Poincare Journal of Analysis & Applications
Vol. 2018(2(I)), 57-63

Notes on lifting of Bertrand curve on tangent space TR^3

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Date of Receiving : 02.02.2018
Date of Revision : 20.12.2018
Date of Acceptance : 23.12.2018

Abstract. In this paper, firstly, we define the Bertrand curve of any curve with respect to the vertical, complete and horizontal lifts on space R^3 to its tangent space $TR^3 = R^6$. Secondly, we examine the Frenet-Serret aparatus $\{T^*(s), N^*(s), B^*(s), \kappa^*, \tau^*\}$ and the Darboux vector W^* of the Bertrand curve α^* according to the vertical, complete and horizontal lifts on TR^3 by depend on the lifting of Frenet-Serret aparatus $\{T(s), N(s), B(s), \kappa, \tau\}$ of the first curve α on space R^3. As a result of this transformation on space R^3 to its tangent space TR^3, we can speak about the features of Bertrand curve of any curve on space TR^3 by looking at the characteristics of the first curve.

1. Introduction

In differentiable gometry, lift method has an important role. Because, it is possible to generalize to differentiable structures on any space (resp. manifold) to extended spaces (resp. extended manifolds) using lift function [4, 5, 6, 7, 8, 10]. Thus, it may be extended the following theorem given on space R^3 to its tangent space TR^3. It is well known that many studies related to the differential geometry of curves have been made. Especially, by establishing relations between the Frenet Frames in mutual points of two curves several theories have been obtained. The best known of the Bertrand curves discovered by J. Bertrand in 1850 are one of the important and interesting topics of classical special curve theory. A Bertrand curve is defined as a special curve which shares its principal normals with another special curve, called Bertrand mate or Bertrant curve partner. If

$$\alpha^* = \alpha + \lambda N, \lambda = constant, \tag{1.1}$$

then (α, α^*) are called Bertrand curve pair. If α and α^* Bertrand curves pair, then $\langle T, T^* \rangle = \cos\theta$ =constant [2, 3]. The definition of $n-$dimensional Bertrand curves

2010 *Mathematics Subject Classification.* 28A51,53A04,57R25.
Key words and phrases. Vector fields; Bertrand curve; vertical lift; complete lift; horizontal lift; tangent space.
The authors are thankful to the referee(s) for their valuable time and suggestions towards the improvement of this paper.
Communicated by: A. Salimov
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in Lorentzian space is given by comparing a well-known Bertrand pair of curves in $n-$dimensional Euclidean space. It shown that the distance between corresponding of Bertrand pair of curves and the angle between the tangent vector fields of these points are constant. It is well-known that, if a curve is diferentiable in an open interval, at each point, a set of mutually orthogonal unit vectors can be constructed. And these vectors are called Frenet frame or moving frame vectors. The rates of these frame vectors along the curve define curvatures of the curves. The set, whose elements are frame vectors and curvatures of a curve α, is called Frenet-Serret aparatus of the curves. Let Frenet vector fields be $T(s), N(s), B(s)$ of α and let the curvature and torsion of the curve $\alpha(s)$ be κ and τ, respectively. The quantities $\{T(s), N(s), B(s), W, \kappa, \tau\}$ are collectively Frenet-Serret aparatus of the curves. Let a rigid object move along a regular curve described parametrically by $\alpha(s)$.

For any unit speed curve α, in terms of the Frenet-Serret aparatus, the Darboux vector can be expressed as [1]

$$W(s) = \tau(s)T(s) + \kappa(s)B(s) \tag{1.2}$$

where curvature and torsion function are defined by $\kappa(s) = \| T^{'}(s) \|$ and $\tau(s) = -\left\langle B^{'}(s), N(s)\right\rangle$. If we consider the normalization of the Darboux, we have

$$\sin\varphi = \frac{\tau}{\|W\|}, \quad \cos\varphi = \frac{\kappa}{\|W\|} \tag{1.3}$$

and

$$C = \sin\varphi T + \cos\varphi B \tag{1.4}$$

where $\measuredangle(W, B) = \varphi$.

Definition 1.1. Let $\alpha : I \to R^3$ and $\alpha^* : I \to R^3$ be the C^2-class differentiable unit speed two curves and let $\{T(s), N(s), B(s)\}$ and $\{T^*(s), N^*(s), B^*(s)\}$ be the Frenet frames of the curves α and α^*, respectively. For $\forall s \in I$, if the principal normal vector N of the curve α is linearly dependent on the principal vector N^* of the curve α^*, then the pair (α, α^*) is said to be Bertrand curves pair.

The relations between the Frenet frames

$$\{T(s), N(s), B(s)\} \text{ and } \{T^*(s), N^*(s), B^*(s)\}$$

are as follows [2]:

$$\begin{aligned} T^*(s) &= \cos\theta T + \sin\theta B, \\ N^*(s) &= N, \\ B^*(s) &= \sin\theta T + \cos\theta B, \end{aligned} \tag{1.5}$$

where $\measuredangle(T, T^*) = \theta$.

Theorem 1.2. *Let (α, α^*) be a Bertrand curves pair on R^3. For the curvatures and the torsions of the Bertrand curves pair (α, α^*) we have* [3]

$$\begin{aligned} \kappa^* &= \frac{\lambda\kappa - \sin^2\theta}{\lambda(1-\lambda\kappa)}, \quad \lambda = constant \\ \tau^* &= \frac{\sin^2\theta}{\lambda^2\tau}. \end{aligned} \tag{1.6}$$

Theorem 1.3. *Let (α, α^*) be a Bertrand curves pair on R^3. For the curvatures and the torsions of the Bertrand curves pair (α, α^*) we have* [2]

$$\begin{aligned} \kappa^* \frac{ds^*}{ds} &= \kappa \cos\theta - \tau \sin\theta, \\ \tau^* \frac{ds^*}{ds} &= \kappa \sin\theta - \tau \cos\theta. \end{aligned} \tag{1.7}$$

By using equation (1.2), we can write Darboux vector belonging to Bertrand mate α^*.

$$W^* = \tau^* T^* + \kappa^* B^* \tag{1.8}$$

If we consider the normalization of the Darboux vector, we have

$$C^* = \sin\varphi^* T^* + \cos\varphi^* B^* \tag{1.9}$$

From the equation (1.3) and (1.7), we can write

$$\sin\varphi^* = \sin(\varphi + \theta) \;\text{ and }\; \cos\varphi^* = \cos(\varphi + \theta) \tag{1.10}$$

where $\|W^*\| = \sqrt{(\kappa^*)^2 + (\tau^*)^2} = \|W\|$ and $\measuredangle(W^*, B^*) = \varphi^*$. By the using (1.5) and (1.10), the final version of the equation (1.9) is as follows

$$C^* = \sin\varphi T + \cos\varphi B \tag{1.11}$$

The paper is structured as follows. Firstly, we define the Bertrand curve with respect to the vertical, complete and horizontal lifts on space R^3 to its tangent space $TR^3 = R^6$. Secondly, we examine the Frenet-Serret aparatus of the Bertrand curve α^* according to the vertical, complete and horizontal lifts on TR^3 by depend on the lifting of Frenet-Serret aparatus $\{T(s), N(s), B(s), \kappa, \tau\}$ of the first curve α on space R^3. Finally, it is revealed the vertical, complete and horizontal lifts of the Darboux vector W^* by depend on the lifting of Frenet-Serret aparatus of the curve α. In addition, we include all special cases the curvature $\kappa^*(s)$ and torsion $\tau^*(s)$ of the Frenet-Serret aparatus $\{T^*(s), N^*(s), B^*(s), \kappa^*, \tau^*\}$ of the Bertrand curve α^* with respect to the vertical, complete and horizontal lifts on space R^3 to its tangent space TR^3.

In this study, all geometric objects will be assumed to be of class C^∞ and the sum is taken over repeated indices. Also, v, c and H denote the vertical, complete and horizontal lifts any differentiable geometric structures defined on R^3 to its tangent space TR^3, respectively.

2. Lift of vector field

The vertical lift of a vector field X on space R^3 to extended space $TR^3(= R^6)$ is the vector field $X^v \in \chi(TR^3)$ given as [4, 10]:

$$X^v(f^c) = (Xf)^v, \; \forall f \in F(R^3) \tag{2.1}$$

The vector field $X^c \in \chi(TR^3)$ defined by

$$X^c(f^c) = (Xf)^c, \; \forall f \in F(R^3) \tag{2.2}$$

is called the complete lift of a vector field X on R^3 to its tangent space TR^3.

The horizontal lift of a vector field X on space R^3 to TR^3 is the vector field $X^H \in \chi(TR^3)$ determined by

$$X^H(f^v) = (Xf)^v, \; \forall f \in F(R^3) \tag{2.3}$$

the general properties of vertical, complete and horizontal lifts of a vector field on R^3 as follows:

Proposition 2.1. Let be functions all $f, g \in F(R^3)$ and vector fields all $X, Y \in \chi(R^3)$. Then it is satisfied the following equalities [8, 9, 10].

$$\begin{aligned}
(X+Y)^v &= X^v + Y^v,\ \ (X+Y)^c = X^c + Y^c,\ (X+Y)^H = X^H + Y^H, \quad (2.4)\\
(fX)^v &= f^v + X^v,\ (fX)^c = f^c X^v + f^v X^c,\ X^v(f^v) = 0, (fg)^H = 0,\\
X^c(f^v) &= X^v(f^c) = (Xf)^v, X^c(f^c) = (Xf)^c, X^H(f^v) = (Xf)^v,\\
\chi(U) &= Sp\left\{\frac{\partial}{\partial x^\alpha}\right\},\ \chi(TU) = Sp\left\{\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^\alpha}\right\},\\
\left(\frac{\partial}{\partial x^\alpha}\right)^c &= \frac{\partial}{\partial x^\alpha}, \left(\frac{\partial}{\partial x^\alpha}\right)^v = \frac{\partial}{\partial y^\alpha},\ \left(\frac{\partial}{\partial x^\alpha}\right)^H = \frac{\partial}{\partial x^\alpha} - \chi\Gamma^\alpha_\beta \frac{\partial}{\partial y^\alpha}.
\end{aligned}$$

where Γ^α_β are Christoper symbols, U and TU are respectively topolgical opens of R^3 and TR^3, $f^v, f^c \in F(TR^3)$, $X^v, Y^v, X^c, Y^c, X^H, Y^H \in \chi(TR^3), 1 \le \alpha, \beta \le 3$.

3. Lifting of the Bertrand curve $\alpha^*(s)$

In this section, we compute the vertical, complete and horizontal lifts of the Bertrand curve $\alpha^*(s)$ with the curvature $\kappa^*(s)$ and torsion $\tau^*(s)$ on space R^3 to TR^3.

3.1. The vertical lifting of the Bertrand curve $\alpha^*(s)$.

Theorem 3.1. *Let $\alpha^*(s)$ be the Bertrand curve defined by (1.1) on space R^3. Then, we get the following equalities with respect to vertical, complete and horizontal lifts on TR^3.*

i) $\alpha^*(s)_1 = (\alpha^*(s))^v = \alpha(s) + \lambda N^v(s)$.
This is general equation for all Bertrand curve $\alpha^*(s)$ on space R^3 to TR^3.
ii) $\alpha^*(s)_2 = (\alpha^*(s))^c = \alpha^c(s)$.
This is general equation for all Bertrand curve $\alpha^*(s)$ on space R^3 to TR^3.
iii) $\alpha^*(s)_3 = (\alpha^*(s))^H = 0$.
This is general equation for all Bertrand curve $\alpha^*(s)$ on space R^3 to TR^3, where $\lambda = cons\tan t$, $\forall s \in I$.

Proof. *i)* For a unit speed curve $\alpha^*(s)_1 = (\alpha^*(s))^v$ on TR^3 with curvature $(\kappa^*(s))^v$ and torsion $(\tau^*(s))^v$ on TR^3. Let f be a function and λ =constant. Then we write $f^{\,v} = f$ and $\lambda^v = \lambda$ with respect to vertical lifts on tangent bunde TR^3 [10]. If we apply vertical lift to both side the equation (1.1), we get

$$\begin{aligned}
(\alpha^*(s))^v &= \alpha^v(s) + \lambda N^v(s))\\
&= \alpha(s) + \rho(s) N^v(s).
\end{aligned}$$

ii) For a unit speed curve $\alpha^*(s)_2 = (\alpha^*(s))^c$ on TR^3 with curvature $(\kappa^*(s))^c$ and torsion $(\tau^*(s))^c$ on TR^3. Let f be a function and λ =constant. Then, we write $\lambda^c = 0$ with respect to complete lifts on tangent bunde TR^3[10]. If we apply complete lift to both side the equation (1.1), we get

$$\begin{aligned}
(\alpha^*(s))^c &= \alpha^c(s) + \lambda^c N^c(s)\\
&= \alpha^c(s)
\end{aligned}$$

iii) For a unit speed curve $\alpha^*(s)_3 = (\alpha^*(s))^H$ on TR^3 with curvature $(\kappa^*(s))^H$ and torsion $(\tau^*(s))^H$ on TR^3. Let f be a function and λ =constant. Then we write $f^{\,H} = 0$ and $\lambda^H = 0$ with respect to horizontal lifts on tangent bunde TR^3 [10]. If we apply horizontal lift to both side the equation (1.1), we get

$$\begin{aligned}(\alpha^*(s))^H &= \alpha^H(s) + \lambda^H N^H(s)\\ &= 0\end{aligned}$$

□

Theorem 3.2. *Let $\alpha^*(s)$ be the Bertrand curve on R^3 and a unit speed curve $\alpha^*(s)_1 = (\alpha^*(s))^v$ on TR^3 with curvature $(\kappa^*(s))^v$ and torsion $(\tau^*(s))^v$ on TR^3. Then, we get the following equalities with respect to vertical lifts on tangent bunde TR^3.*

$$\begin{aligned}(T^*(s))^v &= (\cos\theta)^v T^v + (\sin\theta)^v B^v \qquad (3.1)\\ (N^*(s))^v &= N^v(s)\\ (B^*(s))^v &= (\sin\theta)^v N^v + (\cos\theta)^v B^v\end{aligned}$$

where T, N, B, κ, τ is respectively tangent vector, normal vector, binormal vector, curvature, torsion of the curve $\alpha(s)$.

Proof. For Bertrand curves pair, we write $\langle T, T^*\rangle = \cos\theta$ =constant [2, 3]. So, we get $\langle T^v, (T^*)^v\rangle = (\cos\theta)^v = \cos\theta$. In addition, we know $(N^*(s))^v = N^v(s)$ from the definition of Bertrand curve. Morover, Since the measure of the angle between $(T^*)^v$ and T^v is constant, the measure of the angle between $(B^*(s))^v$ and B^v is also constant. This constant is θ on TR^3. Considering that $(T^*)^v, T^v, (B^*(s))^v$ and B^v are unit length vectors, the equations for $(T^*)^v$ and $(B^*(s))^v$ are obtained immediately. □

Corollary 3.3. Let the curvature κ and torsion τ of the curve $\alpha(s)$ on R^3 are constant or non-constant functions and $\alpha^*(s)$ be the Bertrand curve on R^3. For a unit speed curve $\alpha^*(s)_1 = (\alpha^*(s))^v$ on with curvature $(\kappa^*(s))^v$ and torsion $(\tau^*(s))^v$ on TR^3, we say the curve $(\alpha^*(s))^v$ on TR^3 is similar structure and apperance to R^3 with respect to vertical lifts.

3.2. The complete and horizontal lifting of the Bertrand curve $\alpha^*(s)$.

Theorem 3.4. *Let $\alpha^*(s)$ be Bertrand curve on R^3 and a unit speed curve $\alpha^*(s)_2 = (\alpha^*(s))^c$ on TR^3 with curvature $(\kappa^*(s))^c$ and torsion $(\tau^*(s))^c$ on TR^3. Then, for all curves everytime we say that the complete lift of principal normal vector $N^*(s)$ of the Bertrand curve $\alpha^*(s)$ is equal to the complete lift of the principal normal vector $N(s)$ of the Bertrand curve $\alpha(s)$ on tangent bunde TR^3.*

Proof. Similarly to vertical lifts, the theorem easily proved with respect to complete lift. Because of $\cos\theta$ =constant, we write $(\cos\theta)^c = 0$. From the equation (3.1), we have $T^*(s))^c = 0, (B^*(s))^c = 0$ and $(N^*(s))^c = N^c(s)$ □

where T, N, B, κ, τ is respectively tangent vector, normal vector, binormal vector, curvature, torsion of the curve $\alpha(s)$.

Theorem 3.5. *Let the curvature κ and torsion τ of the curve $\alpha(s)$ be constant or non-constant functions on R^3. Everytime, the horizontal lift of basic normal vector $N^*(s)$ of the Bertrand curve $\alpha^*(s)$ is equal to the horizontal lift of the principal normal vector*

$N(s)$ of the Bertrand curve $\alpha(s)$ on tangent bunde TR^3. (the horizontal lift of basic normal vector $(N^*(s))^H$ linear dependency $T^c(s)$ on TR^3).

where $\alpha^*(s)$ be the Bertrand curve on R^3 and a unit speed curve $\alpha^*(s)_3 = (\alpha^*(s))^H$ on TR^3 with curvature $(\kappa^*(s))^H$ and torsion $(\tau^*(s))^H$ on TR^3

Proof. Let the curvature κ and torsion τ of the curve $\alpha(s)$ be constant or non-constant functions on R^3. For all functions on R^3, we write $f^H = 0$ with respect to horizontal lifts [10]. So, we get $(\cos\theta)^H = (\sin\theta)^H = 0$ and $(N^*(s))^H = N^H(s)$ on TR^3. □

Corollary 3.6. Let $\alpha^*(s)$ be the Bertrand curve with curvature $\kappa^*(s)$ and torsion $\tau^*(s)$ on R^3. Then, respectively, the curvature $(\kappa^*(s))^v$ and torsion $(\tau^*(s))^v$ of the curve $(\alpha^*(s))^v$ is the same as κ^* and τ^* of the Bertrand curve $\alpha^*(s)$ with respect to vertical lifts on TR^3, where κ, τ is respectively curvature and torsion of the curve $\alpha(s)$.

Proof. If we apply a vertical lift to both sides of the equality (1.6) and using the Proposition 2.1, we easily get the following results.

$$\begin{aligned} (\kappa^*)^v &= \frac{\lambda^v\kappa^v - (\sin^2\theta)^v}{\lambda^v(1-\lambda\kappa)^v} = \frac{\lambda\kappa - \sin^2\theta}{\lambda(1-\lambda\kappa)} = \kappa^* \\ (\tau^*)^v &= \frac{(\sin^2\theta)^v}{(\lambda^2)^v\tau^v} = \frac{\sin^2\theta}{\lambda^2\tau} = \tau^*. \end{aligned} \tag{3.2}$$

where λ =constant, $\langle T, T^*\rangle = \cos\theta$ =constant. □

Corollary 3.7. Let $\alpha^*(s)$ be the Bertrand curve with curvature $\kappa^*(s)$ and torsion $\tau^*(s)$ on R^3. Then, the curvature $(\kappa^*(s))^c$ and torsion $(\tau^*(s))^c$ of the curve $(\alpha^*(s))^c$ are already zero with respect to complete lifts on TR^3.

Proof. For all constant functions on R^3, we write $f^c = 0$ with respect to proporties of the complete lifts on TR^3[10]. Using the $\lambda^c = 0$ and $(\cos\theta)^c = 0$, we easily get $(\kappa^*(s))^c = 0$ and $(\tau^*(s))^c = 0$. □

Corollary 3.8. Let $\alpha^*(s)$ be the Bertrand curve with curvature $\kappa^*(s)$ and torsion $\tau^*(s)$ on R^3. Then, the curvature $(\kappa^*(s))^H$ and torsion $(\tau^*(s))^H$ of the curve $(\alpha^*(s))^H$ are already zero with respect to complete lifts on TR^3.

Proof. For all constant or non-constant functions on R^3, we write $f^H = 0$ with respect to proporties of the horizontal lifts on TR^3[10]. So, we can write $(\kappa^*(s))^H = (\tau^*(s))^H = 0$. □

3.3. The Darboux vector of the Bertrand curve with recpect to vertical,complete and horizontal lifts on TR^3. By using equation (1.8), we can write Darboux vector $(W^*)^v$ belonging to $(\alpha^*(s))^v$ with respect to vertical lifts on TR^3.

$$(W^*)^v = (\tau^*)^v(T^*)^v + (\kappa^*)^v(B^*)^v \tag{3.3}$$

If we consider the normalization of the Darboux vector and using the equation (1.3) and (3.2), we can write unit Darboux vector $(C^*)^v$ the following equation with respect to vertical lifts on TR^3.

$$(C^*)^v = (\sin\varphi)^v(T)^v + (\cos\varphi)^v(B)^v \tag{3.4}$$

Corollary 3.9. Let the curvature κ and torsion τ of the curve $\alpha(s)$ on R^3 are constant or non-constant functions and $\alpha^*(s)$ be the Bertrand curve on R^3. For a curve $\alpha^*(s)_2 = (\alpha^*(s))^c$ with curvature $(\kappa^*(s))^c$ and torsion $(\tau^*(s))^c$ on TR^3, we say the unit Darboux vector $(C^*)^c$ on TR^3 is a point everytime with respect to complete lifts on TR^3.

Proof. Using the equation $(\sin\varphi)^c = 0$ and $(\cos\varphi)^c = 0$, we get $(C^*)^c = 0$. □

Corollary 3.10. Let the curvature κ and torsion τ of the curve $\alpha(s)$ on R^3 are constant or non-constant functions and $\alpha^*(s)$ be the Bertrand curve on R^3. For a curve $\alpha^*(s)_3 = (\alpha^*(s))^H$ with curvature $(\kappa^*(s))^H$ and torsion $(\tau^*(s))^H$ on TR^3, we say the unit Darboux vector $(C^*)^H$ on TR^3 is a point everytime with respect to horizontal lifts on TR^3.

Proof. Using the equation $(\sin\varphi)^H = 0$ and $(\cos\varphi)^H = 0$, we get $(C^*)^H = 0$ □

4. conclusion

In this study, using lifting methods, we see that it may be generalized the Bertrand curve $\alpha^*(s)$ and Frenet-Serret aparatus of the Bertrand curve $\alpha^*(s)$ given by (1.5),(1.6),(1.8) with recpect to vertical, complete and horizontal lifts on TR^3. In addition, we can speak about the features of Bertrand curve of any curve on space TR^3 by looking at the characteristics of the first curve α.

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