Poincare Journal of Analysis & Applications
Vol. 3, No. 1 (2016), 1-12

DOI: 10.46753/pjaa.2016.v03i01.001

CONVERGENCE OF ADOMIAN DECOMPOSITION METHOD FOR PDES

SILVIA SEMINARA† AND MARIA INÉS TROPAREVSKY

Date of Receiving	:	25.06.2015
Date of Revision	:	06.10.2015
Date of Acceptance	:	06.10.2015

Abstract. Adomian Decomposition Method (ADM) is a powerful tool for solving general functional equations. It provides a sequence of analytic approximate solutions to a wide range of nonlinear problems. Trying to formulate a general framework for the method, several demonstrations of convergence have been published, with varying degree of success. In this work we state conditions under which the approximate solutions to partial differential equations provided by ADM are merely a reordering of the Taylor polynomials of the solution, and, consequently, no other demonstration of convergence is necessary in those cases.

1. Introduction

Adomian Decomposition Method (ADM) was introduced by George Adomian in the '80s ([2], [3], [4]). It provides analytical approximate solutions for very general nonlinear equations without linearization or simplification. The description of the method is simple, but, because of the wide range of problems it covers, it is difficult to state a general demonstration of convergence. The author gave no formal proofs but provided (without showing the details) a justification based on the composition of Taylor series. He offered lots of examples of application to problems with well known solutions, to show the goodness of the method. In the last two decades, a great quantity of works have been published where it was successfully applied to many different problems. More or less general demonstrations of convergence were also offered.
In this work we state conditions under which the solutions provided by ADM to certain partial differential equations (PDE) are just reorderings of the Taylor polynomials of the solution, as George Adomian posited. We formally prove this issue and show some examples.

Key words and phrases. Nonlinear equations, Approximate analytical solutions, Adomian decomposition method, PDEs.
Communicated by. Virender
†Corresponding author.

2. The Adomian Decomposition Method

Although well known, we briefly describe ADM, in order to settle the notation. We refer to [3] and [4] for a detailed description.
Let us consider a general functional equation of the form

$$F[u(x)] = g(x) \tag{2.1}$$

where $F : X \to Y$ is an operator between two Banach (functional) spaces X and Y, conveniently chosen. The operator F is supposed to be decomposed as

$$F = L + R + N \tag{2.2}$$

where L is a linear and *easily* invertible operator, R is a linear operator and N is non linear one. Thus, from (2.1) and (2.2), we have

$$u = L^{-1}[g(x)] - L^{-1}[R(u)] - L^{-1}[N(u)]. \tag{2.3}$$

In addition, it is assumed that N can be written in the form

$$N[u(x)] = \sum_{n=0}^{+\infty} A_n[u(x)], \tag{2.4}$$

where the A_n's are known as *Adomian's polynomials*, although they are not always polynomials in the usual sense, regarding the independent variable x.
If an analytic solution u is supposed to exist and it is written

$$u(x) = \sum_{n=0}^{+\infty} u_n(x). \tag{2.5}$$

Assuming that - as we will show - each polynomial A_n can be built depending only on $u_0, u_1, \cdots u_n$, from (2.3) we can find a recursive formula for the terms in (2.5):

$$\begin{cases} u_0(x) = & L^{-1}[g(x)] \\ u_n(x) = & -L^{-1}[R(u_{n-1}(x)] - L^{-1}[A_{n-1}(u_0(x), \cdots, u_{n-1}(x)] \qquad \forall n \geq 1, \end{cases} \tag{2.6}$$

Initial or boundary conditions are taken into account when $u_0(x)$ is obtained.
In [4] G. Adomian asserted that the partial sums

$$\varphi_m = \sum_{n=0}^{m} u_n \tag{2.7}$$

are good approximations of the solution, and converge rapidly to it, but gave no formal proof of this statement. He also remarked that this method does not provide a numerical solution, but an approximate analytical one.

Up to here the description of the method is simple, but it is necessary to remark that the hypothesis on the operator F (that is, on L and N) are very restrictive and not easy to be verified in the general case. Nor is it easy to give a general proof of convergence. Also, it has to be noted that in most of the examples offered in the literature the existence of an analytical solution is supposed but not always justified.

Regarding the construction of the Adomian's polynomials A_n, Adomian himself and other authors described different ways to build them up. We refer to [2], [3] and [4] and the improvements proposed in [5], [6], [15] and [18]. In [11] and [12], supposing that N

is *an analytical operator in a proper Banach space*, the author considers an auxiliary parameter ε, stating that

$$N(u,\varepsilon) = N(\sum_{n=0}^{+\infty} \varepsilon^n u_n) = \sum_{n=0}^{+\infty} A_n(u)\varepsilon^n, \tag{2.8}$$

from which

$$A_n(u) = \frac{1}{n!}\frac{d^n}{d\varepsilon^n}[N(\sum_{k=0}^{+\infty} \varepsilon^k u_k)]|_{\varepsilon=0} = \frac{1}{n!}\frac{d^n}{d\varepsilon^n}[N(\sum_{k=0}^{n} \varepsilon^k u_k)]|_{\varepsilon=0} = A_n(u_0,\cdots,u_n) \tag{2.9}$$

which is a general formula for the Adomian polynomials when a unique function u is considered.

Since the 80's, several proofs of the convergence of the partial sums φ_m to the solution have been presented (see, for example, [7], [8], [11], [12], [13], [15] and [16]; we specially highlight [1]) - but some of them are incomplete or imprecise.
In [17] we proved that the approximations provided by ADM in the case of an autonomous problem with an ordinary differential equation of the form

$$\begin{cases} u'(x) = & f(u(x)) \\ u(0) = & u_0 \end{cases}$$

are *exactly* the Taylor polynomials of the solution, if L is chosen in (2.2) as the first derivative with respect to x. For this demonstration we used a formula published in 1855 by Faà di Bruno (see [14]) for the successive derivatives of a composition.
In Section 5 we will prove a similar result for PDE.

3. Examples with partial differential equations

Before proving the convergence of ADM for PDE, we illustrate the application of the method to partial differential equations.

Let us consider the following inital value problem:

$$\begin{cases} u_{tt}(x,t) &=& -2x^2u(u+2tu_t) \qquad \forall t>0 \\ u(x,0) &=& 1 \\ u_t(x,0) &=& 0 \end{cases} \tag{3.1}$$

With the notation of (2.1) and (2.2) we have $F[u] = \frac{\partial^2 u}{\partial t^2} + 2x^2u(u+2t\frac{\partial u}{\partial t})$ and $g(x,t)=0$.
We will choose $L[u] = \frac{\partial^2 u}{\partial t^2}$, $R[u]=0$ and $N[u] = 2x^2u(u+2t\frac{\partial u}{\partial t})$ (note that it is not the only possibility).
Recalling Cauchy-Kowalevskaya Theorem (see, for example, [10]) , the existence of an analytical solution $u(x,t)$ is guaranteed, and we can assume (2.5). Now, $N[u]$ in the form of (2.4) could be obtained by means of Cauchy product of the correponding series of u and its derivatives.
The inverse operator of L is a double integration with respect to t, which we will indicate $L^{-1}[\,.\,] = \int\int[\,.\,]dtdt$.

The recurrence formula (2.6) results, in this case,

$$\begin{cases} u_0(x,t) &= L^{-1}[0] = u_t(x,0).t + u(x,0)|_{t=0} = u(x,0) = 1 \\ u_n(x,t) &= -2x^2 \sum_{k=0}^{n-1} \int_0^t \{\int_0^s u_k(x,z)[u_{n-1-k}(x,z) + 2z\frac{\partial u_{n-1-k}(x,z)}{\partial z}]dz\}ds \qquad \forall n \geq 1 \end{cases}$$

from which, obtaining some terms, we can guess the general expression from regularity:

$$\begin{cases} u_0(x,t) &= 1 \\ u_n(x,t) &= (-1)^n t^{2n} x^{2n} \qquad \forall n \geq 1 \end{cases}$$

We propose

$$u(x,t) = \sum_{n=0}^{\infty} (-1)^n t^{2n} x^{2n} = \frac{1}{1+t^2x^2} \tag{3.2}$$

in $D = \{(x,t) \in \mathbb{R}^2 : |xt| < 1\}$.
Replacing (3.2) in (3.1) we verify *it is* the exact solution to the problem in the region D, and no proof of convergence is necessary.
In this example, ADM gives a closed formula for $u(x,t)$. Observe the aproximate solutions, that is the partial sums φ_n in (2.7), are the Taylor polynomials of $u(x,t)$.

Let us consider another inital value problem:

$$\begin{cases} u_t(t,x) &= u_x + 2tx(t-x)u^2 \\ u(0,x) &= 1 \end{cases} \tag{3.3}$$

With the notation of (2.1) and (2.2) we have $F[u] = \frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} - 2tx(t-x)u^2$ and $g(t,x) = 0$.
We will choose $L[u] = \frac{\partial u}{\partial t}$, $R[u] = -\frac{\partial u}{\partial x}$ and $N[u] = -2tx(t-x)u^2$.
Recalling again Cauchy-Kowalevskaya Theorem, the existence of an analytical solution $u(t,x)$ is guaranteed, an we can assume (2.5). $N[u]$ in the form of (2.4) could be obtained by means of (2.9). The inverse operator of L is the integration with respect to t: $L^{-1}[\,.\,] = \int [\,.\,]dt$.
The recurrence formula (2.6) results, in this case,

$$\begin{cases} u_0(t,x) &= L^{-1}[0] = u(0,x) = 1 \\ u_n(t,x) &= \int_0^t [u_{n-1}(z,x)]_x dz + 2x \int_0^t z(z-x)[A_{n-1}(u_1(z,x),...,u_{n-1}(z,x)]dz \qquad \forall n \geq 1 \end{cases}$$

from which we obtain the following terms:

$$\begin{aligned} u_0(t,x) &= 1 \\ u_1(t,x) &= \tfrac{2t^3x}{3} - t^2x^2 \\ u_2(t,x) &= \tfrac{t^4}{6} - \tfrac{2t^3x}{3} + \tfrac{4t^6x^2}{9} - \tfrac{4t^5x^3}{3} + t^4x^4 \\ u_3(t,x) &= -\tfrac{t^4}{6} + \tfrac{2t^7x}{9} - \tfrac{11t^6x^2}{9} + \tfrac{4t^5x^3}{3} + \tfrac{8t^9x^3}{27} - \tfrac{4t^8x^4}{3} + 2t^7x^5 - t^6x^6 \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

Although, at first sight, no general formula could be guessed in this example, in the following sections we will see how we can guarantee the convergence of the partial sums to the solution of the problem.

4. Adomian polynomials for several variables

In this section we deduce a general formula for the Adomian polynomials when the non linear operator N has two arguments: $N = N[u, v]$.
This expression will be useful to study PDEs in the general case.
We will suppose $u = \sum_{n=0}^{+\infty} u_n$ and $v = \sum_{n=0}^{+\infty} v_n$ and then obtain A_n for the expression

$$N(u, v) = \sum_{n=0}^{+\infty} A_n(u_0, u_1, u_2, ..., u_n, v_0, v_1, v_2, ..., v_n). \tag{4.1}$$

We will consider that N is an analytic function of u and v arround (u_0, v_0).
In this case the Taylor series for N is

$$N(u, v) = \sum_{n=0}^{+\infty} \frac{1}{n!} \sum_{i=0}^{n} \binom{n}{i} \frac{\partial^n N}{\partial u^i \partial v^{n-i}}(u_0, v_0)(u - u_0)^i (v - v_0)^{n-i}$$

so, replacing the series for u and v, we have

$$N(u, v) = \sum_{n=1}^{+\infty} \sum_{i=0}^{n} \frac{1}{i!(n-i)!} \frac{\partial^n N}{\partial u^i \partial v^{n-i}}(u_0, v_0)(\sum_{k=1}^{+\infty} u_k)^i (\sum_{j=1}^{+\infty} v_j)^{n-i} \tag{4.2}$$

The series $(\sum_{k=1}^{+\infty} u_k)^i$ is a sum over all the possible products between i factors u_k, with or without repetitions; in a similar way, $(\sum_{j=1}^{+\infty} v_j)^{n-i}$ is a sum over all the possible products between $n - i$ factors v_j. Counting all these products and reordering the terms of the series (4.2), we can obtain A_n gathering all the terms with grade less or equal to n, only in powers of $u_0, u_1, u_2, ..., u_n, v_0, v_1, v_2, ..., v_n$. The analyticity of N, u and v allow us to reorder the series without affecting their absolute convergence.
Similarly as in the case of one variable, it is $A_0 = N(u_0, v_0)$, while for $n \geq 1$ a possible reordering leads to

$$\begin{aligned} &A_n(u_0, u_1, u_2, ..., u_n, v_0, v_1, v_2, ..., v_n) = \\ &= \sum_{k=1}^{n} \sum_{i=0}^{k} \frac{\partial^k N}{\partial u^{k-i} \partial v^i}(u_0, v_0) \sum_{\substack{i_1, i_2, \dots, i_n \\ j_1, j_2, \dots, j_n}} \frac{(u_1)^{i_1}(u_2)^{i_2} \dots (u_n)^{i_n} (v_1)^{j_1}(v_2)^{j_2} \dots (v_n)^{j_n}}{i_1! i_2! \dots i_n! j_1! j_2! \dots j_n!} \end{aligned} \tag{4.3}$$

where the inner sum is over every solution of the system

$$\left\{ \begin{array}{rl} i_1 + 2i_2 + \cdots + ni_n + j_1 + 2j_2 + \cdots + nj_n = & n \\ i_1 + i_2 + \cdots + i_n = & k - i \\ j_1 + j_2 + \cdots + j_n = & i \end{array} \right. \tag{4.4}$$

with $i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_n \in \mathbb{N}_0$.
Adomian polynomials for three or more variables could be deduced in a similar way.
It is easy to verify that, replacing each u_k in (4.3) by $\varepsilon^k u_k$ and each v_j by $\varepsilon^j v_j$, we obtain $A_n(u, v, \varepsilon) = A_n(u, v)\varepsilon^n$, so, using the parameter ε to group the terms with the

same degree, we get

$$N(u,v,\varepsilon)=N(\sum_{k=0}^{+\infty}\varepsilon^k u_k,\sum_{j=0}^{+\infty}\varepsilon^j v_j)=\sum_{n=0}^{+\infty}A_n(u,v)\varepsilon^n$$

from where a formula for the polynomials is obtained:

$$A_n(u,v)=\frac{1}{n!}\frac{d^n}{d\varepsilon^n}[N(\sum_{k=0}^{+\infty}\varepsilon^k u_k,\sum_{j=0}^{+\infty}\varepsilon^j v_j)]|_{\varepsilon=0} \tag{4.5}$$

The derivative of order n is the derivative of the composition $N(u(\varepsilon),v(\varepsilon))$, which could be obtained by means of Faà di Bruno's formula in its multivariate version (see, for example, [9]):

$$\frac{d^n}{d\varepsilon^n}[N(u(\varepsilon),v(\varepsilon))]|_{\varepsilon=0}$$
$$=\sum_{k=1}^{n}\sum_{i=0}^{k}\sum_{\substack{i_1,i_2,\dots,i_n\\ j_1,j_2,\dots,j_n}}\frac{n!}{i_1!\dots i_n!j_1!\dots j_n!}\frac{\partial^k N}{\partial u^{k-i}\partial v^i}|_{(u(0),v(0))}(\frac{u'(0)}{1!})^{i_1}\dots(\frac{u^{(n)}(0)}{n!})^{i_n}(\frac{v'(0)}{1!})^{j_1}\dots(\frac{v^{(n)}(0)}{n!})^{j_n}$$

The inner sum is, again, over every solution of (4.4).
Considering $u(\varepsilon)=\sum_{k=0}^{+\infty}u_k\varepsilon^k$ and $v(\varepsilon)=\sum_{j=0}^{+\infty}v_j\varepsilon^j$, we arrive to $u_k=\frac{u^{(k)}}{k!}$ and $v_j=\frac{v^{(j)}}{j!}$ so, replacing this expression in (4.6) we obtain again (4.3).
In this case, the expression of N proposed in (4.1), as sum of Adomian polynomials, is nothing but a reordering of Taylor series of N arround (u_0,v_0).

5. Convergence of ADM for PDEs

Let us consider the following initial value problem:

$$\begin{cases} u_t(t,x)= & N[u,u_x] \quad \forall t>0, x\in\Omega \\ u(0,x)= & f(x) \end{cases} \tag{5.1}$$

where $0\in\Omega\subset\mathbb{R}$, initial data $f(x)$ is an analytical function arround $x=0$ and N is a non linear operator given by $N(u,v)$, an analytical function.
Under these hypothesis, Cauchy-Kowalevskaya Theorem guarantees the existence of a unique solution, analytical arround the origin.
With the notation of (2.1) and (2.2), we can choose $F[\,.\,]=-\frac{\partial[\,.\,]}{\partial t}+N[\,.\,]$, $g(\,.\,)\equiv 0$ and $L^{-1}[\,.\,]=\int dt$.
If $u(x,t)=\sum_{n=0}^{+\infty}u_n(x,t)$, the recurrence formula (2.6) leads to

$$\begin{cases} u_0(t,x)= & L^{-1}[0]=f(x) \\ u_n(t,x)= & L^{-1}[A_{n-1}(u_0,\cdots,u_{n-1},u_{0_x},\cdots,u_{n-1_x})] \qquad \forall n\geq 1, \end{cases} \tag{5.2}$$

where the Adomian polynomials A_n are obtained by means of (4.3) or, equivalently, by (4.5).

Some of the terms are

$$\begin{aligned}
u_0 = &\quad f\\
A_0[u_0, u_{0_x}] = &\quad N\\
u_1 = &\quad \int_0^t A_0[u_0, u_{0_x}]ds = tN\\
A_1[u_0, u_1, u_{0_x}, u_{1_x}] = &\quad u_1 N_{10} + u_{1_x} N_{01} =\\
= &\quad t[NN_{10} + N_{10}N_{01}f' + N_{01}^2 f'']
\end{aligned}$$

$$\begin{aligned}
u_2 = &\quad \int_0^t A_1[u_0, u_{0_x}, u_1, u_{1_x}]ds =\\
= &\quad \tfrac{t^2}{2}[NN_{10} + N_{10}N_{01}f' + N_{01}^2 f'']\\
A_2[u_0, u_1, u_2, u_{0_x}, u_{1_x}, u_{2_x}] = &\quad \tfrac{1}{2}N_{20}u_1^2 + N_{11}u_{1_x}u_1 + N_{10}u_2 + \tfrac{1}{2}N_{02}u_{1_x}^2 + N_{01}u_{2_x} =\\
= &\quad \tfrac{t^2}{2}[N_{01}^3 f''' + 3N_{01}^2 N_{02} f''^2 + 3N_{10}N_{01}^2 f'' +\\
&\quad + N_{10}N_{01}N_{11}f'f'' + NN_{10}^2 + N_{10}^2 N_{01} f' +\\
&\quad + N_{10}^2 N_{02} f'^2 + 3NN_{01}N_{11}f'' + 3N_{01}^2 N_{11} f'f'' +\\
&\quad + 2NN_{10}N_{11}f' + N_{10}N_{01}N_{11}f'^2 + N^2 N_{20} +\\
&\quad + NN_{01}N_{20}f' + N_{01}^2 N_{20} f'^2]\\
u_3 = &\quad \int_0^t A_2[u_0, u_{0_x}, u_1, u_{1_x}, u_2, u_{2_x}]ds =\\
= &\quad \tfrac{t^3}{3!}[N_{01}^3 f''' + 3N_{01}^2 N_{02} f''^2 + 3N_{10}N_{01}^2 f'' +\\
&\quad + N_{10}N_{01}N_{11}f'f'' + NN_{10}^2 + N_{10}^2 N_{01} f' +\\
&\quad + N_{10}^2 N_{02} f'^2 + 3NN_{01}N_{11}f'' + 3N_{01}^2 N_{11} f'f'' +\\
&\quad + 2NN_{10}N_{11}f' + N_{10}N_{01}N_{11}f'^2 + N^2 N_{20} +\\
&\quad + NN_{01}N_{20}f' + N_{01}^2 N_{20} f'^2]
\end{aligned}$$

For simplicity, we omited the variables t and x from each u_k and its derivatives, and the variable x from f and its derivatives; N_{ij} is the derivative of N, i times with respect to its first variable and j times with respect to its second variable; N and every derivative of N are evaluated in $(f(x), f'(x))$.

As the solution $u(x,t)$ is analytical arround the origin,

$$u(x,t) = \sum_{n=0}^{+\infty} \sum_{\ell=0}^{n} \frac{\partial^n u}{\partial x^\ell \partial t^{n-\ell}}(0,0) \frac{x^\ell}{\ell!} \frac{t^{n-\ell}}{(n-\ell)!}$$

We can reorder this series because of the absolute convergence:

$$u(x,t) = \sum_{n=0}^{+\infty} [\sum_{j=0}^{n} \frac{\partial^{(j+n)} u}{\partial x^j \partial t^n}(0,0) \frac{x^j}{j!}] \frac{t^n}{n!}$$

Comparing this expression with the terms obtained from (5.2) we can state the following result:

Proposition 5.1. The terms $u_n(x,t)$ for the problem (5.1), generated by the recurrence formula (5.2) based on the Adomian polynomials A_n, verify

$$u_n(x,t) = \frac{t^n}{n!} \sum_{j=0}^{n} \frac{\partial^{(j+n)} u}{\partial x^j \partial t^n}(0,0) \frac{x^j}{j!},$$

and, consequently, the ADM approximations $\varphi_m = \sum\limits_{n=0}^{m} u_n(x,t)$ are partial sums of a reordering of the Taylor series arround the origin of the analytical solution of the problem.

Proof:

As the Taylor series is unique, we can state

$$\frac{t^n}{n!} \sum_{j=0}^{n} \frac{\partial^{(j+n)} u}{\partial x^j \partial t^n}(0,0) \frac{x^j}{j!} = \frac{t^n}{n!} \frac{\partial^n u}{\partial t^n}(x,0)$$

so we will prove, by mathematical induction, that

$$u_n(x,t) = \frac{t^n}{n!} \frac{\partial^n u}{\partial t^n}(x,0) \tag{5.3}$$

For $n = 0$ (5.3) is true because $u_0(x,t) = f(x) = u(x,0)$, initial condition of the problem.

Now we will suppose it is true for $k \leq n$ and we will prove the assertion for $n+1$.

From the differential equation in (5.1) we have

$$\frac{\partial^{(n+1)} u}{\partial t^{n+1}}(x,0) = \frac{\partial^n (N[u(x,t), u_x(x,t)])}{\partial t^n}|_{t=0}$$

Then we will prove that

$$\begin{aligned} u_{n+1}(x,t) = & \int_0^t [A_n(u_0(x,s), \cdots, u_n(x,s), u_{0_x}(x,s), \cdots, u_{n_x}(x,s))] ds = \\ = & \tfrac{t^{n+1}}{(n+1)!} \tfrac{\partial^n (N[u(x,t), u_x(x,t)])}{\partial t^n}|_{t=0} \end{aligned}$$

By (4.3) it is

$$
\begin{aligned}
&A_n(u_0,u_1,u_2,...,u_n,u_{0_x},u_{1_x},u_{2_x},...,u_{n_x})\\
&=\sum_{k=1}^{n}\sum_{i=0}^{k}N_{(k-i)i}\sum_{\substack{i_1,i_2,\dots,i_n\\ j_1,j_2,\dots,j_n}}\frac{(u_1)^{i_1}(u_2)^{i_2}\dots(u_n)^{i_n}(u_{1_x})^{j_1}(u_{2_x})^{j_2}\dots(u_{n_x})^{j_n}}{i_1!i_2!\dots i_n!j_1!j_2!\dots j_n!}
\end{aligned}
$$

where $N_{(k-i)i}$ is the derivative of N, $k-i$ times with respect to its first variable and i times with respect to its second variable, each of them evaluated in $[u_0,u_{0_x}]=[u(0,x),u_x(0,x)]$. The inner sum is over every solution of

$$
\begin{cases}
i_1+2i_2+\cdots+ni_n+j_1+2j_2+\cdots+nj_n= & n\\
i_1+i_2+\cdots+i_n= & k-i\\
j_1+j_2+\cdots+j_n= & i
\end{cases}
$$

with $i_1,i_2,\dots,i_n,j_1,j_2,\dots,j_n\in\mathbb{N}_0$.

By inductive hypothesis it is

$$
\frac{\partial^k u}{\partial t^k}(x,0)=\frac{k!}{t^k}u_k(x,t) \tag{5.4}
$$

for every $k\le n$, then

$$
\begin{aligned}
&A_n(u_0,u_1,u_2,...,u_n,u_{0_x},u_{1_x},u_{2_x},...,u_{n_x})\\
=\;&\sum_{k=1}^{n}\sum_{i=0}^{k}N_{(k-i)i}\sum_{\substack{i_1,i_2,\dots,i_n\\ j_1,j_2,\dots,j_n}}\frac{(\frac{t}{1!}\frac{\partial u}{\partial t})^{i_1}\dots(\frac{t^n}{n!}\frac{\partial^n u}{\partial t^n})^{i_n}(\frac{t}{1!}\frac{\partial u_x}{\partial t})^{j_1}\dots(\frac{t^n}{n!}\frac{\partial^n u_x}{\partial t^n})^{j_n}}{i_1!i_2!\dots i_n!j_1!j_2!\dots j_n!}=\\
=\;&\sum_{k=1}^{n}\sum_{i=0}^{k}\sum_{\substack{i_1,\dots,i_n\\ j_1,\dots,j_n}}\frac{t^{(i_1+2i_2+\dots+ni_n+j_1+2j_2+\dots nj_n)}}{i_1!i_2!\dots i_n!j_1!j_2!\dots j_n!}N_{(k-i)i}\Big(\frac{\frac{\partial u}{\partial t}}{1!}\Big)^{i_1}\dots\Big(\frac{\frac{\partial^n u}{\partial t^n}}{n!}\Big)^{i_n}\Big(\frac{\frac{\partial^2 u}{\partial t\partial x}}{1!}\Big)^{j_1}\dots\Big(\frac{\frac{\partial^{n+1} u}{\partial t^n\partial x}}{n!}\Big)^{j_n}\\
=\;&\frac{t^n}{n!}\sum_{k=1}^{n}\sum_{i=0}^{k}\sum_{\substack{i_1,\dots,i_n\\ j_1,\dots,j_n}}\frac{n!}{i_1!i_2!\dots i_n!j_1!j_2!\dots j_n!}N_{(k-i)i}\Big(\frac{\frac{\partial u}{\partial t}}{1!}\Big)^{i_1}\dots\Big(\frac{\frac{\partial^n u}{\partial t^n}}{n!}\Big)^{i_n}\Big(\frac{\frac{\partial^2 u}{\partial t\partial x}}{1!}\Big)^{j_1}\dots\Big(\frac{\frac{\partial^{n+1} u}{\partial t^n\partial x}}{n!}\Big)^{j_n}
\end{aligned}
$$

where the derivatives of u, following (5.4), *are evaluated in* $(x,0)$.

From this last expression we have

$$
\begin{aligned}
&u_{n+1}(x,t)=\int_0^t[A_n(u_0(x,s),u_1(x,s),...,u_n(x,s),u_{0_x}(x,s)...,u_{n_x}(x,s))]ds=\\
=\;&\int_0^t\frac{s^n}{n!}\Big[\sum_{k=1}^{n}\sum_{i=0}^{k}\sum_{\substack{i_1,\dots,i_n\\ j_1,\dots,j_n}}\frac{n!}{i_1!\dots i_n!j_1!\dots j_n!}N_{(k-i)i}\Big(\frac{\frac{\partial u}{\partial t}}{1!}\Big)^{i_1}\dots\Big(\frac{\frac{\partial^n u}{\partial t^n}}{n!}\Big)^{i_n}\Big(\frac{\frac{\partial^2 u}{\partial t\partial x}}{1!}\Big)^{j_1}\dots\Big(\frac{\frac{\partial^{n+1} u}{\partial t^n\partial x}}{n!}\Big)^{j_n}\Big]ds=\\
=\;&\frac{t^{n+1}}{(n+1)!}\sum_{k=1}^{n}\sum_{i=0}^{k}\sum_{\substack{i_1,\dots,i_n\\ j_1\dots,j_n}}\frac{n!}{i_1!\dots i_n!j_1!\dots j_n!}N_{(k-i)i}\Big(\frac{\frac{\partial u}{\partial t}}{1!}\Big)^{i_1}\dots\Big(\frac{\frac{\partial^n u}{\partial t^n}}{n!}\Big)^{i_n}\Big(\frac{\frac{\partial^2 u}{\partial t\partial x}}{1!}\Big)^{j_1}\dots\Big(\frac{\frac{\partial^{n+1} u}{\partial t^n\partial x}}{n!}\Big)^{j_n}
\end{aligned} \tag{5.5}
$$

Recalling once more Faà di Bruno's formula for multiple variables (see [9]) we have

$$
\begin{aligned}
&\frac{\partial^n(N[u(x,t),u_x(x,t)])}{\partial t^n}\Big|_{t=0}=\\
=\;&\sum_{k=1}^{n}\sum_{i=0}^{k}\sum_{\substack{i_1,i_2,\dots,i_n\\ j_1,j_2,\dots,j_n}}\frac{n!}{i_1!\dots i_n!j_1!\dots j_n!}N_{(k-i)i}\Big(\frac{\frac{\partial u}{\partial t}}{1!}\Big)^{i_1}\dots\Big(\frac{\frac{\partial^n u}{\partial t^n}}{n!}\Big)^{i_n}\Big(\frac{\frac{\partial^2 u}{\partial t\partial x}}{1!}\Big)^{j_1}\dots\Big(\frac{\frac{\partial^{n+1} u}{\partial t^n\partial x}}{n!}\Big)^{j_n}
\end{aligned} \tag{5.6}
$$

where $N_{(k-i)i}$ is evaluated in $[u(0,x), u_x(0,x)]$ and the derivatives of u are evaluated in $(x,0)$.
From (5.6) in (5.5) we have

$$\begin{aligned} u_{n+1}(x,t) = & \int_0^t [A_n(u_0(x,s),\cdots,u_n(x,s),u_{0_x}(x,s),\cdots,u_{n_x}(x,s))]ds = \\ = & \frac{t^{n+1}}{(n+1)!}\frac{\partial^n(N[u(x,t),u_x(x,t)])}{\partial t^n}|_{t=0} \end{aligned}$$

as we wanted to prove. □

Corollary 5.2. The partial sums $\varphi_m = \sum_{n=0}^{m} u_n(x,t)$, generated by ADM recurrence formula (5.2) for the problem (5.1), converge to the solution of the problem.

Proof: as we saw in Proposition 5.1, the approximations are just partial sums of reorderings of Taylor series of the solution, so the convergence is proved. □

Note that we have demonstrated the convergence of ADM for a particular case of PDE with the only hypothesis for N of being analytic *as a function*, and not - as it is usual in the literature - as a functional operator, which is more difficult to be verified.

6. A more general case

We can extend the statement of Proposition 5.1 to more general problems:

$$\begin{cases} \partial_t^k u = N[(\partial_x^\alpha \partial_t^j u)_{|\alpha|+j\leq k, 0\leq j<k}]) \\ \partial_t^j u(x,0) = f_j(x) \qquad 0\leq j<k \end{cases}$$

with N and f_j analytical functions, for which Cauchy-Kowalevskaya guarantees the existence of a unique analytical solution, could be written like (5.1) by means of a change of variables.
Let us consider, for example, the second order problem

$$\begin{cases} u_{tt}(t,x) = & N[u,u_x,u_t,u_{xt},u_{xx}] \quad \forall t>0, x\in\Omega \\ u(0,x) = & f_1(x) \\ u_t(0,x) = & f_2(x), \end{cases}$$

If we replace u by u_1, u_x by u_2 and u_t by u_3 we have

$$\begin{cases} u_{1_t} = & u_3 \\ u_{2_t} = & u_{3_x} \\ u_{3_t} = & N[u_1,u_2,u_3,u_{3_x},u_{2_x}] \\ u_1(0,x) = & f_1(x) \\ u_2(0,x) = & f_1'(x) \\ u_3(0,x) = & f_2(x) \end{cases} \tag{6.1}$$

By replacing (u_1,u_2,u_3) by $\mathbf{u}$, $(N_1,N_2,N_3) = (u_3,u_{3_x},N[u_1,u_2,u_3,u_{3_x},u_{2_x}])$ by $\mathbf{N}[\mathbf{u},\mathbf{u}_x]$ and $(f_1(x),f_1'(x),f_2(x))$ by $\mathbf{f}(x)$, we can write (6.1) as in (5.1).
Simillarly for problems with PDEs of greater order, and problems for which N explicitly depends on t and x.
For example:

$$\begin{cases} u_t(t,x) = & N[t,x,u,u_x] \quad \forall t>0, x\in\Omega \\ u(0,x) = & f(x) \end{cases}$$

could be written as (5.1) if we replace $(u_1, u_2, u_3) = (u, t, x)$ by $\mathbf{u}$, $(f(x), 0, x)$ by $\mathbf{f}$ and $(N[u_1, u_2, u_3, u_{1_x}], 0, 1)$ by $\mathbf{N}$.

7. Conclusions

Adomian's Decomposition Method is a tool introduced in the '80s for solving general nonlinear equations. It provides analytical approximate solutions for a wide range of problems. Trying to formulate a general framework for the method, several proofs of convergence of the approximations to the solution have been published, with varying degree of success. A general demonstration of convergence is difficult, precisely because of the wide range of problems it covers.
In this work we proved that, under appropiate conditions, the approximate solutions to partial differential equations provided by ADM are merely partial sums of reorderings of the Taylor series of the solution, as G. Adomian had posited without formally proving it. Consequently, no additional demonstration of convergence is necessary in those cases. As in [17], we *dusted off* an old formula from Faà di Bruno to build up the proof.
Note we have only required N to be analytic *as a function*, and not, as usual, as a functional operator, which is much more difficult to be verified.

References

[1] A. Abdelrazec and D. Pelinovsky, "Convergence of the Adomian Decomposition Method for Initial-Value Problems", *Numerical Methods for Partial Differential Equations* 27(4), pp. 749-766, 2011.
[2] G. Adomian and G. E. Adomian, "A Global Method for Solution of Complex Systems", *Mathematical Modelling* 5, pp. 251-263, 1984.
[3] G. Adomian, "A Review of the Decomposition Method and Some Recent Results for Nonlinear Equations", *Mathematical and Computer Modelling* 13(7), pp. 17-43, 1990.
[4] G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*. Dordrecht: Kluwer Academic Publishers, 1994.
[5] E. Babolian and Sh. Javadi "New method for calculating Adomian Polynomials", *Applied Mathematics and Computation* 153, pp. 253-259, 2004.
[6] J. Biazar and S. M. Shafiof, "A Simple Algorithm for Calculating Adomian Polynomials",*Int. J. Contemp. Math. Sciences* 2(29), pp. 975-982, 2007.
[7] Y. Cherruault, "Sur la convergence de la méthode d'Adomian", *Revue française d'automatique, d'informatique et de recherche opérationnelle* 22(3), pp. 291-299, 1988.
[8] Y. Cherruault, "Convergence of Adomian?s Method", *Kybernetes* 18(2), pp. 31-38, 1989.
[9] G. Constantine and T. Savits, "A multivariate Faà di Bruno formula with applications", *Transactions of the American Mathematical Society* 348(2), pp. 503-520, 1996.
[10] G. Folland, *Introduction to partial differential equations*. N.J. : Princeton Univ. Press, 1995.
[11] L. Gabet, "The Theoretical Foundation of Adomian Method", *Computers and Mathematics with Applications* 27(12), pp. 41-52, 1994.
[12] L. Gabet, "Esquisse d'une Théorie Décompositionnelle", *Modélisation Mathématique et Analyse Numérique* 30(7), pp. 799-814, 1996.
[13] N. Himoun, K. Abbaoui and Y. Cherruault,"New results on Adomian Method", *Kybernetes* 32(4), pp. 523-539, 2003.
[14] W. P. Johnson,"The Curious History of Faà di Bruno's Formula", *American Mathematical Monthly* 109 (3), pp. 217-234, 2002.
[15] T. Mavoungou and Y. Cherruault, "Convergence of Adomian?s Method and Applications to Non-linear Partial Differential Equations", *Kybernetes* 21(6), pp. 13-25, 1992.

[16] A. Rèpaci, “Nonlinear dynamical systems: on the accuracy of Adomian?s decomposition method”, *Appl. Math. Lett.*, 3 (4), pp. 35-39, 1990.
[17] S. Seminara and M. I. Troparevsky, “Some Remarks on Adomian Decomposition Method”, *Poincare Journal of Analysis and Applications*, 2014(2), pp. 63-70, 2014.
[18] A. M. Wazwaz, “A new algorithm for calculating Adomian polynomials for non- linear operators”, *Appl. Math. Comp.* 111, pp. 53-69, 2000.

(Silvia Seminara) Dep. de Matemática - Facultad de Ingeniería, Universidad de Buenos Aires - Argentina
E-mail address: seminarasilvia@gmail.com

(María Inés Troparevsky) Dep. de Matemática - Facultad de Ingeniería, Universidad de Buenos Aires - Argentina
E-mail address: mitropa@fi.uba.ar