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On example of circular arc approximation by quadratic minimal splines

A.A. Makarov

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Abstract. The paper considers the quadratic minimal splines and local spline approximation scheme with explicit dual approximation functionals. A simple case of circular arc (quarter of a circle) approximation is regarded. Numerical results of spline approximation are presented.

1. Introduction

Circles and circular arcs are widely used in computer-aided design systems. It is known, that it is not possible to represent a circle with a polynomial curve in explicit form, but it can be represented, for example, by a quadratic rational spline. A circular arc does not have a parametric polynomial representation, but it can be represented by using a quadratic rational curve, e. g. Bézier or NURBS. There are a lot of papers devoted to circular arc approximation (see [1, 2, 3] and references therein). Nevertheless the construction of a circular arc approximation with the highest possible accuracy is a very important issue.

This note is motivated by the paper [3] and the reference on the paper [4] given there. Although it is shown [5] that spline curves can approximate with higher order than spline functions, we start with approximation by spline functions constructed from approximation relations by using a complete chain of vectors and a generating vector function $\boldsymbol{\varphi}$ (see, e.g., [6]); such splines have a minimal support and a maximum degree of smoothness ($B_{\boldsymbol{\varphi}}$-splines).

Quadratic polynomial B-splines are the partial case of $B_{\boldsymbol{\varphi}}$-splines generated by the vector function $\boldsymbol{\varphi} = (1, t, t^2)^T$ (see [7]). The establishment of conditions for the sign definiteness of minimal splines is important for practical application in geometric design problems. For example, the quadratic hyperbolic minimal splines generated by vector function $\boldsymbol{\varphi}(t) = (1, \sinh t, \cosh t)^T$ are positive without any

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constraints imposed on the grid (see [8]), unlike trigonometric minimal splines generated by $\boldsymbol{\varphi}(t) = (1, \sin t, \cos t)^T$ that are positive under condition that the fineness characteristics of a grid is less than π (see [9]).

We use a local spline approximation scheme with explicit dual approximation functionals [10], which are linear combinations of values and derivatives of the function to be approximated. The paper presents numerical results of the simplest case of circular arc (quarter of a circle) approximation.

2. On coordinate splines

Let $\mathbb{Z}$ be the set of integers, $\mathbb{Z}_+ := \{j \ : \ j \geq 0, \ j \in \mathbb{Z}\}$, $\mathbb{R}^1$ be the set of reals. The linear vector space of three-dimensional column vectors is denoted by $\mathbb{R}^3$; the vectors of this space are identified with one-column matrices, and usual matrix operations are applied to these vectors. In particular, for two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ the expression $\mathbf{a}^T\mathbf{b}$ denotes the Euclidean inner product of the vectors. The square matrix with columns $\mathbf{a}_0$, $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^3$ (in the indicated order) is denoted by $(\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2)$, and the expression $\det(\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2)$ means its determinant. The vector components are denoted by square brackets and indexed by the integer numbers; for instance, $\mathbf{a} = ([\mathbf{a}]_0, [\mathbf{a}]_1, [\mathbf{a}]_2)^T$. For an arbitrary number $S \in \mathbb{Z}_+$, we denote $C^S[a, b] := \{u \ : \ u^{(i)} \in C[a, b], \, i = 0, 1, \ldots, S\}$, setting $C^0[a, b] = C[a, b]$. We write $\mathbf{u} \in \mathbf{C}^S[a, b]$, if the components of a vector function $\mathbf{u} \in \mathbb{R}^3$ are S times continuously differentiable on a segment $[a, b]$.

On a segment $[a, b] \subset \mathbb{R}^1$ we consider a grid X (with supplementary knots outside the segment $[a, b]$, lying on a certain interval $(\alpha, \beta) \supset [a, b]$):

$$X : x_{-2} \leq x_{-1} \leq a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b \leq x_{n+1} \leq x_{n+2}.$$

Introduce the notation $J_{i,k} := \{i, i+1, \ldots, k\}, i, k \in \mathbb{Z}, i < k$. An ordered set $\mathbf{A} := \{\mathbf{a}_j\}_{j \in J_{-2,n-1}}$ of vectors $\mathbf{a}_j \in \mathbb{R}^3$ is called a *vector chain.* A chain $\mathbf{A}$ is said to be *complete*, if $\det(\mathbf{a}_{j-2}, \mathbf{a}_{j-1}, \mathbf{a}_j) \neq 0$ for all $j \in J_{0,n-1}$.

The union of elementary grid intervals we denoted by $M := \cup_{j \in J_{0,n-1}} (x_j, x_{j+1})$. We denote the linear space of real-valued functions defined on M by $\mathbb{X}(M)$. We set $S_j := [x_j, x_{j+3}], j \in J_{-2,n-1}$.

Consider the three-component (column) vector function $\boldsymbol{\varphi} \ : \ [a, b] \mapsto \mathbb{R}^3$ with components in the space $\mathbf{C}^2[a, b]$ and nonzero Wronskian:

$$\det(\boldsymbol{\varphi}(t), \boldsymbol{\varphi}'(t), \boldsymbol{\varphi}''(t)) \neq 0 \quad \forall \ t \in [a, b].$$

Let $\mathbf{A}$ is a complete vector chain. Suppose that functions $\omega_j \in \mathbb{X}(M), j \in J_{-2,n-1}$, satisfy the identities

$$\begin{aligned} \sum_{j'=k-2}^{k} \mathbf{a}_{j'} \, \omega_{j'}(t) \equiv & \ \ \boldsymbol{\varphi}(t) \quad \forall \, t \in (x_k, x_{k+1}), \, \forall \, k \in J_{0,n-1}, \\ \omega_j(t) \equiv & \ \ 0 \qquad \ \forall \, t \in M \backslash S_j, \, \forall \, j \in J_{-2,n-1}. \end{aligned} \tag{2.1}$$

For each fixed $t \in (x_k, x_{k+1})$ and for all $k \in J_{0,n-1}$ relations (2.1) can be regarded as a system of linear algebraic equations with respect to the unknowns $\omega_j(t)$. By virtue of assumption that vector chain $\mathbf{A}$ is complete, this system is uniquely solvable. It also follows that $\operatorname{supp}\omega_j \subset S_j$, and by Cramer's rule we find

$$\omega_j(t) = \begin{cases} \dfrac{\det(\mathbf{a}_{j-2}, \mathbf{a}_{j-1}, \boldsymbol{\varphi}(t))}{\det(\mathbf{a}_{j-2}, \mathbf{a}_{j-1}, \mathbf{a}_j)}, & t \in (x_j, x_{j+1}), \\ \dfrac{\det(\mathbf{a}_{j-1}, \boldsymbol{\varphi}(t), \mathbf{a}_{j+1})}{\det(\mathbf{a}_{j-1}, \mathbf{a}_j, \mathbf{a}_{j+1})}, & t \in (x_{j+1}, x_{j+2}), \\ \dfrac{\det(\boldsymbol{\varphi}(t), \mathbf{a}_{j+1}, \mathbf{a}_{j+2})}{\det(\mathbf{a}_j, \mathbf{a}_{j+1}, \mathbf{a}_{j+2})}, & t \in (x_{j+2}, x_{j+3}). \end{cases} \tag{2.2}$$

The linear span of the functions $\omega_j(t)$ is called *the space of quadratic minimal coordinate* $(\mathbf{A}, \boldsymbol{\varphi})$*-splines*. We denote it by $\mathbb{S}(X, \mathbf{A}, \boldsymbol{\varphi})$. The identities (2.1) are called *the approximation relations*. The vector function $\boldsymbol{\varphi}$ is called *the generating*.

For a vector-valued function $\boldsymbol{\varphi} \in \mathbf{C}^1[a, b]$ we set

$$\boldsymbol{\varphi}_j := \boldsymbol{\varphi}(x_j), \quad \boldsymbol{\varphi}'_j := \boldsymbol{\varphi}'(x_j), \quad j \in J_{-2,n+2}.$$

As is known [7], if $\mathbf{a}_j = c_j \mathbf{b}_j$ for any $j \in J_{0,n-1}$, where c_j are arbitrary nonzero constants and $\mathbf{b}_j$ are the vectors defined by the formula

$$\mathbf{b}_j := \det(\boldsymbol{\varphi}_{j+2}, \boldsymbol{\varphi}'_{j+2}, \boldsymbol{\varphi}'_{j+1})\boldsymbol{\varphi}_{j+1} - \det(\boldsymbol{\varphi}_{j+2}, \boldsymbol{\varphi}'_{j+2}, \boldsymbol{\varphi}_{j+1})\boldsymbol{\varphi}'_{j+1},$$

then $\omega_j \in C^1[a, b]$ for any $j \in J_{-2,n-1}$.

Let $[\boldsymbol{\varphi}(t)]_0 \equiv 1$ and $c_j := 1/\det(\boldsymbol{\varphi}_{j+2}, \boldsymbol{\varphi}'_{j+2}, \boldsymbol{\varphi}'_{j+1})$. Introduce the vector $\mathbf{a}_j^N := c_j \mathbf{b}_j$, then

$$\mathbf{a}_j^N = \boldsymbol{\varphi}_{j+1} - \frac{\det(\boldsymbol{\varphi}_{j+2}, \boldsymbol{\varphi}'_{j+2}, \boldsymbol{\varphi}_{j+1})}{\det(\boldsymbol{\varphi}_{j+2}, \boldsymbol{\varphi}'_{j+2}, \boldsymbol{\varphi}'_{j+1})}\, \boldsymbol{\varphi}'_{j+1}. \tag{2.3}$$

For the vectors $\mathbf{a}_j^N$, $j \in J_{-2,n-1}$, the zeroth component in componentwise form of the approximation relations (2.1) yields to partition of unity property

$$\sum_{j=-2}^{n-1} \omega_j(t) \equiv 1 \quad \forall\, t \in [a, b].$$

In this case the functions $\omega_j(t)$ are called *the quadratic normalized minimal coordinate* $B_{\boldsymbol{\varphi}}$*-splines*.

Remark 2.1. Observe that for $\boldsymbol{\varphi}(t) = (1, t, t^2)^T$ the functions $\omega_j(t)$ coincide with the known quadratic polynomial B-splines (of the third order).

3. On local spline approximation

Consider a linear vector space $\mathfrak{U}$ of reals and the conjugate space $\mathfrak{U}^*$ of linear functionals f over the space of $\mathfrak{U}$. The value of a functional f on an element $u \in \mathfrak{U}$ is denoted by $\langle f, u\rangle$. For a functional $f \in (C^S)^*$ we write $\operatorname{supp} f \subset [c, d]$ whenever the value $\langle f, u\rangle$ is determined by the values of the function $u \in C^S$ on the interval (c, d).

We say that the system of functionals $\{\nu_i\}_{i\in\mathbb{Z}}$ is *biorthogonal* to a system of functions $\{w_j\}_{j\in\mathbb{Z}}$, if $\langle \nu_i, w_j\rangle = \delta_{i,j}$ for all $i, j \in \mathbb{Z}$, where $\delta_{i,j}$ is Kronecker symbol. The functionals ν_i are said to be *dual* to the functions w_j.

Given a function $u \in C^1[a, b]$. Consider linear functionals μ_j defined by the formula:

$$\langle \mu_j, u\rangle := u(x_{j+1}) - \frac{\det(\boldsymbol{\varphi}_{j+2}, \boldsymbol{\varphi}'_{j+2}, \boldsymbol{\varphi}_{j+1})}{\det(\boldsymbol{\varphi}_{j+2}, \boldsymbol{\varphi}'_{j+2}, \boldsymbol{\varphi}'_{j+1})}\, u'(x_{j+1}). \tag{3.1}$$

As is known [10], the system of linear functionals $\{\mu_j\}_{j\in J_{-2,n-1}}$ is biorthogonal to the system of functions $\{\omega_{j'}\}_{j'\in J_{-2,n-1}}$, i. e.

$$\langle \mu_j, \omega_{j'}\rangle = \delta_{j,j'}.$$

Taking into account the location of the support of the function ω_j for $t \in (x_k, x_{k+1})$, consider the spline

$$\widetilde{u}(t) = \sum_{j=k-2}^{k} \langle \mu_j, u\rangle\, \omega_j(t), \quad t \in (x_k, x_{k+1}), \tag{3.2}$$

functionals μ_j in this case are called the *approximation functionals*.

It is easy to see that the approximation (3.2) possess the sharpness property on the functions $u \in \{[\boldsymbol{\varphi}]_i \;:\; i = 0, 1, 2\}$, i. e.

$$\widetilde{u} \equiv u \quad \text{for} \quad u \in \{[\boldsymbol{\varphi}]_i \;:\; i = 0, 1, 2\}.$$

4. Numerical results

For simplicity we approximate a circular arc (quarter of a circle), defined by the explicit representation $u(t) = \sqrt{1 - t^2},\ t \in [-0.5, 0.5]$. The purpose of this section is to present the numerical results of the approximation by quadratic minimal splines depending on selection of generating vector function $\boldsymbol{\varphi}$ for the above simplest case of circular arc representation.

We construct the approximation $\widetilde{u}_n$ of type (3.2) with approximation functionals (3.1) on segment $[a, b] = [-0.5, 0.5]$ on uniform grid $X_n := \{x_j : x_j = j/n, j = 0, \ldots, n\}$. As an approximation error we use the maximum of the absolute value of the difference between values of approximation $\widetilde{u}_n$ and values of original function u calculated on the ten times smaller grid ($N = 10n$), i. e.

$$E_n = \max_{x_j\in X_N} |\widetilde{u}_n(x_j) - u(x_j)|. \tag{4.1}$$

Let us regard following generating vector functions: $\boldsymbol{\varphi}^B(t) := (1, t, t^2)^T$, $\boldsymbol{\varphi}^H(t) := (1, \sinh t, \cosh t)^T$, $\boldsymbol{\varphi}^P(t) := (1, \sqrt{1-t}, \sqrt{1+t})^T$, and $\boldsymbol{\varphi}^C(t) := (1, t, \sqrt{1-t^2})^T$, corresponding to the splines ω_j, defined by formulas (2.2) and (2.3). The approximation error (4.1) for considered function u depending on selection of generating vector function $\boldsymbol{\varphi}$ and the size n of grid X_n are represented in table 1.

TABLE 1. Approximation error E_n for $u(t) = \sqrt{1-t^2},\ t \in [-0.5, 0.5]$

$\boldsymbol{\varphi}(t)$	$n = 10$	$n = 20$	$n = 30$
$(1, t, t^2)^T$	1.2×10^{-4}	1.6×10^{-5}	5×10^{-6}
$(1, \sinh t, \cosh t)^T$	9.2×10^{-5}	1.3×10^{-5}	4×10^{-6}
$(1, \sqrt{1-t}, \sqrt{1+t})^T$	2.3×10^{-5}	3.1×10^{-6}	9.6×10^{-7}
$(1, t, \sqrt{1-t^2})^T$	≈ 0	≈ 0	≈ 0

Thus, using different generating vector function $\boldsymbol{\varphi}$ leads to improvement of the quality of approximation.

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(A.A. Makarov) St. Petersburg State University, 7/9, Universitetskaya nab., St. Petersburg, 199034, Russia.

E-mail address: `a.a.makarov@spbu.ru`