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# Regular pairs for solving Fredholm integral equations of the second kind

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***Abstract.*** Classical Fredholm equations of the second kind are integral equations that appear in different areas of applied mathematics, sometimes as equivalent formulation of boundary value problems for ordinary differential equations. Numerical solutions to these equations have been often developed in the context of Galerkin type schemes. In particular, Petrov-Galerkin method has been proposed to solve this type of integral equation by projecting on appropriate finite dimensional subspaces. Recently, the notion of regular pair was introduced. It guarantees solvability and numerical stability of the approximation scheme. Different characterizations of the property of being regular are presented in the literature. In the case of Hilbert spaces, it is related to the positive definitiveness of a matrix. In this work we present pairs of simple subspaces $\{X_n, Y_n\}$ that are regular $\forall n \in \mathbb{N}$ and show the goodness of the approximation in two numerical examples.

## 1. Introduction

Classical Fredholm equations of the second kind are integral equations of the form

$$u(t) - \int_a^b k(s,t)u(s)ds = g(t), \qquad t \in [a,b] \tag{1.1}$$

where $u$ is the unknown function in a Banach space $X$; the kernel $k : [a,b] \times [a,b] \to \mathbb{R}$ and the right-hand side $g : [a,b] \to \mathbb{R}$ are given functions.
These equations usually appear as equivalent formulation of value problems for ordinary differential equations, and are named after Erik Ivar Fredholm (1866-1927) for his numerous contributions to the field.
If we introduce the operator $K : X \to X$,

$$Ku(t) = \int_a^b k(s,t)u(s)ds,$$

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we can rewrite (1.1) as

$$u - Ku = g. \tag{1.2}$$

If $k : [a,b] \times [a,b] \to \mathbb{R}$ is a continuous function, $K$ is a linear bounded operator and a sufficient condition to guarantee the existence and uniqueness of a solution of (1.2) is $||K|| < 1$ ([8], p. 23).

Numerical solutions to these equations have been developed in different contexts. In [12] the authors categorize different approximation methods regarding their accuracy and efficiency. Numerical solutions using the mean value theorem were developed in [2], and in [7] for higher dimensional problems. Fredholm fuzzy integral equations of the second kind were solved using neural networks in [6]. In [9], numerical solutions were proposed by means of a modified Simpson's quadrature rule.

In the context of Galerkin type schemes, Petrov-Galerkin Method has been proposed to solve this type of integral equation by projecting on two sequences of appropriate finite dimensional subspaces of the space $X$, $X_n = span\left\{\phi_j^n,\ j = 1, \cdots, d_n\right\} \subset X$ and $Y_n = span\left\{\psi_i^n,\ i = 1, \cdots, d_n\right\} \subset X$, the subspaces of trial and test functions respectively.

In the case of being $X$ a Hilbert space with inner product $\langle\ .\ \rangle$, Petrov-Galerkin method looks for $u_n \in X_n$ such that

$$\langle u_n - Ku_n, v\rangle = \langle g, v\rangle \quad \forall v \in Y_n. \tag{1.3}$$

As $X_n$ and $Y_n$ are finite dimension subspaces, solving (1.3) reduces to solve a linear algebraic equation, represented by a matrix $B$ of order $d_n$:

$$\sum_{j=1}^{d_n} \underbrace{[\langle\phi_j^n, \psi_i^n\rangle - \langle K\phi_j^n, \psi_i^n\rangle]}_{B_{ij}} c_j^n = \langle g, \psi_i^n\rangle \qquad \forall i = 1, ..., d_n \tag{1.4}$$

In [3] and [4], the authors proved that, if $K : X \to X$ is a compact linear operator not having 1 as an eigenvalue, and the pair $\{X_n, Y_n\}$ is a *regular pair* in a sense that will be detailed in the next section, equation (1.3) has a unique solution $u_n$ that satisfies

$$||u - u_n|| \leq c \inf_{x^* \in X_n} ||x^* - u||, \tag{1.5}$$

for $u$ the exact solution of (1.1). In this way, solvability and numerical stability of the approximation scheme are guaranteed. Different characterizations of the property of being "regular" are presented in the literature. Bounds on the condition number of the matrix associated to the resulting linear problem are also reported. The importance of this result is that, if the pair of sequences of subspaces is a regular one, the accuracy of the approximations $u_n$ to the unique solution $u$ of (1.1) does not depend formally on $Y_n$, as can be noted in (1.5). The idea is, then, to build regular pairs with simple test function subspaces $Y_n$, easy to handle, knowing that good convergence of the method is preserved.

With this aim, construction of discontinuous orthogonal multiwavelets is shown in [1] and [3]. In [10], a particular choice of subspaces and some examples are developed following [1].

In this work, we construct regular pairs $\{X_n, Y_n\}$ of subspaces of $L^2([0,1])$, following the definition introduced in [3] and [4], to solve Fredholm equations of the second kind, using piecewise polynomial functions. In the case of $Y_n$, the test functions, piecewise constant

functions are chosen to simplify computation. We prove regularity of the pair $\{X_n, Y_n\}$ for every $n$. The calculations are simple and the convergence is good. We present numerical examples to show the performance of the Petrov-Galerkin approximation scheme.
In the next section we review the Petrov-Galerkin method and prove the equivalence of the definitions of regular pair. In Section 3 we present our own choice of $\{X_n, Y_n\}$. Two numerical examples are shown in Section 4. Finally, we state some conclusions.

## 2. The Petrov-Galerkin method with regular pairs

Let $X$ be a Hilbert space, with inner product $\langle \, . \, \rangle$ and $|| \, . \, ||$ the associated norm. Consider $K : X \to X$ a compact linear operator and the related equation

$$u - Ku = g \tag{2.1}$$

for $g \in X$ fixed.
Suppose that 1 is not an eigenvalue of $K$, that is, 0 is the only solution of the homogeneous equation associated to (2.1), then, if there exists any solution of (2.1), it is unique ([8], p. 38). Since for a compact linear operator every eigenvalue $\lambda$ of $K$ satisfies $|\lambda| \leq ||K||$, if $||K|| < 1$ existence and uniqueness of a solution for (2.1) are guaranteed.
Once the existence and uniqueness are guaranteed, we look for the unique $u \in X$ satisfying (2.1) or, at least, a *good* approximation of it.
For each $n \in \mathbb{N}$, let $X_n \subset X$ and $Y_n \subset X$ be subspaces of $X$ of finite dimensions, verifying $dim(X_n) = dim(Y_n)$. Petrov-Galerkin method for equation (2.1) is a numerical method for finding $u_n \in X_n$ such that

$$\langle u_n - Ku_n, v\rangle = \langle g, v\rangle \quad \forall v \in Y_n. \tag{2.2}$$

The goal is to establish conditions under which

($i_1$) (2.2) has a unique solution $u_n \in X_n$ for each $n \in \mathbb{N}$ and
($i_2$) $||u_n - u|| \xrightarrow[n\to\infty]{} 0$, where $u$ is the unique solution of (2.1).

It is easy to see ([4], p. 409) that a condition ensuring the existence of a unique solution $u_n \in X_n$ for (2.2) is

$$X_n^{\perp} \cap Y_n = \{0\}. \tag{2.3}$$

Related to condition ($i_2$), from [8] we can expect convergence only if, for each $x \in X$, there exists a sequence $\{x_n, n \in \mathbb{N}\}$, with $x_n \in X_n$, so that $x_n \xrightarrow[n\to\infty]{} x$.
In the rest of this section, and the next one, we will follow [3] and [4] to precise the details of the method.
The sequences of subspaces of $X$, $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$, are chosen verifying the following denseness condition:

(H) given $x \in X$: $\exists x_n \in X_n, \exists y_n \in Y_n$, for $n = 1, 2, ...$ , so that $x_n \xrightarrow[n\to\infty]{} x$ and $y_n \xrightarrow[n\to\infty]{} x$.

**2.1. Regular pairs.** We will denote by $\{X_n, Y_n\}$ the sequences of subspaces $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ of $X$.
According to [4], $\{X_n, Y_n\}$ is said to be a *regular pair* if there exists a linear operator $\Pi_n : X_n \to Y_n$ so that

I) $\Pi_n X_n = Y_n$;
II) $\exists C_1 \in \mathbb{R} / \forall x \in X_n : ||x|| \leq C_1 \langle x, \Pi_n x\rangle^{\frac{1}{2}}$;
III) $\exists C_2 \in \mathbb{R} / \forall x \in X_n : ||\Pi_n x|| \leq C_2 ||x||$.

Conditions I) and II) guarantee (2.3): indeed, if $v \in X_n^{\perp} \cap Y_n$ and $\Pi_n X_n = Y_n$, then $v = \Pi_n w$ for $w \in X_n$. If II) holds, $||w|| \leq C_1 \langle w, \Pi_n w\rangle^{\frac{1}{2}} = C_1 \langle w, v\rangle^{\frac{1}{2}} = 0$, so $v = \Pi_n 0 = 0$.
The following theorem (see [4]) completes what we need to guarantee the existence of a unique approximation in each subspace $X_n$, and the convergence of the method:

**Theorem 2.1.** *Let $X$ be a Banach space and $K : X \to X$ a compact linear operator. Assume that $1$ is not an eigenvalue of the $K$. Suppose that $X_n$ and $Y_n$ satisfy condition $(H)$ and $\{X_n, Y_n\}$ is a regular pair. Then, there exists $n_0 > 0$ such that, for $n > n_0$, equation $\langle u_n - K u_n, v\rangle = \langle g, v\rangle \quad \forall v \in Y_n$ has unique solution $u_n \in X_n$ for any given $g \in X$, that satisfies $||u_n - u|| \leq c \inf_{x^* \in X_n} ||x^* - u||$ where $u \in X$ is the unique solution of equation $u - Ku = g$ and $c > 0$ is a constant independent of $n$.*

Finally, following [3], regular pairs can be characterized by means of the so called "correlation matrix".
Supose $\{\phi_i^n,\ i = 1, \cdots, d_n\}$ and $\left\{\psi_j^n,\ j = 1, \cdots, d_n\right\}$ are, respectively, basis of $X_n$ and $Y_n$, and let define the following $d_n \times d_n$ matrices:

$$\begin{aligned} G(X_n) &:= [\langle \phi_i^n, \phi_j^n\rangle] \\ G(Y_n) &:= [\langle \psi_i^n, \psi_j^n\rangle] \\ G(X_n, Y_n) &:= [\langle \phi_i^n, \psi_j^n\rangle] \\ G_+(X_n, Y_n) &:= \frac{1}{2}[G(X_n, Y_n) + G(Y_n, X_n)] \end{aligned}$$

Let consider the real case. $G(X_n)$ and $G(Y_n)$ are simetric and positive definite, as they are the gramian matrices of the inner products in $X_n$ and $Y_n$ respectively; $G(X_n, Y_n)$ is named "correlation matrix", and $G_+(X_n, Y_n)$ is its simetric part.
As it is mentioned in [3], we will prove the following

**Proposition 2.2.** It is enough for $\{X_n, Y_n\}$ to be a regular pair that the matrix $G_+(X_n, Y_n)$ is positive definite.

*Proof.* As $G(X_n)$ is positive definite:

$$0 < \lambda_{min} ||x||^2 \leq x^T G(X_n) x \leq \lambda_{max} ||x||^2 \quad \forall x \in X_n, x \neq 0,$$

if $\lambda_{min}$ and $\lambda_{max}$ are the minimum and maximum (real) eigenvalues of $G(X_n)$.
Similarly, for $G(Y_n)$,

$$0 < \nu_{min} ||x||^2 \leq x^T G(Y_n) x \leq \nu_{max} ||x||^2 \quad \forall x \in X_n, x \neq 0,$$

for $\nu_{min}$ and $\nu_{max}$ the minimum and maximum eigenvalues of $G(Y_n)$.
In the same way, if $G_+(X_n, Y_n)$ is positive definite,

$$0 < \mu_{min} ||x||^2 \leq x^T G_+(X_n, Y_n) x \leq \mu_{max} ||x||^2 \quad \forall x \in X_n, x \neq 0.$$

Now, for $\sigma = \frac{\lambda_{max}}{\mu_{min}}$ and $\sigma' = \frac{\nu_{max}}{\lambda_{min}}$,

$$0 < x^T G(X_n) x \leq \sigma x^T G_+(X_n, Y_n) x \quad \forall x \in X_n, x \neq 0$$

and

$$0 < x^T G(Y_n)x \leq \sigma' x^T G(X_n)x \qquad \forall x \in X_n, x \neq 0.$$

Let define $\Pi_n : X_n \to Y_n \;\; / \;\; \Pi_n \phi_i^n = \psi_i^n$, for $i = 1, \cdots, d_n$. Clearly, $\Pi_n X_n = Y_n$.
If $x \in X_n$, with $x = \sum_{i=1}^{d_n} \alpha_i \phi_i$, it is $\Pi_n x = \sum_{i=1}^{d_n} \alpha_i \psi_i$ and

$$\begin{aligned} ||x||^2 &= \alpha^T G(X_n)\alpha \leq \sigma \alpha^T G_+(X_n, Y_n)\alpha = \sigma \alpha^T \tfrac{1}{2}[G(X_n, Y_n) + G(Y_n, X_n)]\alpha = \\ &= \sigma \langle x, \sum_{i=1}^{d_n} \alpha_i \psi_i \rangle = \underbrace{\sigma}_{C_1^2} \langle x, \Pi_n x \rangle \end{aligned}$$

Additionally,

$$||\Pi_n x||^2 \;\; = \alpha^T G(Y_n)\alpha \leq \sigma' \alpha^T G(X_n)\alpha = \underbrace{\sigma'}_{C_2^2} ||x||$$

and the conditions I), II) and III) for $\{X_n, Y_n\}$ to be a regular pair are fulfilled.$\diamondsuit$

## 3. Regular pair $\{X_n, Y_n\}$ in $L^2([0,1])$

Now we will choose sequences of finite dimensional subspaces, $\{X_n, n \in \mathbb{N}\}$ and $\{Y_n, n \in \mathbb{N}\}$, denses in $X = L^2[0,1]$, such that $\{X_n, Y_n\}$ is a regular pair.

**3.1. The $S_n^m$ sets.** Let $S_n^m$ be the subspace of sectionally defined functions, consisting of polynomials of degree less than $m$ on each subinterval of $[0,1]$ of the form $I_{j,n} = (\frac{j}{2^n}, \frac{j+1}{2^n})$, for $j = 0, \ldots, 2^n - 1$. For instance, $S_n^2$ is the set of polynomials of degree 0 or 1 on each subtinterval $I_{j,n}$ for $j = 0, \ldots, 2^n - 1$, and $S_n^1$ is the set of piecewise constant functions on the same subintervals.
The dimension of $S_n^m$ is $m2^n$ and they form an increasing sequence,

$$S_0^m \subset S_1^m \subset \cdots S_n^m \subset \cdots .$$

Let $S^m = \cup_{n=0}^{\infty} S_n^m$. It contains the polynomials of degree less than $m$ in any of the subintervals of the form $I_{j,n} = (\frac{j}{2^n}, \frac{j+1}{2^n})$ for some $n$ and $j = 0, \ldots, 2^n - 1$.
Note that step functions on the subintervals $(\frac{j}{2^n}, \frac{j+1}{2^n}) \subset [0,1]$ are functions of $S^1$. Then, since $C_0([0,1])$ (continuous functions with compact support on $[0,1]$) can be approximated by these step functions and $C_0([0,1])$ is dense in $L^2([0,1])$, it results that the closure of $S_m$, $\overline{S^m}$, is $L^2([0,1])$.
Let consider the Legendre polynomials of degree $l$, $P_l(x)$, originally defined in $[-1,1]$, adapted to an interval $(a,b)$, i.e. $L_l^{(a,b)}(x) = P_l(\frac{2x-b-a}{b-a})\chi_{(a,b)}$, being $\chi_{(a,b)}$ the characteristic function of the interval $(a,b)$. For example, if $a = 0, b = \frac{1}{2}$: $L_0^{(0,\frac{1}{2})}(x) = 1$, $L_1^{(0,\frac{1}{2})}(x) = (4x-1)\chi_{(0,\frac{1}{2})}$ and $L_2^{(0,\frac{1}{2})}(x) = \frac{3(4x-1)^2-1}{2}\chi_{(0,\frac{1}{2})}$.
Afterwards we normalize them, obtaining $p_l^{(a,b)}(x) = \frac{L_l^{(a,b)}(x)}{\|L_l^{(a,b)}\|_2}$.
For given $m \geq 1$ and $n$, we construct basis for $S_n^m$, considering this polynomials on adequate subintervals. We divide the interval $[0,1]$ in $2^n$ subintervals $I_{j,n}$, for $j = 0, \cdots, 2^n - 1$, and adapt and normalize the Legendre polynomials of degree $l$, for $l = 0, \cdots, m-1$, to the intervals $I_{j,n}$. In that way we obtain $m2^n$ polynomials of degree less than $m$, with support on one of the subintervals $I_{j,n}$: $p_l^{(\frac{j}{2^n}, \frac{j+1}{2^n})}$, $j = 0, ..., 2^n - 1$,

$l = 0, \cdots, m-1$.
In order to simplify the notation we rename: $p_l^{i,n} = p_l^{(\frac{i}{2^n}, \frac{i+1}{2^n})}$ and $q_l^{j,n} = p_l^{i,n+1}$.

**3.2. Choice of $X_n$ and $Y_n$.** Let

$$X_n = S_n^2 = span\left\{p_0^{0,n}, p_1^{0,n}, p_0^{1,n}, p_1^{1,n}, \cdots, p_0^{2^n-1,n}, p_1^{2^n-1,n}\right\},$$

so $dim(X_n) = 2.2^n$, and

$$Y_n = S_{n+1}^1 = span\left\{q_0^{0,n}, q_0^{1,n}, \cdots, q_0^{2^{n+1}-1,n}\right\},$$

and $dim(Y_n) = 1.2^{n+1} = dim(X_n)$.
As we have chosen orthogonal polynomials, condition $X_n^{\perp} \cap Y_n = \{0\}$ is fulfilled, and the uniqueness of the solution of each Petrov-Galerkin equation (2.2) is guaranteed .
With the notation of Section 2, we rename the basis: $\phi_i^n(x) = p_0^{\frac{i-1}{2},n}(x)$ if $i$ is odd, $\phi_i^n(x) = p_1^{\frac{i-2}{2},n}(x)$ if $i$ is even, and $\psi_j^n(x) = q_0^{j,n}(x)$ for $X_n$ and $Y_n$ respectively. Thus, $X_n = span\{\phi_i^n, i = 1, \cdots, 2^{n+1}\}$ and $Y_n = span\{\psi_j^n, j = 1, \cdots, 2^{n+1}\}$.
In order to show that $\{X_n, Y_n\}$ is a regular pair, we recall Proposition 2.2.
The correlation matrix

$$G(X_n, Y_n) = [\langle \phi_i, \psi_j \rangle] = [\int_0^1 \phi_i(x)\psi_j(x)dx] \tag{3.1}$$

is a $2^{n+1} \times 2^{n+1}$ block matrix, composed by square blocks of dimension $2 \times 2$ on the principal diagonal. Each block is identical and independent of $n$:

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \end{bmatrix}.$$

Then, $G_+(X_n, Y_n) = \frac{1}{2}[G(X_n, Y_n) + G(Y_n, X_n)]$ is, in turn, a block matrix, with positive definite blocks on its principal diagonal,

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}-\sqrt{\frac{3}{2}}}{4} \\ \frac{\sqrt{2}-\sqrt{\frac{3}{2}}}{4} & \frac{\sqrt{3}}{2\sqrt{2}} \end{bmatrix}$$

and we can conclude that the pair is regular.
Although the dimension of the subspaces may be large, the choice is justified since computations are easy because of the simple structure of the test functions subspace. In addition, in the numerical experiments, we observe that a small $n$ is enough to obtain good accuracy.
We can generalize the regular pair choosing $X_n = S_n^{2m}$ and $Y_n = S_{n+1}^m$ , $m \in \mathbb{N}$. In this case the correlation matrix is a $2^{n+1}m \times 2^{n+1}m$ block matrix with blocks of dimension $2m \times 2m$.

## 4. Numerical Examples

In this section we illustrate the performance of the Petrov-Galerkin approximation considering the proposed basis for the regular pair of subspaces of test and trial functions, $X_n = S_n^2$ and $Y_n = S_{n+1}^1$, in two typical examples.
For each example we plot the real solution $u$ to de Fredholm equation together with the

approximate $u_3$ on $[0,1]$. In order to appreciate the accuracy of the approximation, the error in a small subinterval is also shown. Tables showing the error $|u(t)-u_4(t)|^2$ for some $t \in [0,1]$ are presented. The values of $||u-u_n||_{L^2([0,1])}$ for $n=3,4$ and 5 are also offered.

**4.1. Example 1.** For the equation

$$u(t) - \int_0^1 e^{t-2s} u(s)\, ds = \sin(t), \qquad u \in L^2([0,1]) \tag{4.1}$$

the exact solution is $u(t) = \sin(t) + e^{t+1}(\frac{1}{5} - \frac{\cos(1)+2\sin(1)}{5e^2})$.
The linear operator $K : L^2([0,1]) \to L^2([0,1])$, $Ku(t) = \int_0^1 k(s,t)u(s)ds = \int_0^1 e^{t-2s}u(s)\, ds$ is compact and $\|K\| \leq (\int_0^1\int_0^1 k^2(s,t)dsdt)^{\frac{1}{2}} \leq 0.89 < 1$, thus 1 is not an eigenvalue of $K$ and the convergence of the Petrov-Galerkin method is guaranteed.
Note that since the integral operator has a degenerate kernel, computations are simple. As it can be seen in Figure 1, on the left, the real solution and the approximation for $n=3$ are actually indistinguishable in $[0,1]$. In the same figure, on the right, a zoom of a small interval where the difference can be appreciated.

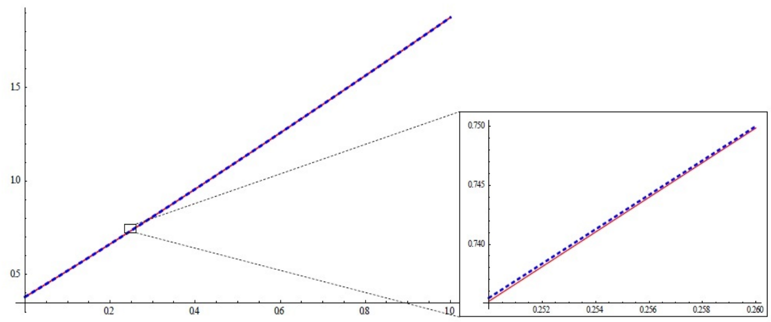


FIGURE 1. $u$ and $u_3$ for Example 1

For $n=4$, in Table 4.1 the values of the error $|u(t)-u_4(t)|^2$ for $t = \frac{j}{10}$, $\quad j = 0, \cdots 10$ are displayed. The quadratic error $||u-u_n||_{L^2([0,1])} = [\int_0^1 |u_n(t)-u(t)|^2 dt]^{1/2}$ for $n=3,4$ and 5 are $2.7\ 10^{-8}$, $1.7\ 10^{-9}$ and $1.07\ 10^{-10}$ respectively.

**4.2. Example 2.** For the equation

$$u(t) - \int_0^1 \frac{e^{st}}{2} u(s)\, ds = te^t - \frac{1+te^{t+1}}{2(1+t)^2} \qquad u \in L^2([0,1]) \tag{4.2}$$

the exact solution is $u(t) = te^t$.
The linear operator $K : L^2([0,1]) \to L^2([0,1])$, $\quad Ku(t) = \int_0^1 k(s,t)u(s)ds = \frac{1}{2}\int_0^1 e^{st}u(s)\, ds$ is compact and $\|K\| \leq (\int_0^1\int_0^1 k^2(s,t)dsdt)^{\frac{1}{2}} \leq 0.68 < 1$, thus 1 is not an eigenvalue of $K$ and the convergence of the Petrov-Galerkin method is guaranteed.
Note that in this case the kernel is not degenerate.

| $t$ | $\|u(t) - u_4(t)\|^2$ |
|---|---|
| 0 | $1.418 \cdot 10^{-8}$ |
| 0.1 | $0.010 \cdot 10^{-8}$ |
| 0.2 | $0.212 \cdot 10^{-8}$ |
| 0.3 | $0.001 \cdot 10^{-8}$ |
| 0.4 | $0.001 \cdot 10^{-8}$ |
| 0.5 | $0.213 \cdot 10^{-8}$ |
| 0.6 | $0.034 \cdot 10^{-8}$ |
| 0.7 | $0.003 \cdot 10^{-8}$ |
| 0.8 | $0.003 \cdot 10^{-8}$ |
| 0.9 | $0.047 \cdot 10^{-8}$ |

TABLE 1. $|u(t) - u_4(t)|^2$ for Example 1

It can be seen in Figure 2 that, once again, the real solution and the approximation for $n = 3$ are indistinguishable in $[0, 1]$. In the same figure, on the right, a zoom of a small interval where the difference can be appreciated.

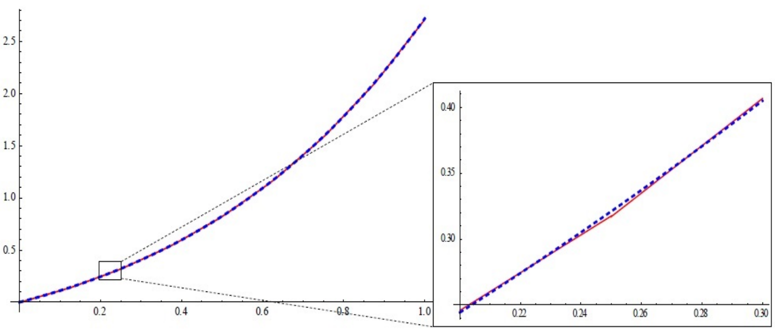


FIGURE 2. $u$ and $u_3$ for Example 2

In Table 2, the values of the error $|u(t) - u_4(t)|^2$ for $t = \frac{j}{10}$, $j = 0, \cdots, 10$ are displayed for $n = 4$. The quadratic error for $n = 3, 4$ and 5 are 2.8 $10^{-3}$, 6.9 $10^{-4}$ and 5.6 $10^{-8}$ respectively.

## 5. Conclusions and future work

We adapted Legendre polynomials to build simple sequences of subspaces of finite dimension and applied Petrov-Galerkin method to solve Fredholm integral equations of the second kind. Trial and test functions are easy to handle, for instance test functions can be chosen as piecewise constant functions on [0,1]. For this particular choice of subspaces, the property of regularity that guarantees the convergence of the approximate solutions to the exact one can be readily proven.

| $t$ | $\lvert u(t) - u_4(t)\rvert^2$ |
|---|---|
| 0 | $0.455 \cdot 10^{-6}$ |
| 0.1 | $0.110 \cdot 10^{-6}$ |
| 0.2 | $0.001 \cdot 10^{-6}$ |
| 0.3 | $0.001 \cdot 10^{-6}$ |
| 0.4 | $0.265 \cdot 10^{-6}$ |
| 0.5 | $1.923 \cdot 10^{-6}$ |
| 0.6 | $0.457 \cdot 10^{-6}$ |
| 0.7 | $0.006 \cdot 10^{-9}$ |
| 0.8 | $0.057 \cdot 10^{-9}$ |
| 0.9 | $1.052 \cdot 10^{-6}$ |

TABLE 2. $|u(t) - u_4(t)|^2$ for Example 2

Although the dimension $d_n$ of the subspaces may be large, computations are simple because of the structure of the test functions subspace. We illustrate the performance in two typical numerical examples where considerable precision is obtained, even for small $d_n$.

The next step in our work is to apply the same regular pair we used here to develope an iterated Petrov-Galerkin method for improving the order of convergence.

## References


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