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# Matrix-valued wave packet Bessel sequences and symmetric frames in $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$

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***Abstract.*** We consider matrix-valued wave packet systems in the matrix-valued function space $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ $(d, s, r \geq 1)$. Some results on matrix-valued wave packet Bessel sequences have been extensively discussed in view to generate frames from Bessel sequences in $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$. Necessary conditions for matrix-valued Bessel sequences in terms of an estimate of a series related to the Fourier transform of the matrix-valued wave packet function are given. The frame property of a matrix-valued symmetric wave packet function in higher dimensions is discussed.

## 1. Introduction

Antolín and Zalik in [1], introduced and studied matrix-valued wavelets for the matrix-valued function space $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$. The matrix-valued function space $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ is related to video imaging. Xia and Suter [29] classified and constructed vector-valued (matrix-valued) wavelets with sampling property. They showed that certain linear combinations of known scalar-valued wavelets may yield multiwavelets. Inspired by the work in [1], Jyoti, Deepshikha, Vashisht and Verma [18] studied matrix-valued wave packet frames in $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, where $s$ and $r$ are positive integers. They discussed an interplay between wave packet frames in the Lebesgue space $L^2(\mathbb{R}^d)$ and matrix-valued wave packet frames for the function space $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$. The first purpose of this paper is to show some relation between the series related to the Fourier

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transform of the matrix-valued wave packet function and Bessel bound. The second purpose is to discuss extension problem of frame theory in the context of extension of matrix-valued Bessel sequences to dual frames for $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$. By using the technique of Christensen, Kim and Kim, the extension problem of matrix-valued Bessel sequences in $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ to frames for $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ is discussed. Finally, we show that, as in the case of standard wave packet frames [28, Theorem 1], the frame properties of matrix-valued symmetric wave packet frames associated with a given matrix-valued wave packet frame for the underlying function space are preserved.

**1.1. Relation to existing work.** Duffin and Schaeffer [13] introduced the concept of a frame for separable Hilbert spaces, while addressing some difficult problems from the theory of non-harmonic Fourier series. Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle ., .\rangle$; and norm induced by the inner product $\langle ., .\rangle$ is given by $\|f\| = \sqrt{\langle f, f\rangle}$, $f \in \mathcal{H}$. A family $\{f_k\}_{k=1}^{\infty} \subset \mathcal{H}$ is called a *frame* for $\mathcal{H}$, if there exist positive scalars $A_o \le B_o < \infty$ such that for all $f \in \mathcal{H}$,

$$A_o\|f\|^2 \le \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \le B_o\|f\|^2. \tag{1.1}$$

The scalars $A_o$ and $B_o$ are called *lower frame bound* and *upper frame bound*, respectively. If it is possible to choose $A_o = B_o$, then we say that $\{f_k\}_{k=1}^{\infty}$ is a $A_o$*-Parseval frame* (or $A_o$*-tight frame*). If only upper inequality in (1.1) holds, then we say that $\{f_k\}_{k=1}^{\infty}$ is a *Bessel sequence* with *Bessel bound* $B_o$. If $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$, then $S : \mathcal{H} \to \mathcal{H}$ given by $Sf = \sum_{k=1}^{\infty}\langle f, f_k\rangle f_k$ is a bounded, linear and invertible operator on $\mathcal{H}$, and is called the frame operator. This gives the *reconstruction formula* of each vector $f \in \mathcal{H}$, $f = SS^{-1}f = \sum_{k=1}^{\infty}\langle S^{-1}f, f_k\rangle f_k$. Thus, each vector has an explicit series expansion which need not be unique. For application of frames and related concepts we refer to book of Casazza and Kutyniok [2], Christensen [5] and Han [16]. Very recent work on discrete frames of translates and discrete wavelet frames, and their duals in finite dimensional spaces can be found in [10, 11]. The theory of iterated function systems, quantum mechanics and wavelets is emerging in important applications in frame theory, see [1, 12, 14, 15, 16, 21, 22, 26, 30, 31, 32] and many references therein.

Cordoba and Fefferman [7] introduced the concept of wave packet system by applying certain collection of dilations, modulations and translations to the Gaussian function in the study of some classes of singular integral operators. In mathematical physics, a wave packet is a function $\Psi$ defined on a domain $\Omega \subset \mathbb{R}^d$ which is "well localized in phase space". That is, $\Psi$ and its Fourier transform $\widehat{\Psi}$ are both concentrated in reasonably small sets. The wave packets over the real line are generated from classical dilations, translations, and modulations of a given window (or scaling) function. Labate, Weiss, and Wilson [23] adopted the same expression to describe, more generally, any collection of functions which are obtained by applying the same operations to a finite family of functions in $L^2(\mathbb{R}^d)$. Gabor systems, wavelet systems and the Fourier transform of wavelet systems are special cases of wave packet systems. Lacey and Thiele [24, 25] gave applications of wave packet systems in boundedness of the Hilbert transforms. The wave packet systems and related frame properties have been studied by several authors, see [3, 8, 9, 18, 19, 20] and references therein.

This work has its motivation in the study of matrix-valued wavelets [1] by Antolín and Zalik in the function space $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$. We recall that the standard norm of the function space $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ is not induced by the vector-valued inner product, see (2.1). Further, in Hilbert frame inequality (1.1), the norm of $\mathcal{H}$ is induced by the inner product on $\mathcal{H}$. This makes study for matrix-valued frames in $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ an interesting investigation. Jyoti and Vashisht [18] introduced matrix-valued wave packet frames in $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$. It is showed in [18] that the frame properties, in general, cannot be carried from $L^2(\mathbb{R}^d)$ to the matrix-valued function space $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ and vice-versa. The WH-packets and sums of matrix-valued wave packet frames in $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ are studied in [19, 20]. In the present work, we study matrix-valued Bessel sequences and symmetric frames in $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$. Different necessary and sufficient conditions for matrix-valued wave packet Bessel sequences are given. Some sufficient conditions on a pair of window functions in the matrix-valued function space $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ for matrix-valued wave packet Bessel sequences to be frames for $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ are given.

**1.2. Overview and main results.** We organize the paper as follows. Section 2 recalls basic notations and definitions to make the paper self-contained. The main results are given in Section 3. A necessary condition in Theorem 3.1 for matrix-valued Bessel sequences in $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ in terms of the estimate of a series related to the Fourier transform of the matrix-valued wave packet function initiates Section 3. Theorem 3.2 gives estimate of the matrix-valued and corresponding wave packet functions in terms of Bessel bounds, and Theorem 3.4 extends sufficient condition for wave packet systems in $L^2(\mathbb{R}^d)$ in terms of an estimate of a series associated with a given wave packet system to matrix-valued Bessel sequences in $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$. An application for Theorem 3.4 can be looked in Example 3.5. By now we are ready to derive sufficient conditions to discuss extension of Bessel sequences to frames or dual frames for the underlying space. The results proving sufficient conditions to study the extension of a pair of WHMV wave packet Bessel sequences to dual WHMV wave packet frames for $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ are given in Theorem 3.6 and Theorem 3.8. Section 4 gives credit to frame properties of matrix-valued symmetric wave packet frames associated with a given matrix-valued wave packet frame for the underlying function space. Theorem 4.2 shows that as in the case of standard wave packet frames, frame properties can be extended to matrix-valued wave packet frames in higher dimensions. The proof of this result also uses Lemma 4.1 which gives a property of the Frobenius norm on the matrix-valued function space.

## 2. Preliminaries

As is standard, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of all integers, positive integers, real numbers and complex numbers, respectively. Symbol $\mathbf{O}$ denotes the zero-matrix of finite order, and for $s \in \mathbb{N}$, $\mathbf{O}_{s\times s}$ and $\mathbf{I}_{s\times s}$ denote the zero matrix and the identity matrix of order $s$-by-$s$, respectively. Matrix-valued functions are denoted by bold letters. Let $s, r \in \mathbb{N}$ be arbitrary, but fixed. We consider the matrix-valued function space

$$L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})=\Big\{\mathbf{f}(\xi):\ f_{ij}(\xi)\in L^2(\mathbb{R}^d),\ \xi\in\mathbb{R}^d,\ 1\le i\le s,\ 1\le j\le r\Big\},$$

where

$$\mathbf{f}(\xi)=\begin{bmatrix} f_{11}(\xi) & f_{12}(\xi) & \dots & f_{1r}(\xi)\\ f_{21}(\xi) & f_{22}(\xi) & \dots & f_{2r}(\xi)\\ & & \dots & \\ f_{s1}(\xi) & f_{s2}(\xi) & \dots & f_{sr}(\xi)\end{bmatrix}.$$

The Frobenius norm on $L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$ is given by

$$\|\mathbf{f}\|=\Big(\sum_{\substack{1\le i\le s\\1\le j\le r}}\int_{\mathbb{R}^d}|f_{ij}(\xi)|^2d\xi\Big)^{\frac{1}{2}}.$$

The function space $L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$ is a Banach space with respect to the Frobenius norm.

The integration of a matrix-valued function $\mathbf{f}\in L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$ is given by

$$\int_{\mathbb{R}^d}\mathbf{f}(\xi)=\begin{bmatrix} \int_{\mathbb{R}^d}f_{11}(\xi) & \int_{\mathbb{R}^d}f_{12}(\xi) & \dots & \int_{\mathbb{R}^d}f_{1r}(\xi)\\ \int_{\mathbb{R}^d}f_{21}(\xi) & \int_{\mathbb{R}^d}f_{22}(\xi) & \dots & \int_{\mathbb{R}^d}f_{2r}(\xi)\\ & \dots & \dots & \\ \int_{\mathbb{R}^d}f_{s1}(\xi) & \int_{\mathbb{R}^d}f_{s2}(\xi) & \dots & \int_{\mathbb{R}^d}f_{sr}(\xi)\end{bmatrix}.$$

For any matrix-valued functions $\mathbf{f},\mathbf{g}\in L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$, we write

$$\langle\mathbf{f},\mathbf{g}\rangle=\int_{\mathbb{R}^d}\mathbf{f}(\xi)\mathbf{g}^*(\xi), \tag{2.1}$$

where $*$ denotes the transpose and the complex conjugate. Note that $\langle\bullet,\bullet\rangle$ is matrix-valued and therefore it is not an inner product. It has the following properties.

(i) For every $\mathbf{f},\mathbf{g}\in L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$,

$$\langle\mathbf{f},\mathbf{g}\rangle=\langle\mathbf{g},\mathbf{f}\rangle^*.$$

(ii) For every $\mathbf{f},\mathbf{g},\mathbf{h}\in L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$ and every $M_1,M_2\in\mathcal{M}_s(\mathbb{C})$, where $\mathcal{M}_s(\mathbb{C})$ denotes the complex vector space of all $s\times s$ complex matrices

$$\langle M_1\mathbf{f}+M_2\mathbf{g},\mathbf{h}\rangle=M_1\langle\mathbf{f},\mathbf{h}\rangle+M_2\langle\mathbf{g},\mathbf{h}\rangle.$$

It is also readily seen that

$$\|\mathbf{f}\|^2=\mathrm{trace}\langle\mathbf{f},\mathbf{f}\rangle.$$

Two functions $\mathbf{f},\mathbf{g}\in L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$ are said to be orthogonal if $\langle\mathbf{f},\mathbf{g}\rangle=\mathbf{O}_{s\times s}$. Let $\mathbb{V},\mathbb{W}$ be two closed subspaces of $L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$ such that $\mathbb{W}\subset\mathbb{V}$, then the orthogonal complement of $\mathbb{W}$ in $\mathbb{V}$ is the closed subspace defined by

$$\mathbb{W}^\perp=\{\mathbf{h}\in\mathbb{V}:\langle\mathbf{h},\mathbf{f}\rangle=\mathbf{O}_{s\times s},\ \text{for all }\mathbf{f}\in\mathbb{W}\}.$$

A sequence $\{\mathbf{f}_k\}_{k\in\mathbb{N}}$ in $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ is said to be *orthonormal*, if

$$\langle \mathbf{f}_k, \mathbf{f}_l\rangle = \begin{cases} \mathbf{I}_{s\times s}, k = l, \\ \mathbf{O}_{s\times s}, k \neq l. \end{cases} \tag{2.2}$$

Moreover, a sequence $\{\mathbf{f}_k\}_{k\in\mathbb{N}} \subset L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ is an orthonormal basis for $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ if it satisfies (2.2) and every $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ can be written as

$$\mathbf{f} = \sum_{k\in\mathbb{N}} \langle \mathbf{f}, \mathbf{f}_k\rangle \mathbf{f}_k.$$

The Fourier transform of a matrix-valued function $\mathbf{f} \in L^1(\mathbb{R}^d, \mathbb{C}^{s\times r}) \cap L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, denoted by $\hat{\mathbf{f}}$, is defined by

$$\hat{\mathbf{f}}(\gamma) = \int_{\mathbb{R}^d} \mathbf{f}(x) e^{-2\pi i x\cdot\gamma} dx, \ \gamma \in \mathbb{R}^d.$$

By $GL_d(\mathbb{R})$ we denote the set of all invertible $d$-by-$d$ matrices over $\mathbb{R}$. Let $a, b \in \mathbb{R}^d$ and let $C$ be a real $d$-by-$d$ matrix. We consider operators $T_a, \ E_b, D_C : L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}) \to L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ given by

$$\begin{aligned} &T_a\mathbf{f}(\xi) = \mathbf{f}(\xi - a) \quad \text{(Translation by } a); \\ &D_C\mathbf{f}(\xi) = |detC|^{\frac{1}{2}} \mathbf{f}(C\xi) \quad \text{(Dilation by } C); \\ &E_b\mathbf{f}(\xi) = e^{2\pi i b\cdot\xi}\mathbf{f}(\xi), \quad \text{(Modulation by } b), \end{aligned}$$

where $b\cdot\xi$ is the standard inner product of $b$ and $\xi$ in the Euclidean space $\mathbb{R}^d$.

We consider the space $\ell^2(\mathbb{Z}^{d+2}, \mathcal{M}_s(\mathbb{C}))$ given by

$$\ell^2(\mathbb{Z}^{d+2}, \mathcal{M}_s(\mathbb{C})) := \Big\{ \{M_k\}_{k\in\mathbb{Z}^{d+2}} \subset \mathcal{M}_s(\mathbb{C}) : \sum_{k\in\mathbb{Z}^{d+2}} \|M_k\|^2 < \infty \Big\}.$$

The space $\ell^2(\mathbb{Z}^{d+2}, \mathcal{M}_s(\mathbb{C}))$ is a Hilbert space with related norm given by

$$\|\{M_k\}_{k\in\mathbb{Z}^{d+2}}\| = \Big( \sum_{k\in\mathbb{Z}^{d+2}} \|M_k\|^2 \Big)^{\frac{1}{2}}.$$

Let $\Psi \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ be given by

$$\Psi(\xi) = \begin{bmatrix} \psi_{11}(\xi) & \psi_{12}(\xi) & \dots & \psi_{1r}(\xi) \\ \psi_{21}(\xi) & \psi_{22}(\xi) & \dots & \psi_{2r}(\xi) \\ \dots & \dots & \dots & \dots \\ \psi_{s1}(\xi) & \psi_{s2}(\xi) & \dots & \psi_{sr}(\xi) \end{bmatrix}. \tag{2.3}$$

Functions $\psi_{in}, \ 1 \leq i \leq s, \ 1 \leq n \leq r$, are called *atoms* of $\Psi$. Let $\Lambda_1, \Lambda_2 \subset \mathbb{R}$ and $\Lambda^d \subset \mathbb{R}^d$ be countable sets. Let $\{A_j\}_{j\in\Lambda_1}$ be a sequence in $GL_d(\mathbb{R})$, $B \in GL_d(\mathbb{R})$ and $\{C_m\}_{m\in\Lambda_2} \subset \mathbb{R}^d$. A system of the form

$$\mathcal{W}(\Psi) \equiv \{D_{A_j} T_{Bk} E_{C_m} \Psi\}_{j\in\Lambda_1, m\in\Lambda_2, k\in\Lambda^d}$$

is called a *Weyl-Heisenberg matrix-valued wave packet system* in $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$. Throughout the paper, we consider $\Lambda_1 = \Lambda_2 = \mathbb{Z}$ and $\Lambda^d = \mathbb{Z}^d$. For the sake of simplicity, we denote $(A^T)^{-1}$ by $A^\sharp$.

**Definition 2.1.** [18] A family $\mathcal{W}(\Psi) = \{D_{A_j} T_{Bk} E_{C_m} \Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \subset L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ is called a *Weyl-Heisenberg matrix-valued wave packet frame* (*WHMV wave packet frame*, in short) for $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, if there exist positive scalars $L_\Psi \leq U_\Psi < \infty$ such that

$$L_\Psi \|\mathbf{f}\|^2 \leq \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\| \left\langle D_{A_j} T_{Bk} E_{C_m} \Psi, \mathbf{f} \right\rangle \right\|^2 \leq U_\Psi \|\mathbf{f}\|^2 \tag{2.4}$$

holds for every $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$.

The scalars $L_\Psi$ and $U_\Psi$ are called *lower wave packet frame bound* and *upper wave packet frame bound*, respectively. In this case we say that $\Psi$ is a matrix-valued wave packet function. If upper inequality in (2.4) is satisfied, then we say that $\mathcal{W}(\Psi)$ is a *WHMV wave packet Bessel sequence* with *wave packet Bessel bound* $U_\Psi$. For $s = r = 1$, $\mathcal{W}(\Psi) = \{D_{A_j} T_{Bk} E_{C_m} \Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ is the ordinary wave packet frame for $L^2(\mathbb{R}^d)$.

The operator $T_\Psi : \ell^2(\mathbb{Z}^{d+2}, \mathcal{M}_s(\mathbb{C})) \to L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ defined by

$$T_\Psi : \{M_{j,k,m}\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \mapsto \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} M_{j,k,m} D_{A_j} T_{Bk} E_{C_m} \Psi$$

is called the *pre-frame operator* (or *synthesis operator*). The *analysis operator* $\Theta_\Psi : L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}) \to \ell^2(\mathbb{Z}^{d+2}, \mathcal{M}_s(\mathbb{C}))$ is given by

$$\Theta_\Psi : \mathbf{f} \mapsto \Big\{ \langle \mathbf{f}, D_{A_j} T_{Bk} E_{C_m} \Psi \rangle \Big\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}.$$

The composition $S_\Psi = T_\Psi \Theta_\Psi : L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}) \to L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ is called the *frame operator* associated with $\mathcal{W}(\Psi)$, and is given by

$$S_\Psi : \mathbf{f} \mapsto \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \langle \mathbf{f}, D_{A_j} T_{Bk} E_{C_m} \Psi \rangle D_{A_j} T_{Bk} E_{C_m} \Psi, \ \mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}).$$

$S_\Psi$ is a well-defined bounded linear operator.

Inequalities in (2.4) can be written as

$$A_\Psi \mathrm{trace}\langle \mathbf{f}, \mathbf{f} \rangle \leq \mathrm{trace}\langle S_\Psi \mathbf{f}, \mathbf{f} \rangle \leq B_\Psi \mathrm{trace}\langle \mathbf{f}, \mathbf{f} \rangle, \text{ for all } \mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}).$$

This gives

$$0 \leq \mathrm{trace}\langle (I - {B_\Psi}^{-1} S_\Psi) \mathbf{f}, \mathbf{f} \rangle \leq \mathrm{trace}\Big\langle \Big( \frac{B_\Psi - A_\Psi}{B_\Psi} \Big) \mathbf{f}, \mathbf{f} \Big\rangle, \ \mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}). \tag{2.5}$$

The frame operator $S_\Psi$ associated with the WHMV wave packet frame $\mathcal{W}(\Psi)$ is an invertible operator on $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$.

We end the section by recording different necessary and sufficient conditions for Bessel sequences in $L^2(\mathbb{R}^d)$ which can be found in [3].

**Proposition 2.2.** [3] *Let $\{D_{A_j}T_{Bk}E_{C_m}\psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ be a Bessel sequence in $L^2(\mathbb{R}^d)$ with bound $U$. Then,*

$$\sum_{j,m\in\mathbb{Z}} |\hat{\psi}(({A_j}^T)^{-1}\gamma - C_m)|^2 \leq U|detB| \ a.e.$$

*Proof.* Immediate consequence of an intermediate step (equation (6)) in the proof of [3, Corollary 3.3] and [3, Proposition 3.4]. □

**Proposition 2.3.** [3] *Let $\{D_{A_j}T_{Bk}E_{C_m}\psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ be a wave packet system in $L^2(\mathbb{R}^d)$ such that*

$$U := \frac{1}{|detB|} \sup_{\gamma\in\mathbb{R}^d} \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\hat{\psi}(({A_j}^T)^{-1}\gamma - C_m)\hat{\psi}(({A_j}^T)^{-1}\gamma - C_m - (B^T)^{-1}k)| < \infty.$$

*Then $\{D_{A_j}T_{Bk}E_{C_m}\psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ is a Bessel sequence in $L^2(\mathbb{R}^d)$ with bound $U$.*

## 3. Matrix-valued Wave Packet Bessel sequences

We start with a necessary condition for matrix-valued Bessel sequences in $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ in terms of the estimate of a series related to the Fourier transform of the matrix-valued wave packet function.

**Theorem 3.1.** *Let $\{D_{A_j}T_{Bk}E_{C_m}\Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ be a Bessel sequence in $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ with bound $U_\Psi$ where*

$$\Psi = \begin{bmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1r} \\ \psi_{21} & \psi_{22} & \cdots & \psi_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ \psi_{s1} & \psi_{s2} & \cdots & \psi_{sr} \end{bmatrix}.$$

*Then,*

$$\sum_{j,m\in\mathbb{Z}} \|\hat{\Psi}(({A_j}^T)^{-1}\gamma - C_m)\|^2 \leq srU_\Psi|detB| \ a.e.$$

*Proof.* Let $i, n$ $(1 \leq i \leq s, 1 \leq n \leq r)$ and $f \in L^2(\mathbb{R}^d)$ be arbitrary, but fixed.

Choose $\mathbf{h}(\bullet) = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \underbrace{f(\bullet)}_{in^{\text{th}}\ \text{place}} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{bmatrix} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$.

Then

$$\begin{aligned}\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left|\left\langle D_{A_j}T_{Bk}E_{C_m}\psi_{in}, f\right\rangle\right|^2 &\leq \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\|\left\langle D_{A_j}T_{Bk}E_{C_m}\Psi, \mathbf{h}\right\rangle\right\|^2 \\ &\leq U_\Psi\|\mathbf{h}\|^2 \\ &= U_\Psi\|f\|^2.\end{aligned}$$

This implies that for each $i, n$ $(1 \leq i \leq s, 1 \leq n \leq r)$, the family $\{D_{A_j}T_{Bk}E_{C_m}\psi_{in}\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ is a Bessel sequence for $L^2(\mathbb{R}^d)$ with wave packet Bessel bound $U_\Psi$. By Proposition 2.2, we have

$$\sum_{j,m\in\mathbb{Z}} |\hat{\psi_{in}}((A_j{}^T)^{-1}\gamma - C_m)|^2 \leq U_\Psi|detB| \text{ a.e. } (1 \leq i \leq s, 1 \leq n \leq r).$$

Therefore

$$\begin{aligned}&\sum_{j,m\in\mathbb{Z}} \|\hat{\Psi}((A_j{}^T)^{-1}\gamma - C_m)\|^2 \\ &= \sum_{\substack{1\leq i\leq s \\ 1\leq n\leq r}} \sum_{j,m\in\mathbb{Z}} |\hat{\psi_{in}}((A_j{}^T)^{-1}\gamma - C_m)|^2 \\ &\leq srU_\Psi|\det B| \text{ a.e.}\end{aligned}$$

This concludes the proof. □

The next result gives estimate of the matrix-valued and corresponding wave packet functions in terms of Bessel bounds. In quantum physics, this type of estimate is known as the "majorization of energy of window functions". The reason is that the square of the norm of a signal is called its energy, and here it is dominated by Bessel bounds.

**Theorem 3.2.** *Let $\{D_{A_j}T_{Bk}E_{C_m}\Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ be a Bessel sequence in $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ with Bessel bound $U_\Psi$ . Then,*

*(i)* $\|\Psi\|^2 \leq sU_\Psi$ .

*(ii)* $\|\psi_{in}\|^2 \leq U_\Psi$, *for every* $1 \leq i \leq s, 1 \leq n \leq r$.

*Proof.* $(i)$: Since

$$\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\|\left\langle D_{A_j}T_{Bk}E_{C_m}\Psi, \mathbf{f}\right\rangle\right\|^2 \leq U_\Psi\|\mathbf{f}\|^2 \text{ for all } \mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}),$$

for any (but fixed) $j_o, m_o \in \mathbb{Z}, k_o \in \mathbb{Z}^d$, we have

$$\left\|\left\langle D_{A_{j_o}} T_{Bk_o} E_{C_{m_o}} \Psi, D_{A_{j_o}} T_{Bk_o} E_{C_{m_o}} \Psi\right\rangle\right\|^2 \leq U_\Psi \|D_{A_{j_o}} T_{Bk_o} E_{C_{m_o}} \Psi\|^2. \quad (3.1)$$

We compute

$$\begin{aligned}
\|\Psi\|^4 &= \|D_{A_{j_o}} T_{Bk_o} E_{C_{m_o}} \Psi\|^4 \\
&= |\text{trace}\langle D_{A_{j_o}} T_{Bk_o} E_{C_{m_o}} \Psi, D_{A_{j_o}} T_{Bk_o} E_{C_{m_o}} \Psi\rangle|^2 \\
&\leq s\left\|\left\langle D_{A_{j_o}} T_{Bk_o} E_{C_{m_o}} \Psi, D_{A_{j_o}} T_{Bk_o} E_{C_{m_o}} \Psi\right\rangle\right\|^2 \\
&\leq sU_\Psi \|D_{A_{j_o}} T_{Bk_o} E_{C_{m_o}} \Psi\|^2 \ \Big(\text{by } (3.1)\Big) \\
&= sU_\Psi \|\Psi\|^2.
\end{aligned}$$

This proves (i).

*(ii)*: From the first step of the proof of Theorem 3.1, for each $1 \leq i \leq s, 1 \leq n \leq r$, the family $\{D_{A_j} T_{Bk} E_{C_m} \psi_{in}\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ is a Bessel sequence in $L^2(\mathbb{R}^d)$ with bound $U_\Psi$ . That is, for each $f \in L^2(\mathbb{R}^d)$, we have

$$\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left|\left\langle D_{A_j} T_{Bk} E_{C_m} \psi_{in}, f\right\rangle\right|^2 \leq U_\Psi \|f\|^2 \ \Big(1 \leq i \leq s, 1 \leq n \leq r\Big).$$

Therefore, for fixed $j', m' \in \mathbb{Z}, k' \in \mathbb{Z}^d$, we have

$$\left|\left\langle D_{A_{j'}} T_{Bk'} E_{C_{m'}} \psi_{in}, D_{A_{j'}} T_{Bk'} E_{C_{m'}} \psi_{in}\right\rangle\right|^2 \leq U_\Psi \|D_{A_{j'}} T_{Bk'} E_{C_{m'}} \psi_{in}\|^2,$$

which implies

$$\|\psi_{in}\|^2 \leq U_\Psi \quad \Big(1 \leq i \leq s, 1 \leq n \leq r\Big).$$

Hence, (ii) is proved.

□

A matrix-valued wave packet system in $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ may not be a Bessel sequence. This is justified in the following example.

**Example 3.3.** Let $\psi \in L^2(\mathbb{R})$ be given by $\psi = \chi_{[0,1]}$. Then $\{E_m T_k \psi\}_{m,k\in\mathbb{Z}}$ and hence $\{T_k E_m \psi\}_{m,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$, see [5, p. 96]. Let $\Psi \in L^2(\mathbb{R}, \mathbb{C}^{2\times 1})$ be given by $\Psi = \begin{bmatrix}\psi \\ 0\end{bmatrix}$. We will show that $\{D_{2^j} T_k E_m \Psi\}_{j,m,k\in\mathbb{Z}}$ is not a Bessel sequence for $L^2(\mathbb{R}, \mathbb{C}^{2\times 1})$. Let if possible, $\{D_{2^j} T_k E_m \Psi\}_{j,m,k\in\mathbb{Z}}$

be a Bessel sequence with bound $U_\Psi$. Then, for any fixed $\mathbf{f}_o = \begin{bmatrix} f_{11} \\ f_{21} \end{bmatrix} \in L^2(\mathbb{R}, \mathbb{C}^{2\times 1})$, we compute

$$
\begin{aligned}
&\sum_{j,m,k\in\mathbb{Z}} \left\| \left\langle D_{2^j} T_k E_m \Psi, \mathbf{f}_o \right\rangle \right\|^2 \\
&= \sum_{j,m,k\in\mathbb{Z}} \left[ \left| \langle T_k E_m \psi, D_{2^{-j}} f_{11} \rangle \right|^2 + \left| \langle T_k E_m \psi, D_{2^{-j}} f_{21} \rangle \right|^2 \right] \\
&= \sum_{j\in\mathbb{Z}} \left[ \|D_{2^{-j}} f_{11}\|^2 + \|D_{2^{-j}} f_{21}\|^2 \right] \\
&= \sum_{j\in\mathbb{Z}} \left[ \|f_{11}\|^2 + \|f_{21}\|^2 \right] \\
&= \sum_{j\in\mathbb{Z}} \|\mathbf{f}_o\|^2 \\
&> U_\Psi \|\mathbf{f}_o\|^2,
\end{aligned}
$$

which is a contradiction.

Christensen and Rahimi gave a sufficient condition for wave packet systems in $L^2(\mathbb{R}^d)$ in terms of an estimate of a series associated with a given wave packet system, see Proposition 2.3. In the following result, we observed that the Bessel property is preserved in higher dimensions. This extends Proposition 2.3 to matrix-valued Bessel sequences in the function space $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$.

**Theorem 3.4.** *Let $\{D_{A_j} T_{Bk} E_{C_m} \Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ be a WHMV wave packet system in $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ such that*

$$
\begin{aligned}
U_\Psi := &\max_{\substack{1\le i\le s \\ 1\le n\le r}} \frac{1}{|detB|} \times \\
&\sup_{\gamma\in\mathbb{R}^d} \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\hat{\psi}_{in}(({A_j}^T)^{-1}\gamma - C_m) \hat{\psi}_{in}(({A_j}^T)^{-1}\gamma - C_m - (B^T)^{-1}k)| < \infty.
\end{aligned}
$$

*Then, $\{D_{A_j} T_{Bk} E_{C_m} \Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ is a WHMV wave packet Bessel sequence with Bessel bound $B_o = rsU_\Psi$.*

*Proof.* For any $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, we compute

$$\begin{aligned}
&\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\| \left\langle D_{A_j} T_{Bk} E_{C_m} \Psi, \mathbf{f} \right\rangle \right\|^2 \\
&= \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \Big[ \big| \sum_{n=1}^{r} \int_{\mathbb{R}^d} D_{A_j} T_{Bk} E_{C_m} \psi_{1n} \overline{f_{1n}} \big|^2 + \cdots \\
&\qquad\qquad + \big| \sum_{n=1}^{r} \int_{\mathbb{R}^d} D_{A_j} T_{Bk} E_{C_m} \psi_{sn} \overline{f_{1n}} \big|^2 \\
&\quad + \big| \sum_{n=1}^{r} \int_{\mathbb{R}^d} D_{A_j} T_{Bk} E_{C_m} \psi_{1n} \overline{f_{2n}} \big|^2 + \cdots + \big| \sum_{n=1}^{r} \int_{\mathbb{R}^d} D_{A_j} T_{Bk} E_{C_m} \psi_{sn} \overline{f_{2n}} \big|^2 + \cdots \\
&\quad + \big| \sum_{n=1}^{r} \int_{\mathbb{R}^d} D_{A_j} T_{Bk} E_{C_m} \psi_{1n} \overline{f_{sn}} \big|^2 + \cdots + \big| \sum_{n=1}^{r} \int_{\mathbb{R}^d} D_{A_j} T_{Bk} E_{C_m} \psi_{sn} \overline{f_{sn}} \big|^2 \Big] \\
&\le \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} r \Big[ \sum_{n=1}^{r} \big| \int_{\mathbb{R}^d} D_{A_j} T_{Bk} E_{C_m} \psi_{1n} \overline{f_{1n}} \big|^2 + \cdots \\
&\qquad\qquad + \sum_{n=1}^{r} \big| \int_{\mathbb{R}^d} D_{A_j} T_{Bk} E_{C_m} \psi_{sn} \overline{f_{1n}} \big|^2 \\
&\quad + \sum_{n=1}^{r} \big| \int_{\mathbb{R}^d} D_{A_j} T_{Bk} E_{C_m} \psi_{1n} \overline{f_{2n}} \big|^2 + \cdots + \sum_{n=1}^{r} \big| \int_{\mathbb{R}^d} D_{A_j} T_{Bk} E_{C_m} \psi_{sn} \overline{f_{2n}} \big|^2 + \cdots \\
&\quad + \sum_{n=1}^{r} \big| \int_{\mathbb{R}^d} D_{A_j} T_{Bk} E_{C_m} \psi_{1n} \overline{f_{sn}} \big|^2 + \cdots + \sum_{n=1}^{r} \big| \int_{\mathbb{R}^d} D_{A_j} T_{Bk} E_{C_m} \psi_{sn} \overline{f_{sn}} \big|^2 \Big] \\
&= \sum_{\substack{j,m\in\mathbb{Z} \\ k\in\mathbb{Z}^d}} r \sum_{i=1}^{s} \sum_{n=1}^{r} \Big[ \big| \int_{\mathbb{R}^d} D_{A_j} T_{Bk} E_{C_m} \psi_{in} \overline{f_{1n}} \big|^2 \quad + \big| \int_{\mathbb{R}^d} D_{A_j} T_{Bk} E_{C_m} \psi_{in} \overline{f_{2n}} \big|^2 \\
&\quad + \cdots + \big| \int_{\mathbb{R}^d} D_{A_j} T_{Bk} E_{C_m} \psi_{in} \overline{f_{sn}} \big|^2 \Big] \\
&= \sum_{\substack{j,m\in\mathbb{Z} \\ k\in\mathbb{Z}^d}} r \sum_{i=1}^{s} \sum_{n=1}^{r} \sum_{i=1}^{s} \Big| \int_{\mathbb{R}^d} D_{A_j} T_{Bk} E_{C_m} \psi_{in} \overline{f_{in}} \Big|^2 \\
&\le r \sum_{i=1}^{s} \sum_{n=1}^{r} \sum_{i=1}^{s} \int_{\mathbb{R}^d} |\hat{f}_{in}(\gamma)|^2 \times \\
&\quad \frac{1}{|detB|} \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\hat{\psi}_{in}(({A_j}^T)^{-1}\gamma - C_m) \hat{\psi}_{in}(({A_j}^T)^{-1}\gamma - C_m - (B^T)^{-1}k)| d\gamma
\end{aligned}$$

$$\leq rs\Big(\max_{\substack{1\leq i\leq s\\1\leq n\leq r}} \frac{1}{|detB|}\times$$

$$\sup_{\gamma\in\mathbb{R}^d} \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\hat{\psi}_{in}((A_j{}^T)^{-1}\gamma - C_m)\hat{\psi}_{in}((A_j{}^T)^{-1}\gamma - C_m - (B^T)^{-1}k)|\Big)\|\mathbf{f}\|^2.$$

Hence $\{D_{A_j}T_{Bk}E_{C_m}\Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ is a WHMV wave packet Bessel sequence with the required Bessel bound. $\square$

The following example illustrates Theorem 3.4.

**Example 3.5.** Consider the wave packet system $\{D_{2^j}T_kE_{d_m}\psi\}_{j,k\in\mathbb{Z},m\in\mathbb{Z}\setminus\{0\}}$, where $\psi\in L^2(\mathbb{R})$ is a Shannon-type scaling function defined by $\hat{\psi}=\chi_{[-\frac{1}{4},\frac{1}{4}]}$ and $d_m=\mathrm{sgn}(m)\Big(2^{|m|}-\frac{3}{4}\Big)$ for $m\in\mathbb{Z}\setminus\{0\}$. Then

$$\sup_{\gamma\in\mathbb{R}} \sum_{j,k\in\mathbb{Z},m\in\mathbb{Z}\setminus\{0\}} |\hat{\psi}(2^{-j}\gamma-d_m)\hat{\psi}(2^{-j}\gamma-d_m-k)|=1,$$

see [6] for the details. Let $\Psi\in L^2(\mathbb{R},\mathbb{C}^{2\times 2})$ be defined as

$$\Psi=\begin{bmatrix}\psi_{11} & \psi_{12}\\ \psi_{21} & \psi_{22}\end{bmatrix}, \text{where } \psi_{11}=\psi_{12}=\psi_{21}=\psi_{22}=\psi.$$

Then,

$$\max_{\substack{1\leq i\leq 2\\1\leq n\leq 2}} \sup_{\gamma\in\mathbb{R}} \sum_{j,k\in\mathbb{Z},m\in\mathbb{Z}\setminus\{0\}} |\hat{\psi_{in}}(2^{-j}\gamma-d_m)\hat{\psi_{in}}(2^{-j}\gamma-d_m-k)|=1.$$

Hence, by Theorem 3.4, the system $\{D_{2^j}T_kE_{d_m}\Psi\}_{j,k\in\mathbb{Z},m\in\mathbb{Z}\setminus\{0\}}$ is a WHMV wave packet Bessel sequence in $L^2(\mathbb{R},\mathbb{C}^{2\times 2})$ with Bessel bound $B_o=4$.

To discuss a problem in frame theory related to extension of Bessel sequences to frames or dual frames for Hilbert spaces, Christensen et al. [4] studied extension of Bessel sequences to frames. Deepshikha and Vashisht studied extension problem for standard wave packet frames and Hilbert frames in [8, 9]. The frame property, in general, cannot be carried from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$, see [18], it would be interesting to extend WHMV wave packet Bessel systems to frames in higher dimensions. The next result provides sufficient conditions for the extension of a pair of WHMV wave packet Bessel sequences to dual WHMV wave packet frames for $L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$. This is an extension of [4, Lemma 4.1] to wave packet frames in higher dimensions.

**Theorem 3.6.** *Let* $\{D_{A^j}T_{Bk}E_{Cm}\Psi_1\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d}$, $\{D_{A^j}T_{Bk}E_{Cm}\widetilde{\Psi_1}\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d}$ *be WHMV wave packet Bessel sequences in* $L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$ *with pre-frame* *operators* $T_{\Psi_1}$ *and* $T_{\widetilde{\Psi_1}}$ *and analysis operators* $\Theta_{\Psi_1}$ *and* $\Theta_{\widetilde{\Psi_1}}$*, respectively. Let* $I$ *be the identity operator on* $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$*. Assume there exists* $\Phi \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ *such that*

(i) $\{D_{A^j}T_{Bk}E_{Cm}\Phi\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d}$ *is a WHMV wave packet frame for* $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ *with a dual* $\{D_{A^j}T_{Bk}E_{Cm}\widetilde{\Phi}\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d}$.

(ii) $T_{\Psi_1}\Theta_{\widetilde{\Psi_1}}T_{Bk}E_{Cm}\Phi = T_{Bk}E_{Cm}T_{\Psi_1}\Theta_{\widetilde{\Psi_1}}\Phi,\ k, m \in \mathbb{Z}^d$.

*Let* $\Psi_2, \widetilde{\Psi_2} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ *be such that* $\Psi_2 = (I - T_{\Psi_1}\Theta_{\widetilde{\Psi_1}})\Phi$ *and* $\widetilde{\Psi_2} = \widetilde{\Phi}$*.*

*Then, the families* $\{D_{A^j}T_{Bk}E_{Cm}\Psi_1\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d} \bigcup \{D_{A^j}T_{Bk}E_{Cm}\Psi_2\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d}$ *and* $\{D_{A^j}T_{Bk}E_{Cm}\widetilde{\Psi_1}\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d} \bigcup \{D_{A^j}T_{Bk}E_{Cm}\widetilde{\Psi_2}\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d}$ *form dual WHMV wave packet frames for* $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$*.*

*Proof.* For all $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, we have

$$\begin{aligned} T_{\widetilde{\Psi_1}}\Theta_{\Psi_1}\mathbf{f} &= T_{\widetilde{\Psi_1}}\big(\{\langle \mathbf{f}, D_{A^j}T_{Bk}E_{Cm}\Psi_1\rangle\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d}\big) \\ &= \sum_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d} \langle f, D_{A^j}T_{Bk}E_{Cm}\Psi_1\rangle D_{A^j}T_{Bk}E_{Cm}\widetilde{\Psi_1}. \end{aligned}$$

Since $\{D_{A^j}T_{Bk}E_{Cm}\Phi\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d}$ and $\{D_{A^j}T_{Bk}E_{Cm}\widetilde{\Phi}\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d}$ form a pair of dual frames for $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, we have, for each $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$

$$\begin{aligned} (I - T_{\widetilde{\Psi_1}}\Theta_{\Psi_1})\mathbf{f} &= \sum_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d} \langle (I - T_{\widetilde{\Psi_1}}\Theta_{\Psi_1})\mathbf{f}, D_{A^j}T_{Bk}E_{Cm}\Phi\rangle D_{A^j}T_{Bk}E_{Cm}\widetilde{\Phi} \\ &= \sum_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d} \langle \mathbf{f}, (I - T_{\Psi_1}\Theta_{\widetilde{\Psi_1}})D_{A^j}T_{Bk}E_{Cm}\Phi\rangle D_{A^j}T_{Bk}E_{Cm}\widetilde{\Phi}. \end{aligned} \tag{3.2}$$

It is easy to see that $D_{A^j}$ commutes with $T_{\Psi_1}\Theta_{\widetilde{\Psi_1}}$. By using condition (ii), we have

$$\begin{aligned} (I - T_{\Psi_1}\Theta_{\widetilde{\Psi_1}})D_{A^j}T_{Bk}E_{Cm}\Phi &= D_{A^j}(I - T_{\Psi_1}\Theta_{\widetilde{\Psi_1}})T_{Bk}E_{Cm}\Phi \\ &= D_{A^j}T_{Bk}E_{Cm}(I - T_{\Psi_1}\Theta_{\widetilde{\Psi_1}})\Phi. \end{aligned} \tag{3.3}$$

For any $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, we have

$$(I - T_{\widetilde{\Psi_1}}\Theta_{\Psi_1})\mathbf{f} = \mathbf{f} - \sum_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d} \langle f, D_{A^j}T_{Bk}E_{Cm}\Psi_1\rangle D_{A^j}T_{Bk}E_{Cm}\widetilde{\Psi_1}. \tag{3.4}$$

By using (3.2), (3.3), (3.4), and $\Psi_2 = (I - T_{\Psi_1}\Theta_{\widetilde{\Psi_1}})\Phi, \quad \widetilde{\Psi_2} = \widetilde{\Phi}$, for all $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, we have

$$\mathbf{f} = \sum_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d} \langle \mathbf{f}, D_{A^j}T_{Bk}E_{Cm}\Psi_1\rangle D_{A^j}T_{Bk}E_{Cm}\widetilde{\Psi_1} +$$

$$\sum_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d} \langle f, D_{A^j}T_{Bk}E_{Cm}\Psi_2\rangle D_{A^j}T_{Bk}E_{Cm}\widetilde{\Psi}_2.$$

Hence the system $\{D_{A^j}T_{Bk}E_{Cm}\Psi_1\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d} \bigcup \{D_{A^j}T_{Bk}E_{Cm}\Psi_2\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d}$ and $\{D_{A^j}T_{Bk}E_{Cm}\widetilde{\Psi_1}\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d} \bigcup \{D_{A^j}T_{Bk}E_{Cm}\widetilde{\Psi_2}\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d}$ form dual *WHMV* wave packet frames for $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$. □

Conditions given in Theorem 3.6 are sufficient but not necessary. This is justified in the following example.

**Example 3.7.** Let $\psi \in L^2(\mathbb{R})$ be such that $\widehat{\psi}(\xi) = \chi_{[1,2]}(\xi), \xi \in \mathbb{R}$. Let $\mathbb{Z}_+ = \{j \in \mathbb{Z} : j \geq 0\}$ and $S = \{(j,0), j \in \mathbb{Z}_+\} \bigcup \{(j,-3), j \in \mathbb{Z}_+\} \bigcup \{(0,-1),(0,-2)\}$. Then, the wave packet system $\{D_{2^j}T_kE_m\psi, k \in \mathbb{Z}, (j,m) \in S\}$ is an orthonormal basis for $L^2(\mathbb{R})$, see [17, p. 144].

Consider

$$\Psi_1 = \begin{bmatrix} \psi & 0 \\ 0 & \psi \end{bmatrix} \in L^2(\mathbb{R}, \mathbb{C}^{2\times 2}).$$

Then, $\{D_{2^j}T_kE_m\Psi_1, k \in \mathbb{Z}, (j,m) \in S\}$ is an orthonormal system and for any $\mathbf{f} \in L^2(\mathbb{R}, \mathbb{C}^{2\times 2})$, we have

$$\mathbf{f} = \sum_{k\in\mathbb{Z},(j,m)\in S} \langle \mathbf{f}, D_{2^j}T_kE_m\Psi_1\rangle D_{2^j}T_kE_m\Psi_1.$$

That is, $\{D_{2^j}T_kE_m\Psi_1, k \in \mathbb{Z}, (j,m) \in S\}$ is an orthonormal basis for $L^2(\mathbb{R}, \mathbb{C}^{2\times 2})$. In particular, we have a pair of WHMV wave packet Bessel sequences $\{D_{2^j}T_kE_m\Psi_1, k \in \mathbb{Z}, (j,m) \in S\}$ and $\{D_{2^j}T_kE_m\widetilde{\Psi}_1, k \in \mathbb{Z}, (j,m) \in S\} = \{D_{2^j}T_kE_m\Psi_1, k \in \mathbb{Z}, (j,m) \in S\}$ in $L^2(\mathbb{R}, \mathbb{C}^{2\times 2})$ with pre-frame operators $T_{\Psi_1}$ and $T_{\widetilde{\Psi_1}}$(say), respectively. Further, for any $\mathbf{f} \in L^2(\mathbb{R}, \mathbb{C}^{2\times 2})$, we have

$$\begin{aligned} T_{\Psi_1}\Theta_{\widetilde{\Psi_1}}(\mathbf{f}) &= T_{\Psi_1}(\{\langle \mathbf{f}, D_{2^j}T_kE_m\widetilde{\Psi_1}\rangle\}_{k\in\mathbb{Z},(j,m)\in S}) \\ &= \sum_{k\in\mathbb{Z},(j,m)\in S} \langle \mathbf{f}, D_{2^j}T_kE_m\widetilde{\Psi_1}\rangle D_{2^j}T_kE_m\Psi_1 \\ &= \mathbf{f}. \end{aligned}$$

Thus, $T_{\Psi_1}\Theta_{\widetilde{\Psi_1}} = I$, i.e., $I - T_{\Psi_1}\Theta_{\widetilde{\Psi_1}} = 0$.

Choose $\Psi_2 = \begin{bmatrix} \psi & 0 \\ 0 & 0 \end{bmatrix}$, $\widetilde{\Psi}_2 = 0$. Then,

$$\{D_{2^j} T_k E_m \Psi_1\}_{k\in\mathbb{Z},(j,m)\in S} \bigcup \{D_{2^j} T_k E_m \Psi_2\}_{k\in\mathbb{Z},(j,m)\in S}$$

and $\{D_{2^j} T_k E_m \widetilde{\Psi_1}\}_{k\in\mathbb{Z},(j,m)\in S} \bigcup \{D_{2^j} T_k E_m \widetilde{\Psi_2}\}_{k\in\mathbb{Z},(j,m)\in S}$ constitute dual WHMV wave packet frames for $L^2(\mathbb{R}, \mathbb{C}^{2\times 2})$. But, for any $\Phi \in L^2(\mathbb{R}, \mathbb{C}^{2\times 2})$, $\Psi_2 \neq (I - T_{\Psi_1}\Theta_{\widetilde{\Psi_1}})\Phi = 0$.

The next result gives sufficient conditions for the extension of a pair of WHMV wave packet Bessel sequences to WHMV wave packet dual frames for $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, where the window function $\Phi$ belongs to a special class of functions. First we introduce a notation: for $\mathcal{Y} \subset L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, we write $W_C(\mathcal{Y}) = \overline{\text{span}}\{E_{Cm}\Phi : \Phi \in \mathcal{Y}, m \in \mathbb{Z}^d\}$.

**Theorem 3.8.** *Let* $\{D_{A^j} T_{Bk} E_{Cm} \Psi_1\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d}$, $\{D_{A^j} T_{Bk} E_{Cm} \widetilde{\Psi_1}\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d}$ *be WHMV wave packet Bessel sequences in* $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ *with pre-frame operators* $T_{\Psi_1}$ *and* $T_{\widetilde{\Psi_1}}$ *and analysis operators* $\Theta_{\Psi_1}$ *and* $\Theta_{\widetilde{\Psi_1}}$*, respectively, where the matrix* $A$ *has integer entries and* $A^T C = C A^T$*. Assume there exists* $\Phi \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ *such that*

(*i*) $\{D_{A^j} T_{Bk} E_{Cm} \Phi\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d}$ *is a WHMV wave packet frame for* $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ *with a dual* $\{D_{A^j} T_{Bk} E_{Cm} \widetilde{\Phi}\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d}$,
(*ii*) $T_{\Psi_1}\Theta_{\widetilde{\Psi_1}} T_{Bk}\Phi = T_{Bk} T_{\Psi_1}\Theta_{\widetilde{\Psi_1}}\Phi$, *and*
(*iii*) $\Phi \in [W_C(\{D_{A^j} T_{Bk} E_{Cm} \widetilde{\Psi_1} : j > 0, k, m \in \mathbb{Z}^d\})]^{\perp}$.

*Then, there exist* $\Psi_2, \widetilde{\Psi_2} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ *such that the families*

$$\{D_{A^j} T_{Bk} E_{Cm} \Psi_1\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d} \bigcup \{D_{A^j} T_{Bk} E_{Cm} \Psi_2\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d}$$

*and*

$$\{D_{A^j} T_{Bk} E_{Cm} \widetilde{\Psi_1}\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d} \bigcup \{D_{A^j} T_{Bk} E_{Cm} \widetilde{\Psi_2}\}_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d}$$

*form dual* WHMV *wave packet frames for* $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$.

*Proof.* Since $\Phi \in [W_C(\{D_{A^j} T_{Bk} E_{Cm} \widetilde{\Psi_1} : j > 0, k, m \in \mathbb{Z}^d\})]^{\perp}$, we have

$$\langle \Phi, E_{Cm'} D_{A^j} T_{Bk} E_{Cm} \widetilde{\Psi_1} \rangle = 0 \text{ for all } j > 0, k, m, m' \in \mathbb{Z}^d. \tag{3.5}$$

By using (3.5), we compute

$$
\begin{aligned}
T_{\Psi_1}\Theta_{\widetilde{\Psi_1}}E_{Cm'}\Phi &= \sum_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d}\langle E_{Cm'}\Phi, D_{A^j}T_{Bk}E_{Cm}\widetilde{\Psi_1}\rangle D_{A^j}T_{Bk}E_{Cm}\Psi_1\\
&= \sum_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d}\langle \Phi, E_{-Cm'}D_{A^j}T_{Bk}E_{Cm}\widetilde{\Psi_1}\rangle D_{A^j}T_{Bk}E_{Cm}\Psi_1\\
&= \sum_{j\leq 0,k,m\in\mathbb{Z}^d}\langle \Phi, E_{-Cm'}D_{A^j}T_{Bk}E_{Cm}\widetilde{\Psi_1}\rangle D_{A^j}T_{Bk}E_{Cm}\Psi_1\\
&= E_{Cm'}E_{-Cm'}\sum_{j\leq 0,k,m\in\mathbb{Z}^d}\langle \Phi, E_{-Cm'}D_{A^j}T_{Bk}E_{Cm}\widetilde{\Psi_1}\rangle D_{A^j}T_{Bk}E_{Cm}\Psi_1\\
&= E_{Cm'}\sum_{j\leq 0,k,m\in\mathbb{Z}^d}\langle \Phi, E_{-Cm'}D_{A^j}T_{Bk}E_{Cm}\widetilde{\Psi_1}\rangle E_{-Cm'}D_{A^j}T_{Bk}E_{Cm}\Psi_1\\
&= E_{Cm'}\sum_{j\leq 0,k,m\in\mathbb{Z}^d}\langle \Phi, D_{A^j}E_{-A^{j\sharp}Cm'}T_{Bk}E_{Cm}\widetilde{\Psi_1}\rangle D_{A^j}E_{-A^{j\sharp}Cm'}T_{Bk}E_{Cm}\Psi_1\\
&= E_{Cm'}\sum_{j\leq 0,k,m\in\mathbb{Z}^d}\langle \Phi, \exp\left(2\pi i(-A^{j\sharp}Cm')\cdot Bk\right)D_{A^j}T_{Bk}E_{-A^{j\sharp}Cm'}E_{Cm}\widetilde{\Psi_1}\rangle\\
&\qquad \times \exp\left(2\pi i(-A^{j\sharp}Cm')\cdot Bk\right)D_{A^j}T_{Bk}E_{-A^{j\sharp}Cm'}E_{Cm}\Psi_1\\
&= E_{Cm'}\sum_{j\leq 0,k,m\in\mathbb{Z}^d}\langle \Phi, D_{A^j}T_{Bk}E_{-A^{j\sharp}Cm'}E_{Cm}\widetilde{\Psi_1}\rangle D_{A^j}T_{Bk}E_{-A^{j\sharp}Cm'}E_{Cm}\Psi_1\\
&= E_{Cm'}\sum_{j\leq 0,k,m\in\mathbb{Z}^d}\langle \Phi, D_{A^j}T_{Bk}E_{C(m-A^{j\sharp}m')}\widetilde{\Psi_1}\rangle D_{A^j}T_{Bk}E_{C(m-A^{j\sharp}m')}\Psi_1\\
&= E_{Cm'}\sum_{j\in\mathbb{Z},k,m\in\mathbb{Z}^d}\langle \Phi, D_{A^j}T_{Bk}E_{Cm}\widetilde{\Psi_1}\rangle D_{A^j}T_{Bk}E_{Cm}\Psi_1\\
&= E_{Cm'}T_{\Psi_1}\Theta_{\widetilde{\Psi_1}}\Phi.
\end{aligned}
$$

The result now follows from Theorem 3.6. $\square$

## 4. Matrix-valued Symmetric Wave Packet Frames

Wu and Cao [27] and Wang and Wu [28] studied symmetric wave packet frames with several generators about origin in higher dimensions (in the Lebesgue space $L^2(\mathbb{R}^2)$). For a finite set $\{g^m\}_{m=1}^{M} \subset L^2(\mathbb{R}^2)$, the following new functions in $L^2(\mathbb{R}^2)$) were considered in [28]:

$$
g_1^m(x) = \frac{g^m(x) + g^m(-x)}{2} \quad \text{and} \quad g_2^m(x) = \frac{g^m(x) - g^m(-x)}{2}, \quad m \in \{1, \dots, M\}.
$$

It is proved in [28] that if $\Big\{D_2^j E_\ell T_k g^m : j\in\mathbb{Z}, k,\ell\in\mathbb{Z}^2, m\in\{1,2,\ldots,M\}\Big\}$ is a frame for $L^2(\mathbb{R}^2)$ with frame bounds $L_o, U_o$, then the wave packet system

$$\Big\{D_2^j E_\ell T_k g_1^m \cup D_2^j E_\ell T_k g_2^m : j\in\mathbb{Z}, k,\ell\in\mathbb{Z}^2, m\in\{1,2,\ldots,M\}\Big\}$$

is a symmetric or antisymmetric frame about origin with same frame bounds. We recall that the frame properties of wave packet system for $L^2(\mathbb{R}^d)$ cannot be carried to the higher dimension space $L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$ and vice-versa. This motivated us to study frame property of matrix-valued symmetric wave packet frames. Theorem 4.2 shows that, as in the case of standard wave packet frames [28, Theorem 1], frame properties of matrix-valued symmetric wave packet frames associated with a given matrix-valued wave packet frame for the underlying function space are preserved. For completeness we include its proof. First we state a lemma which gives a property of the Frobenius norm; and will be used in Theorem 4.2.

**Lemma 4.1.** *For $M_1, M_2 \in \mathcal{M}_s(\mathbb{C})$, the Frobenius norm satisfies the following property:*

$$\|M_1+M_2\|^2 = \|M_1\|^2 + \|M_2\|^2 + trace({M_1}^*M_2) + trace(M_1{M_2}^*).$$

**Theorem 4.2.** *Let $\{D_{A_j}T_{Bk}E_{C_m}\Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ be a* WHMV wave packet frame *for $L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$ with frame bounds $L_\Psi, U_\Psi$. Then,* $\{D_{A_j}T_{Bk}E_{C_m}\Psi_1\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}\bigcup\{D_{A_j}T_{Bk}E_{C_m}\Psi_2\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ *is a* WHMV wave packet frame *for $L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$ with the frame bounds $L_\Psi, U_\Psi$, where $\Psi_1,\Psi_2\in L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$ are defined as*

$$\Psi_1(\xi) = \frac{\Psi(\xi)+\Psi(-\xi)}{2}, \quad \Psi_2(\xi) = \frac{\Psi(\xi)-\Psi(-\xi)}{2}, \quad \xi\in\mathbb{R}^d.$$

.

*Proof.* For any $\mathbf{f}\in L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$, we compute

$$\begin{aligned}
&\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \Big\|\Big\langle D_{A_j}T_{Bk}E_{C_m}\Psi_1, \mathbf{f}\Big\rangle\Big\|^2 \\
&= \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \Big\|\Big\langle D_{A_j}T_{Bk}E_{C_m}\frac{\Psi(\cdot)}{2}, \mathbf{f}\Big\rangle + \Big\langle D_{A_j}T_{Bk}E_{C_m}\frac{\Psi(-\cdot)}{2}, \mathbf{f}\Big\rangle\Big\|^2
\end{aligned}$$

$$= \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\|\left\langle D_{A_j}T_{Bk}E_{C_m}\frac{\Psi(\cdot)}{2},\mathbf{f}\right\rangle\right\|^2 + \sum_{\substack{j,m\in\mathbb{Z},\\ k\in\mathbb{Z}^d}} \left\|\left\langle D_{A_j}T_{Bk}E_{C_m}\frac{\Psi(-\cdot)}{2},\mathbf{f}\right\rangle\right\|^2$$
$$+ \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \operatorname{trace}\Big(\left\langle D_{A_j}T_{Bk}E_{C_m}\frac{\Psi(\cdot)}{2},\mathbf{f}\right\rangle^*\left\langle D_{A_j}T_{Bk}E_{C_m}\frac{\Psi(-\cdot)}{2},\mathbf{f}\right\rangle\Big)$$
$$+ \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \operatorname{trace}\Big(\left\langle D_{A_j}T_{Bk}E_{C_m}\frac{\Psi(\cdot)}{2},\mathbf{f}\right\rangle\left\langle D_{A_j}T_{Bk}E_{C_m}\frac{\Psi(-\cdot)}{2},\mathbf{f}\right\rangle^*\Big). \quad (4.1)$$

Similarly, for any $\mathbf{f}\in L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$, we have

$$\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\|\left\langle D_{A_j}T_{Bk}E_{C_m}\Psi_2,\mathbf{f}\right\rangle\right\|^2$$
$$= \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\|\left\langle D_{A_j}T_{Bk}E_{C_m}\frac{\Psi(\cdot)}{2},\mathbf{f}\right\rangle\right\|^2 + \sum_{\substack{j,m\in\mathbb{Z},\\ k\in\mathbb{Z}^d}} \left\|\left\langle D_{A_j}T_{Bk}E_{C_m}\frac{\Psi(-\cdot)}{2},\mathbf{f}\right\rangle\right\|^2$$
$$- \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \operatorname{trace}\Big(\left\langle D_{A_j}T_{Bk}E_{C_m}\frac{\Psi(\cdot)}{2},\mathbf{f}\right\rangle^*\left\langle D_{A_j}T_{Bk}E_{C_m}\frac{\Psi(-\cdot)}{2},\mathbf{f}\right\rangle\Big)$$
$$- \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \operatorname{trace}\Big(\left\langle D_{A_j}T_{Bk}E_{C_m}\frac{\Psi(\cdot)}{2},\mathbf{f}\right\rangle\left\langle D_{A_j}T_{Bk}E_{C_m}\frac{\Psi(-\cdot)}{2},\mathbf{f}\right\rangle^*\Big). \quad (4.2)$$

From (4.1) and (4.2), we have

$$\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\|\left\langle D_{A_j}T_{Bk}E_{C_m}\Psi_1,\mathbf{f}\right\rangle\right\|^2 + \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\|\left\langle D_{A_j}T_{Bk}E_{C_m}\Psi_2,\mathbf{f}\right\rangle\right\|^2$$
$$= 2\sum_{\substack{j,m\in\mathbb{Z},\\ k\in\mathbb{Z}^d}} \left\|\left\langle D_{A_j}T_{Bk}E_{C_m}\frac{\Psi(\cdot)}{2},\mathbf{f}\right\rangle\right\|^2 + 2\sum_{\substack{j,m\in\mathbb{Z},\\ k\in\mathbb{Z}^d}} \left\|\left\langle D_{A_j}T_{Bk}E_{C_m}\frac{\Psi(-\cdot)}{2},\mathbf{f}\right\rangle\right\|^2$$
$$= \frac{1}{2}\sum_{\substack{j,m\in\mathbb{Z},\\ k\in\mathbb{Z}^d}} \left\|\left\langle D_{A_j}T_{Bk}E_{C_m}\Psi(\cdot),\mathbf{f}\right\rangle\right\|^2 + \frac{1}{2}\sum_{\substack{j,m\in\mathbb{Z},\\ k\in\mathbb{Z}^d}} \left\|\left\langle D_{A_j}T_{Bk}E_{C_m}\Psi,\mathbf{f}(-\cdot)\right\rangle\right\|^2$$
$$\geq \frac{1}{2}L_\Psi\|\mathbf{f}\|^2 + \frac{1}{2}L_\Psi\|\mathbf{f}(-\cdot)\|^2$$
$$= L_\Psi\|\mathbf{f}\|^2,\ \mathbf{f}\in L^2(\mathbb{R}^d,\mathbb{C}^{s\times r}). \quad (4.3)$$

Similarly

$$\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\|\left\langle D_{A_j}T_{Bk}E_{C_m}\Psi_1,\mathbf{f}\right\rangle\right\|^2 + \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\|\left\langle D_{A_j}T_{Bk}E_{C_m}\Psi_2,\mathbf{f}\right\rangle\right\|^2$$
$$\leq U_\Psi\|\mathbf{f}\|^2,\ \mathbf{f}\in L^2(\mathbb{R}^d,\mathbb{C}^{s\times r}). \quad (4.4)$$

The proof now follows from (4.3) and (4.4). $\square$

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