{2\pi} |f(e^{it})| dt.$$

An analogue of this theorem has been proved by E. Hille and J. D. Tamarkin [29]:

Theorem B. *If the function $f(z)$ belongs to the Hardy space $H\left(R_+^2\right)$ in the upper half-plane $R_+^2 = \{z \in C : Im z > 0\}$ and $\hat{f}(x)$ is the Fourier transform of its boundary function $f(x)$ on real axis, then*

$$\int_R \frac{\left|\hat{f}(x)\right|}{x} dx \leq \frac{1}{2} \int_{-\infty}^{+\infty} |f(x)|\, dx.$$

An extension of Theorem **B** to the Hardy space $H^p(R), 0 < p \leq 1,$ is known:

Theorem C. *If $f \in H^p(R), 0 < p \le 1$, then*

$$\int_R \left|\hat{f}(x)\right|^p |x|^{p-2}\,dx \le C_p \|f\|^p_{H^p(R)},$$

where the constant $C_p > 0$ does not depend on the function f (see [11]*, p.342).*

Problem. *What are the least constants in right-hand sides of the inequalities in the theorems **A** - **C**?*

N.R. Ladhawala [32] proved dyadic analogue of Theorem **A** in the following form.

Theorem D. *If the function f belongs to dyadic Hardy space $H([0,1))$, then*

$$\sum_{n=1}^{+\infty} \frac{\left(\hat{f}(n)\right)}{n} \le 12\sqrt{2}\,\|f\|_{H[0,1)},$$

where $\hat{f}(n)$ are Walsh-Fourier coefficients of the function f.

Dyadic analogue of the theorem **B** is the following result.

Theorem 4.1. *If $f \in H(R_+)$, then $\int_{R_+} \frac{|\tilde{f}(x)|}{x}dx \le 50\sqrt{2}\,\|f\|_{H_d(R_+)}$. (Dyadic Hardy inequality on R_+).*

Problem. *What is the least constant in right-hand side of this inequality?*

Theorem 4.1 was proved in our paper [20]. S.S. Volosivets proved an analog of this theorem for dyadic Hardy space $H^p(R_+), 0 < p < 1$ (see [49], Theorem 12).

Definition 4.2. *Let the functions $(f * W_n^\alpha)^*(x)$ be defined by the equality*

$$(f * W_n^\alpha)^*(x) \equiv \int_{R_+} (f * W_n^\alpha\tilde{)}(t)\,\psi_x(t)\,dt, n \in N, \alpha > 0.$$

If for a function $f \in L(R_+)$ there exists the limit

$$J_\alpha^*(f)(x) \equiv \lim_{n\to+\infty} (f * W_n^\alpha)^*(x),$$

which is uniform on R_+, then we say that the function f has uniform modified dyadic integral (UMDI) $J_\alpha^*(f)(x)$ of order α.

It is not difficult to show that $J_1^*(f)(x) = (J_1)(f)(x)$ a.e. on R_+. Indeed, taking in account that the equality

$$(f * W_n^\alpha)^\sim(x) = \tilde{f}(x)\,\tilde{W}_n^\alpha(x) = \begin{cases} 0, 0 \le x < 2^{-n}; \\ \tilde{f}(x)/x, x \ge 2^{-n}, \end{cases}$$

holds almost every on R_+, we conclude that

$$(f * W_n^1)^*(x) \equiv \int_{t \ge 2^{-n}} \frac{\tilde{f}(t)}{t}\psi_x(t)\,dt = \int_{t\ge 2^{-n}} (J_1(t)\tilde{)}\psi_x(t)\,dt$$

a.e. on R_+.

By the Theorem 1.10 the sequence in right-hand side of this equality has finite limit a.e. on R_+ as $n \to +\infty$, and we have $J_1^*(f)(x) = (J_1)(f)(x)$ a.e. on R_+.

Theorem 4.3. *Each function $f \in H(R_+)$ has UMDI of first order which coincides with MSDI of first order a.e., i.e. $J_1^*(f) = J_1(f)$ a.e. More precisely, the operator $J_1^* : H(R_+) \to C_W(R_+)$ is bounded and*

$$\|J_1^*(f)\|_{C_W(R_+)} \leq 100\sqrt{2}\,\|f\|_{H(R_+)}\,.$$

The proof of this theorem is based on the lemma:

Lemma 4.4. *The functions $\psi(x, m2^{-n})\,\mathrm{X}_{[0,2^n)}(x)$, $m \in N, n \in Z$, belong to the space $H(R_+)$ and their linear hull L is dense in this space.*

Let us introduce the operator $\tilde{J}_\alpha(f) \equiv J_\alpha(\tilde{f})$. It is well defined on the subspace L of the space $H(R_+)$.

Theorem 4.5. *The operator $\tilde{J}_1 : L \to L(R_+)$ is bounded and his operator norm not exceed $100\sqrt{2}$. This operator may be extended continuously on the space $H(R_+)$ and we have*

$$\left\|\tilde{J}_1\right\|_{H(R_+)\to L(R_+)} \leq 100\sqrt{2}\,\|f\|_{H(R_+)}\,.$$

Theorems 4.3 and 4.5 were published in our paper [21].

5. Dyadic analogue of Tauberian theorem of Wiener and related topics

The following theorem of N. Wiener is known (see [52] or [53]).

Theorem A. *Let the function $K_1 \in L(R)$ is such, that its Fourier transform*

$$\hat{K}_1(x) = \int_R K_1(y)\exp(-ixy)\,dy$$

does not equal to zero at each $x \in R = (-\infty, +\infty)$. If

$$\lim_{x\to\infty}\int_R K_1(x-y)\,f(y)\,dy = C(f)\int_R K_1(y)\,dy, \tag{5.1}$$

for some function $f \in L^\infty(R)$, where $C(f)$ is some constant, then

$$\lim_{x\to\infty}\int_R K_2(x-y)\,f(y)\,dy = C(f)\int_R K_2(y)\,dy \tag{5.2}$$

for all functions $K_2 \in L(R)$.

It is not difficult to show that if $\hat{K}_1$ has a zero on R, then the condition (5.2) does not follow from (5.1).

N. Wiener proved also the following theorems.

Theorem B. *Let $f \in L(R)$ is a given function. Then linear hull of the set $\{f(\cdot + y) : y \in R\}$ is dense in the space $L(R)$ iff $\hat{f}(x) \neq 0$ on R.*

Theorem C. *Let $f \in L^2(R)$.Then the linear hull of the set $\{f(\cdot + y) : y \in R_+\}$ is dense in the space $L^2(R_+)$, iff $\tilde{f}(x) \neq 0$ for almost all $x \in R_+$.*

There exist following dyadic analogues of the Theorems **A - C**:

Theorem 5.1. *Let the function $K_1 \in L(R_+)$ is such that $\inf\limits_{x\in[0,a]} \left|\tilde{K}_1(x)\right| > 0$ for every $a \geq 0$. If*

$$\lim_{x\to+\infty} \int_{R_+} K(x \oplus y) f(y)\, dy = C(f) \int_{R_+} K(y)\, dy$$

for some function $f \in L^\infty(R_+)$ and $K = K_1$, then this limit exists for all functions $K \in L(R_+)$.

Theorem 5.2. *Let f be a function from the space $L(R_+)$. Then the linear hull of the set $\{f(\cdot \oplus y) : y \in R_+\}$ is dense in the space $L(R_+)$, iff $\inf\limits_{x\in[0,a]} \left|\tilde{f}(x)\right| > 0$ for all $a \geq 0$.*

Theorem 5.3. *Let $f \in L^2(R_+)$.Then the linear hull of the set $\{f(\cdot \oplus y) : y \in R_+\}$ is dense in the space $L^2(R_+)$, iff $\tilde{f}(x) \neq 0$ for almost all $x \in R_+$.*

These results are dyadic analogues of the results of N. Wiener mentioned above.
The results of N. Wiener were generalized on the spaces $L(G)$ and $L^2(G)$, where G is locally compact abelian group.
We consider the group R_+ with the topology generated by the base of the neighborhoods $[0, 2^{-n})_{n\in Z}$ of the zero. This group is not locally compact.
Therefore, our results formulated above are not follow from the generalizations mentioned above.

As a corollary from Theorems 5.2 and 5.4 one can obtain

Theorem 5.4. *Let be given a function $f \in L[0,1)$ or $f \in L^2[0,1)$. Then the linear hull of the set $\{f(\cdot \oplus y) : y \in [0,1)\}$ is dense in the space $L[0,1)$ (respectively in the space $L^2[0,1)$), iff*

$$\hat{f}(n) \equiv \int_0^1 f(x)\, w_n(x)\, dx \neq 0$$

for all $n \in Z_+$, where $\{w_n(x)\}_{n=0}^\infty$ is orthonormal Walsh system.

Theorems 5.1 - 5.4 were published in our paper [22]. Similar results on dyadic field were obtained in [23].

6. Dyadic Hardy and Hardy-Littlewood operators on the spaces $H_d(R_+)$ and $BMO_d(R_+)$

The operators

$$H(f)(x) = \int_x^{+\infty} \frac{f(t)}{t} dt, \tag{6.1}$$

$$B(f)(x) = \frac{1}{x} \int_0^x f(t)\, dt. \tag{6.2}$$

are called Hardy and Hardy-Littlewood operators respectively.

For these operators the inequalities of Hardy

$$\|H(f)\|_{L^p(R_+)} \leq p \|f\|_{L^p(R_+)} \left(1 \leq p < \infty\right), \tag{6.3}$$

$$\|B(f)\|_{L^p(R_+)} \leq \frac{p}{p-1} \|f\|_{L^p(R_+)} \left(1 < p \leq \infty\right) \tag{6.4}$$

are well known (see Theorems 327 and 328 in the book [28]).

It has been proved by E. Landau that the constants p and $p/(p-1)$ in right-hand side of these inequalities are the best possible.

The inequality (6.4) is invertible for positive decreasing functions. Indeed, for such functions the reverse inequality

$$\|B(f)\|_{L^p(R_+)} \geq \left(\frac{p}{p-1}\right)^{1/p} \|f\|_{L^p(R_+)} \,(1 < p \leq \infty) \tag{6.5}$$

holds and the constant in right-hand side of this inequality is the best possible (see [33] - [34]).

We define dyadic analogues of the operators (6.1) and (6.2) as follows.

For $x \in R_+ = [0, +\infty)$ let us denote by $n = n(x)$ the integer, which satisfies the inequalities $2^n \leq x < 2^{n+1}$. After that for a function $f \in L^p(R_+)$ we set

$$H_d(f) = \sum_{m=n+1}^{+\infty} 2^{-m} \int_{2^m}^{2^{m+1}} f(t)\, dt, \tag{6.6}$$

and

$$B_d(f)(x) = 2^{-n} \int_0^{2^n} f(t)\, dt, \tag{6.7}$$

for $1 < p \leq \infty$. We call the operators (6.6) and (6.7) dyadic Hardy and Hardy-Littlewood operator respectively. It is obvious that

$$|H_d(f)(x)| \leq 2H(|f|)(x), |B_d(f)(x)| \leq 2B(|f|)(x), x \in R_+.$$

Therefore from (6.3) and (6.4) we obtain

Theorem 6.1. *If $f \in L^p(R_+)$, then*

$$\|H_d(f)\|_{L^p(R_+)} \leq 2p \|f\|_{L^p(R_+)} \,(1 \leq p < \infty),$$

$$\|B_d(f)\|_{L^p(R_+)} \leq \frac{2p}{p-1} \|f\|_{L^p(R_+)} \,(1 < p \leq \infty).$$

Problem. *What are the least constants in right-hand side of these inequalities*?

For positive decreasing functions it is not difficult to show that

$$B_d(f)(x) \geq \frac{1}{2} B(f)(x), x > 0.$$

Therefore, by the inequality (6.5) for positive decreasing functions we have

$$\|B_d(f)\|_{L^p(R_+)} \geq \frac{1}{2}\left(\frac{p}{p-1}\right)^{1/p} \|f\|_{L^p(R_+)} \,(1 < p \leq \infty).$$

Problem. *What is the greatest constant in right-hand side of this inequality*?

Dyadic Hardy space $H_d(R_+)$ was defined in Section 4. Now let us define the space $BMO_d(R_+)$ of functions of dyadic bounded mean oscillaton on R_+.

Definition 6.2. *A function $f \in L_{loc}(R_+)$ belongs to the space $BMO_d(R_+)$, if*

$$\|f\|_* = \sup_{I\in\Delta} \frac{1}{|I|} \int_I |f(x) - f_I|\, dx < +\infty,$$

where $f_I = \frac{1}{|I|}\int_I f(t)\, dt$ and Δ denotes the set of all dyadic intervals on R_+.

Finally we define dyadic space $VMO_d(R_+)$ of functions of dyadic vanishing mean oscillation on R_+.

Definition 6.3. *A function $f \in L_{loc}(R_+)$ belongs to the space $VMO_d(R_+)$, if two conditions are satisfied:*

$$\lim_{\varepsilon\to+0} \sup_{0<|I|\le\varepsilon} K(f, I) = 0, \ \lim_{\varepsilon\to+\infty} \sup_{|I|\ge\varepsilon} K(f, I) = 0,$$

where $I \in \Delta$ and $K(f, I) = \inf\limits_{c\in R} \frac{1}{|I|}\int_I |f(x) - c|\, dx$.

Theorem 6.4. *The operator H_d is bounded in the space $H_d(R_+)$ and its norm does not exceed 2.*

Theorem 6.5. *The operator B_d is bounded in the space $BMO_d(R_+)$ and its norm does not exceed 8.*

The proof of these two theorems see, for example, in our book [26].
The last two theorems are dyadic analogues of the known results for the operators (6.1) and (6.2). The boundedness of the operators (6.1) and (6.2) on *ordinary* Hardy space $H(R)$ and on *ordinary* space $BMO(R)$ respectively has been proved by D. V. Giang and F. Moricz in [12].
Another proof of these results was given after that in our paper [16].

Theorem 6.6. *The operator (6.7) is bounded in the space $VMO_d(R_+)$ and its operator norm does not exceed 8.*

The theorems 6.2-6.4 where published in our paper [19]. The analogues of Theorems 6.4 and 6.6 for local Hardy and BMO spaces where proved in the paper [49].

Theorem 6.7. *The operator $H_d : H_d(R_+) \to H_d(R_+)$ is bounded and its operator norm does not exceed 2.*

Problem. *To find the norms of the operators in the Theorems 6.4-6.7?*
Let us note that in more general setting Theorem 6.7 was proved by S.S. Volosivets [49].

Theorem 6.8. *The space $H_d(R_+)$ does not invariant relative to the operator B_d.*

We denote by $BMO_d^0(R_+)$ the subspace of those functions $f \in BMO_d(R_+)$ for which the equality $\int_{R_+} f(x)\, dx = 0$ is valid. Then the following theorem is true.

Theorem 6.9. *The operator $H_d : BMO_d^0(R_+) \to BMO_d(R_+)$ does not bounded.*

For Hardy and Hardy-Littlewood operators (6.1) and (6.2) the following equalities

$$\widehat{H(f)}_c = B\left(\hat{f}_c\right)(1 \le p \le 2), \ \widehat{B(f)}_c = H\left(\hat{f}_c\right)(1 < p \le 2). \tag{6.8}$$

are known. Here $\hat{f}_c$ denotes cosine- Fourier transform of the function$f$.
The similar equalities for sine-Fourier transform are also valid.

For $p = 2$ these equalities have been proved by E.C. Titchmarsh [47] and for other values of $p \in (1, 2]$ by the author [15].
The equalities (6.8) without improving the conditions on the function were written also by R. Bellman in [1]. But Bellman does not refered to the book by E.C. Titchmarsh mentioned above.

The dyadic analogues of the equalities (6.8) are contained in the following theorem.

Theorem 6.10. *If $f \in L^p(R_+)$, then the following equalities hold*

$$H_d(\tilde{f}) = B_d\left(\tilde{f}\right)(1 \le p \le 2), B_d(\tilde{f}) = H_d\left(\tilde{f}\right)(1 < p \le 2). \tag{6.9}$$

Here $\tilde{f}$ denotes the Walsh-Fourier transform of the function f.

The last theorem was published in our paper [18]. The analogue of this theorem for distributions was proved in the paper [49].
The other dyadic versions of operators H_d and B_d, were introduced by the author in the paper [17].

Let us set

$$H_w(f)(x) = \int_{R_+} f(y)\left(\frac{D(y,t)}{t}\right)^{\sim}(x)\,dy, \tag{6.10}$$

and

$$B_w(f)(x) = \int_{R_+} f(y)\left(\frac{D(x,t)}{t}\right)^{\sim}(y)\,dy, \tag{6.11}$$

where

$$D(x,t) = \int_0^t \psi(x,y)\,dy.$$

We note that Walsh-Fourier transform in (6.10) and (6.11) is taken with respect to the argument t.
If we substitute

$$D_c(x,y) = \int_0^y \cos xt dt = \frac{\sin xy}{x}$$

instead of $D(x,y)$ and cosine Fourier transform instead of Walsh-Fourier transform in (6.10) and (6.11), then we obtain Hardy operator $H(f)$ and Hardy-Littlewood operator $B(f)$ respectively.

Theorem 6.11. *If $f \in L^p(R_+), 1 < p \le 2$, then*

$$H_w(\tilde{f}) = (B(f))\tilde{}, \; B_w(\tilde{f}) = (H_w(f))\tilde{}. \tag{6.12}$$

We note at last that dyadic analogues of Hardy and Hardy-Littlewood operators defined on the interval $[0,1)$ were considered by T. Eisner [8]. In [8] analogues of Hardy and Bellman transforms for trigonometric series are studied.
T. Eisner and F. Schipp [9] introduced a dyadic analogue of the Hardy operator H on R_+ as follows. They denote this operator by C and call it the Cesàro operator. The operator C is introduced by showing that for every function $f \in L(R_+)$ there exists a unique function $Cf \equiv g \in L(R_+)$ such that

$$\tilde{g}(x) = \frac{1}{x}\int_0^x \tilde{f}(u)\,du,$$

where $\tilde{f}$ denotes Walsh- Fourier transform of the function f.
The operator C may be extended uniquely on $L^p(R_+)$, $1 \le p < +\infty$. It is proved that the linear operator C is bounded in $L^p(R_+)$ for $1 \le p < \infty$ and it is unbounded for $p = \infty$.
The integral representation for the operator C is also obtained by T. Eisner and F. Schipp [9].
Let us note that F. Weisz [51] studied dyadic differentiation of functions from martingale Hardy spaces.

7. Some problems

The following theorem by A. Beurling [3] is known.

Theorem A. *Let $f \in L(R)$ and*

$$\iint_{R^2} \left| f(x) \hat{f}(y) \right| \exp(|xy|) dx dy < +\infty,$$

where $\hat{f}$ is Fourier transform of the function f, i.e.

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) \exp(-ixy)\, dy.$$

Then $f = 0$.

Corollary. *If for a function $f \in L(R)$ its Fourier transform $\hat{f}(x)$ equals to zero in some neighborhoods of 0 and ∞, then $f = 0$.*

Problem 1. *To prove a dyadic analogue of Theorem **A** or Corollary.*

The following theorem is known.

Theorem B. *For every set $E \subset [0, 2\pi]$ having Lebesgue measure equal to zero there exists a continuous 2π-periodic function whose trigonometric Fourier series diverges on the set E.*

V. M. Bugadze [5] proved the following theorem.

Theorem C. *For every set $E \subset [0, 1]$ having Lebesgue measure equal to zero there exists a bounded measurable function, whose Fourier series with respect to Walsh-Paley system diverges on the set E.*

K. R. Bitsadze [4] generalized this theorem on Fourier series with respect to multiplicative systems with bounded generating sequences.

Problem 2. *Is the assertion of the Theorem **C** for $W-$continuous function valid or not?*
Let $f^{[1]}(x) \ge 0$ (or $f^{[1]}(x) > 0$) at each point $x \in E$, where $E = [0, 1)$ or $E = [0, +\infty)$ and $f^{[1]}(x)$ is the Butzer-Wagner dyadic derivative of the function f.

Problem 3. *What properties has the function f satisfying the conditions mentioned above on the set E?*

Let the function f is nondecreasing (or strong increasing) on the set E, where $E = [0, 1)$ or $E = [0, +\infty)$. Assume that there exists dyadic derivative $f^{[1]}(x)$ at each point $x \in E$.
Problem 4. *What properties has this derivative $f^{[1]}(x)$ on the set E?*

It is well known that if $f''(x) \geq 0$ *(or* $f''(x) > 0$*)* for each point $x \in (a, b)$, then the function f is convex (or strong convex) on the interval (a, b).

Problem 5. *Let $f^{[2]}(x) \geq 0$ (or $f^{[2]}(x) > 0$) at each point $x \in E$, where $E = [0, 1)$ or $E = [0, +\infty)$, where $f^{[2]}(x)$ is the Butzer-Wagner dyadic derivative of second order of the function f. What properties has the function f on the set E?*

Problem 6. *Let the function f be convex on the set E, where $E = [0, 1)$ or $E = [0, +\infty)$. Assume that there exists dyadic derivative $f^{[2]}(x)$ at each point $x \in E$. What properties has this derivative $f^{[2]}(x)$ on the set E?*

It is known that each function f of bounded variation on the segment $[a, b]$ is differentiable almost everywhere on $[a, b]$.

Problem 7. *Let $f \in V[0, 1]$,* i.e. *the function f* has bounded variation on the interval $[0, 1]$. *Is it true that dyadic derivative $f^{[1]}(x)$ exists almost everywhere on $[0, 1]$?*

Let us define the notion of the function of bounded fluctuation of order $p \in [1, \infty)$ on $[0, 1)$. To this goal we introduce the notations

$$osc\,(f, [a, b)) = \sup_{x,y \in [a,b)} |f(x) - f(y)| \ , \quad I_j^{(n)} = \left[\frac{j-1}{2^n}, \frac{j}{2^n}\right)$$

and set

$$k_p(f, n) = \left(\sum_{j=1}^{2^n} \left(osc\left(f, I_j^{(n)}\right)\right)^p\right)^{1/p} \ , \quad k_p(f, 0) = osc\,(f, [0, 1)) \, , \quad 1 \leq p < \infty.$$

If

$$F_p(f) \equiv \sup_{n \in Z_+} k_p(f, n) < \infty,$$

where $Z_+ = \{0, 1, ...\}$, then the function f to be said of bounded dyadic fluctuation of order p on $[0, 1)$. This notion was introduced by C. W. Onneweer and D. Waterman [35].

Problem 8. *Let be given a function f of bounded dyadic fluctuation of order 1 on* $[0, 1)$. *Dues there exist the Butzer-Wagner derivative $f^{[1]}(x)$ almost everywhere on* $[0, 1)$?

The problems 4-8 were published by us in the book [46], p. 326.

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