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# ON $Q$-ALGEBRAS AND SPECTRAL ALGEBRAS

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***Abstract***. T. Palmer gave the definition of a spectral seminorm and some equivalent statements for the seminormed algebras to be spectral. Some of these statements have been studied in the locally $m$-convex case and in the locally pseudoconvex case. In this paper we give some analogous results for the locally $m$-pseudoconvex algebras. Moreover, different proofs for some known results are given.

## 1. Introduction

In 1947 I. Kaplansky introduced in [9] the notion of $Q$-algebra (more exactly of $Q$-ring), that is a topological algebra $E$ in which the set $Qinv(E)$ of quasi-invertible elements of $E$ is open. Examples of $Q$-algebras can be found in [5] and [6]. It is well known that every Banach algebra is also a $Q$-algebra. The importance of $Q$-algebras has been given, for example in [7], [8],[10]. Nowadays $Q$-algebras are involved in modern differential geometry (see [12, Chapter XI]) and play a significant role in PDE's theory, operator theory, and the $\Psi^*$-quantization (see [7, p. 73]).

In 1987 V.Mascioni [13] gave several equivalent conditions for a unital complex normed algebra to be a $Q$-algebra. Later on T. W. Palmer introduced in [14, p. 294] the notion of a *spectral algebra* as an algebra $E$ in which there exists a seminorm (*spectral seminorm*) $p$ such that $r(x) \leqslant p(x)$ for each $x \in E$, here $r(x)$ denotes the spectral radius of $x$. He gave there analogous results to V. Mascioni for seminormed algebras and M. Fragoulopoulou in [6, p. 85, Theorem 3.1] for locally $m$-convex algebras.

Several characterizations of locally $m$-pseudoconvex $Q$-algebras are given in the present paper.

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## 2. Preliminaries

Let $E$ be a unital topological algebra over the field of complex numbers $\mathbb{C}$, with separately continuous multiplication (in short, a *topological algebra*). If the underlying topological linear space of $E$ is locally pseudoconvex, then $E$ is called a *locally pseudoconvex algebra*. In this case its topology can be defined by a family $\mathcal{P} = \{p_\alpha : \alpha \in \Lambda\}$ of non-homogeneous seminorms (a seminorm $p$ is a non-homogeneous if there exists a number $0 < \rho \leqslant 1$, (the homogenity index of $p$) such that $p(\lambda x) = |\lambda|^\rho p(x)$ for each $x \in E$ and $\lambda \in \mathbb{C}$). A non-homogeneous seminorm $p$ with homogeneous index $\rho$ is called a *$\rho$-seminorm.* When every $p_\alpha$ in the family $\mathcal{P}$ satisfies the condition

$$p_\alpha(xy) \leqslant p_\alpha(x)p_\alpha(y) \text{ for each } \alpha \in \Lambda,$$

$E$ is called *locally $m$-pseudoconvex.*

If $E$ is an algebra over $\mathbb{C}$, the spectrum $\sigma(x)$ of $x$ is given by

$$\sigma(x) = \{\lambda \in \mathbb{C} : -\lambda^{-1}x \notin Qinv(E)\} \cup \{0\}$$

(when $E$ is unital algebra, then by

$$\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda e \text{ is not invertible}\})$$

and the *spectral radius* of $x$ is defined by

$$r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}.$$

**Definition 2.1.** A *spectral seminorm* on an algebra $E$ is a submultiplicative seminorm $p$ such that

$$r(x) \leqslant p(x) \text{ for every } x \in E,$$

If a spectral seminorm is a norm, then it is called a *spectral norm.* An algebra $E$ with a spectral seminorm (spectral norm) is called a *spectral seminormed algebra* (*spectral normed algebra*) and $E$ is called *spectral* if some spectral seminorm can be defined on it.

In the next theorem, due to Mascioni, we present some equivalent conditions for complex normed algebras with unit to be a $Q$-algebra (see [13, p. 11]).

**Theorem 2.2.** *Let $(E, \|\cdot\|)$ be a complex normed algebra with unit $e$. Then the following conditions are equivalent*[1]

1) *$E$ is a $Q$-algebra.*
2) *If $x \in E$ and $\|e - x\| < 1$, then $x \in Inv(E)$*[2].
3) *If $x \in E$ and $\|x\| < 1$, then $\sum x^n$ converges in $E$.*
4) $r(x) = \lim\limits_n \|x^n\|^{\frac{1}{n}} = \inf\limits_n \|x^n\|^{\frac{1}{n}}$ *for every $x \in E$.*
5) $\sup\limits_{\|x\|=1} r(x) < \infty$.
6) $r(x) \leqslant \|x\|$ *for every $x \in E$.*

There is an analogous result to Theorem 2.2 in the spectral algebra context ([14, Theorem 3.1], [15, p. 212, Theorem 2.2.5]). For the sake of completeness, we give it here.

**Theorem 2.3.** *The following conditions are equivalent for a submultiplicative seminorm $p$ on an algebra $E$.*

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[1] Here, and later on, $r(x)$ denotes the spectral radius of $x$.

[2] $Inv(E)$ denotes the set of invertible elements of $E$

1) *$p$ is a spectral seminorm.*
2) *For $x \in E$, $p(x) < 1$ implies that $x$ is quasi-invertible in $E$.*
3) *The set of quasi-invertible elements of $E$ has nonempty interior with respect to $p$.*
4) *The set of quasi-invertible elements of $E$ is open with respect to $p$, ($E$ is a $Q$-algebra).*
5) *There is a positive number $C$ satisfying*
   $$r(x) \leqslant Cp(x) \text{ for every } x \in E.$$
6) $r(x) = \lim_{n\to\infty} p(x^n)^{\frac{1}{n}}$*, for every $x \in E$.*

From the previous theorem follows that a submultiplicative seminorm $p$ on an algebra $E$ is a spectral seminorm if and only if $E$ is a $Q$-algebra with respect to $p$.

**Remark 2.4.** We can consider only the case of a topological algebra $E$ with unit $e$, since by [11, p. 48]

$$x \circ y = 0 \Leftrightarrow (e - x)(e - y) = e \text{ for every } x, y \in E.$$

Then, $Inv(E)$, the set of invertible elements in $E$, is equal to $\{e\} - Qinv(E)$. From that, $Inv(E)$ is open if and only if $Qinv(E)$ is open.

Some of the equivalences in the next result are already known, (see [5, p. 333, Proposition 7.4.20]). We prove them for the sake of completeness and give an alternative proof to 4) implies 5).

**Theorem 2.5.** *The following conditions are equivalent for a submultiplicative $\rho$-seminorm $p$ on an algebra $E$.*

1) *$p$ is a $\rho$-spectral seminorm*[3].
2) *For $x \in E$, $p(x) < 1$ implies that $x$ is quasi-invertible in $E$.*
3) *The set of quasi-invertible elements of $E$ has nonempty interior with respect to $p$.*
4) *The set of quasi-invertible elements of $E$ is open with respect to $p$, ($E$ is a $Q$-algebra).*
5) *There is a positive number $C$ such that*
   $$r(x)^\rho \leqslant Cp(x) \text{ for every } x \in E.$$
6) $r(x)^\rho = \lim_{n\to\infty} p(x^n)^{\frac{1}{n}}$*, for every $x \in E$.*

*Proof.* The proofs of 1) $\Rightarrow$ 2), 2) $\Rightarrow$ 3) and 3) $\Rightarrow$ 4) are analogous to the proof of ([15, p. 212, Theorem 2.2.5].

4) $\Rightarrow$ 5) Since $\theta_E$, the zero element of $E$, is quasi-invertible, there exists a $\varepsilon > 0$ such that $p(x) < \varepsilon$ implies that $x$ is quasi-invertible. If $|\lambda| > 1$, then $|\lambda|^\rho > 1$ and

$$p(\lambda^{-1}x) = |\lambda|^{-\rho} p(x) < \varepsilon$$

which implies that $\lambda^{-1}x$ is quasi-invertible. Hence, $r(x) \leqslant 1$ and it implies that $r(x)^\rho \leqslant 1$.

[3] A $\rho$-seminorm $p$ on an algebra $E$ is called $\rho$-spectral if $r(x)^\rho \leqslant p(x)$ for every $x \in E$.

We consider now $x \in E$, such that $p(x) \neq 0$ and

$$a = \frac{x}{\left(\frac{2}{\varepsilon}\right)^{\frac{-1}{\rho}} p(x)^{\frac{1}{\rho}}}.$$

Then,

$$p(a) = \frac{p(x)}{\left(\frac{2}{\varepsilon}\right)^{-1} p(x)} = \frac{\varepsilon}{2} < \varepsilon$$

and hence $r(a)^\rho \leqslant 1$. Now, as $x = \left(\frac{\varepsilon}{2}\right)^{\frac{1}{\rho}} p(x)^{\frac{1}{\rho}} a$, then

$$r(x)^\rho = \frac{\varepsilon}{2} p(x) r(a)^\rho.$$

We conclude that $r(x)^\rho \leqslant \frac{\varepsilon}{2} p(x)$ for every $x \in E$ such that $p(x) \neq 0$.

If $x \in E$ and $p(x) = 0$, then $0 = p(mx)$ for every $m > 0$. Let $M = \sup\limits_{p(z) \leqslant 1} r(z)$ for $z \in E$; since $mr(x) = r(mx) \leqslant M$, we conclude that $r(x) \leqslant \frac{M}{m}$ for every $m > 0$; this means $r(x) = 0$.

Thus, 5) holds.

5) $\Rightarrow$ 6) Since $r(x^n) = r(x)^n$, then

$$r(x)^\rho = (r(x^n)^{\frac{1}{n}})^\rho = (r(x^n)^\rho)^{\frac{1}{n}} \leqslant (Cp(x^n))^{\frac{1}{n}}$$

and it follows that $r^\rho(x) \leqslant \lim\limits_{n \to \infty} p(x^n)^{\frac{1}{n}}$. To prove the converse inequality, we use the following result (see [5, p. 326, Theorem 7.4.6]):

If $E$ is a $\rho$-Banach algebra, then $r(x)^\rho = \lim\limits_{n \to \infty} p(x^n)^{\frac{1}{n}}$.

We consider now $\overline{E}$, the completion of $E$, the inclusion $\tau : E \to \overline{E}$, $\overline{\sigma}(\tau(x))$, the spectrum of $\tau(x)$ in $\overline{E}$ and $\overline{r}(\tau(x))$ the spectral radius of $\tau(x)$ in $\overline{E}$. Then $\overline{E}$ is a $\rho$-Banach algebra and since $\overline{\sigma}(\tau(x)) \subset \sigma(x)$, we have

$$\lim_{n \to \infty} p((\tau(x))^n)^{\frac{1}{n}} = \overline{r}(\tau(x))^\rho \leqslant r(x)^\rho.$$

6) $\Rightarrow$ 1) Since $p$ is submultiplicative, then $r(x)^\rho \leqslant p(x)$. □

## 3. On locally $m$-pseudoconvex $Q$-algebras

We shall give an analogous result to Theorem 2.2, in the case of a locally $m$-pseudoconvex algebra.

Let $(E, \{p_\alpha : \alpha \in \Lambda\})$ be a locally $m$-pseudoconvex algebra with unit. Let us denote by $\tau$ the topology induced on $E$ by this family and by $\tau_{p_\alpha}$ the one induced only by $p_\alpha$ on $E$. For each $\alpha \in \Lambda$, let $E_\alpha = E/kerp_\alpha$ and define on $E_\alpha$ the norm $p'_\alpha$ by $p'_\alpha(\overline{x}) = p_\alpha(x)$, where $\overline{x} = x + kerp_\alpha$ and denote by $\pi_\alpha : E \to E_\alpha$, the quotient map.

Remember that if $E$ is an unital topological algebra, a net $(x_\lambda)_{\lambda \in \Lambda}$ is called *invertibly convergent* if there exists $x \in E$ such that $(xx_\lambda)_{\lambda \in \Lambda}$ and $(x_\lambda x)_{\lambda \in \Lambda}$ converges to $e$. An algebra is *invertive* if every invertibly convergent sequence is convergent.

**Theorem 3.1.** *Let* $(E, \{p_\alpha : \alpha \in \Lambda\})$ *be a (complex) locally m-pseudoconvex algebra with unit e. Then the conditions* 1), 4), 5) *and* 6) *are equivalent,* 3) *implies* 2) *and* 2) *implies* 1)*:*

1) *E is a Q-algebra;*
2) $\exists\ \alpha \in \Lambda$ *such that* $p_\alpha(e - x) < 1 \Rightarrow x \in Inv(E)$*;*
3) $\exists\ \alpha \in \Lambda$ *such that* $p_\alpha(x) < 1 \Rightarrow \sum_{n=0}^{\infty} x^n$ *converges in E;*
4) $\exists\ \alpha \in \Lambda$ *such that* $r(x)^{\rho_\alpha} = \lim_n\ p_\alpha(x^n)^{\frac{1}{n}} = \inf_n\ p_\alpha(x^n)^{\frac{1}{n}}$ *for every* $x \in E$*;*
5) $\exists\ \alpha \in \Lambda$ *such that* $\sup_{p_\alpha(x) \leqslant 1} r(x) < \infty$*;*
6) $\exists\ \alpha \in \Lambda$ *such that* $r(x)^{\rho_\alpha} \leqslant p_\alpha(x)$ *for all* $x \in E$*, that is,* $(E, p_\alpha)$ *is a* $\rho_\alpha$*-spectral algebra.*

*Moreover, if E is invertive m-(ρ-convex) (this is a locally m-pseudoconvex algebra with the same non homogenity indexes ρ of all pseudoseminorms), then we have that* 4) *implies* 3)*. In this case the conditions* 1), 2), 3), 4), 5) *and* 6) *are equivalent.*

*Proof.* 1) $\Rightarrow$ 5) Recall that a local subbase of the origin is given by the sets

$$\{x \in E : p_\alpha(x) < \delta\}, \quad \alpha \in \Lambda, \delta > 0,$$

see [5, p. 195, Lemma 4.4.2]. So, since $e \in Inv(E)$ and $Inv(E)$ is an open set by hypothesis, then[4] there exists an $\alpha \in \Lambda$ and $\delta > 0$ such that if $p_\alpha(x - e) < \delta$, then $x \in Inv(E)$. We consider $\lambda \in \mathbb{C}$ such that $|\lambda| > \left(\frac{p_\alpha(x)}{\delta}\right)^{\frac{1}{\rho_\alpha}}$. Then

$$p_\alpha\left(\left(e - \frac{x}{\lambda}\right) - e\right) = p_\alpha\left(\frac{x}{\lambda}\right) = \frac{1}{|\lambda|^{\rho_\alpha}} p_\alpha(x) < \delta,$$

and by hypothesis, $e - \frac{x}{\lambda} \in Inv(E)$, which implies that $\lambda e - x \in Inv(E)$. Hence $r(x) \leqslant \left(\frac{p_\alpha(x)}{\delta}\right)^{\frac{1}{\rho_\alpha}}$, so $r(x) \leqslant \left(\frac{1}{\delta}\right)^{\frac{1}{\rho_\alpha}}$, if $p_\alpha(x) \leqslant 1$.

5) $\Rightarrow$ 6) Let $x \in E$, $x \neq \theta$ and $\alpha \in \Lambda$ be such that

$$M_\alpha = \sup_{p_\alpha(x) \leqslant 1} r(x) < \infty.$$

If $p_\alpha(x) = 0$, then $0 = p_\alpha(mx)$ for every $m > 0$. Since $mr(x) = r(mx) \leqslant M_\alpha$, we conclude that $r(x) \leqslant \frac{M_\alpha}{m}$ for every $m > 0$. This means that $r(x) = 0$. Now, if $p_\alpha(x) \neq 0$, then

$$p_\alpha\left(\frac{x}{p_\alpha(x)^{\frac{1}{\rho_\alpha}}}\right) = 1$$

which implies that

$$\frac{1}{p_\alpha(x)} r(x)^{\rho_\alpha} = r\left(\frac{x}{p_\alpha(x)^{\frac{1}{\rho_\alpha}}}\right)^{\rho_\alpha} \leqslant M_\alpha^{\rho_\alpha}.$$

[4] We can assume here that the family $\{p_\alpha : \alpha \in \Lambda\}$ is saturated, otherwise we can saturate it.

Thus $r(x)^{\rho_\alpha} \leqslant M_\alpha^{\rho_\alpha} p_\alpha(x)$. Therefore $(E, p_\alpha)$ satisfies the condition 5) of Theorem 2.5. Then $p_\alpha$ is a $\rho_\alpha$-spectral seminorm (by Theorem 2.5, 1). Hence $(E, p_\alpha)$ is $\rho_\alpha$-spectral.

**Remark 3.2.** Note that the condition 6) of Theorem 2.5 is equivalent for $(E, p_\alpha)$ to be a $Q$-algebra. As $\tau_{p_\alpha} \subseteq \tau$, then $(E, \tau)$ is also a $Q$-algebra. So, we have that the statements 1), 5) and 6) are equivalent.

6) $\Rightarrow$ 4) Let $\alpha$ be such as in the statement 6). Applying Theorem 2.5 to $E$ with respect to the seminorm $p_\alpha$, we have that 6) implies

$$r(x)^{\rho_\alpha} = \lim_n \; p_\alpha(x^n)^{\frac{1}{n}}$$

for every $x \in E$. Now, we shall prove that $r(x)^{\rho_\alpha} = \inf_n \; p_\alpha(x^n)^{\frac{1}{n}}$.

On the one hand, since $r(x) \geqslant r_{E_\alpha}(\pi_\alpha(x))$ then by 6)

$$r_{E_\alpha}(\pi_\alpha(x))^{\rho_\alpha} \leqslant r(x)^{\rho_\alpha} \leqslant p_\alpha(x) = p'_\alpha(\pi_\alpha(x)) \text{ for every } x \in E.$$

and as $\pi_\alpha$ is onto we conclude that $r_{E_\alpha}(\bar{y})^{\rho_\alpha} \leqslant p'_\alpha(\bar{y})$ for every $\bar{y} \in E_\alpha$. Now, by Theorem 2.5, $r_{E_\alpha}(\bar{y})^{\rho_\alpha} = \lim_n \; p'_\alpha((\bar{y})^n)^{\frac{1}{n}}$ for every $\bar{y} \in E_\alpha$. As $\pi_\alpha(x) \in E_\alpha$ for every $x \in E$, then

$$\begin{aligned} r_{E_\alpha}(\pi_\alpha(x))^{\rho_\alpha} &= \lim_n \; p'_\alpha((\pi_\alpha(x))^n)^{\frac{1}{n}} \\ &= \limsup p'_\alpha((\pi_\alpha(x))^n)^{\frac{1}{n}} \\ &= \inf_n \Big( \sup_{k \geqslant n} \; p'_\alpha(\pi_\alpha(x)^k)^{\frac{1}{k}} \Big) \end{aligned}$$

for every $x \in E$. Since

$$\sup_{k \geqslant n} \Big( p_\alpha(x^k)^{\frac{1}{k}} \Big) \geqslant p_\alpha(x^n)^{\frac{1}{n}}, \text{ for every } x \in E$$

then we have that for every $x \in E$,

$$\begin{aligned} r(x)^{\rho_\alpha} \geqslant r_{E_\alpha}(\pi_\alpha(x))^{\rho_\alpha} &= \inf_n \Big( \sup_{k \geqslant n} \; p'_\alpha(\pi_\alpha(x)^k)^{\frac{1}{k}} \Big) \\ &= \inf_n \Big( \sup_{k \geqslant n} \; p_\alpha(x^k)^{\frac{1}{k}} \Big) \\ &\geqslant \inf_n \; p_\alpha(x^n)^{\frac{1}{n}}. \end{aligned}$$

On the other hand, due to [5, Proposition 4.8.3, (ii)], we have that

$$\big(r(x)^{\rho_\alpha}\big)^n = r(x^n)^{\rho_\alpha} \leqslant p_\alpha(x^n)$$

(for every $n$), therefore $r(x)^{\rho_\alpha} \leqslant p_\alpha(x^n)^{\frac{1}{n}}$ (for every $n$) and hence we have that $r(x)^{\rho_\alpha} \leqslant \inf_n \; p_\alpha(x^n)^{\frac{1}{n}}$. So, $r(x)^{\rho_\alpha} = \inf_n \; p_\alpha(x^n)^{\frac{1}{n}}$.

4) $\Rightarrow$ 3) Let $\alpha \in \Lambda$ be as in 4) and $x \in E$ be such that $p_\alpha(x) < 1$. For every $\gamma \in \Lambda$, we denote by $\overline{E_\gamma}$ the completion of the algebra $E_\gamma$ (a $\rho_\gamma$-Banach algebra) and the spectral radius for $\overline{E_\gamma}$, by $r_{\overline{E_\gamma}}$.

Since

$$r(x)^\rho \geqslant r_{E_\gamma}(\pi_\gamma(x))^\rho \geqslant r_{\overline{E_\gamma}}(\pi_\gamma(x))^\rho = \lim_n \ p'_\gamma(\pi_\gamma(x)^n)^{\frac{1}{n}} = \lim_n \ p_\gamma(x^n)^{\frac{1}{n}}$$

for every $\gamma \in \Lambda$, then $\sup\limits_\beta \ \lim\limits_n \ p_\beta(x^n)^{\frac{1}{n}} \leqslant r(x)^\rho$. Now, since $p_\alpha$ is submultiplicative, then by 4), we have that $r(x)^\rho \leqslant p_\alpha(x)$, and by hypothesis $p_\alpha(x) < 1$ we conclude that $\sup\limits_\beta \ \lim\limits_n \ p_\beta(x^n)^{\frac{1}{n}} < 1$. Therefore, there exists $m < 1$ such that

$$\lim_n \ p_\beta(x^n)^{\frac{1}{n}} < m < 1$$

for every $\beta \in \Lambda$. Since $\lim\limits_n \ p_\beta(x^n) < m^n$ with $m < 1$ for each $\beta$, then $x^n \to 0$. If now $E$ is invertive then by [3, Proposition 3.7 (1)], the series $\sum_{n=0}^{\infty} x^n$ converges in $E$.

Moreover, we have that

$$(\sum_{k=0}^{n} x^k)(e-x) = \sum_{k=0}^{n}(x^k)(e-x) = \sum_{k=0}^{n}(x^k - x^{k+1}) = e - x^{n+1} \longrightarrow e.$$

Thus, $(x-e)$ is invertible in $E$ and $(x-e)^{-1} = \sum_{k=0}^{\infty} x^k$.

3) $\Rightarrow$ 2) Let $\alpha \in \Lambda$ be as in 3) and $x \in E$ be such that $p_\alpha(e-x) < 1$. Then, $\sum_{n=0}^{\infty}(e-x)^n$ converges in $E$ and $x^{-1} = (e-(e-x))^{-1}$; that is, $x \in Inv(E)$.

2) $\Rightarrow$ 1) Let $\alpha \in \Lambda$ be as in 2), $x \in Inv(E)$ and $y \in E$ be such that

$$p_\alpha(x-y) < \frac{1}{p_\alpha(x^{-1})},$$

then

$$p_\alpha(e - x^{-1}y) \leqslant p_\alpha(x^{-1})p_\alpha(x-y) < 1,$$

and by 2), $x^{-1}y$ is invertible in $(E, p_\alpha)$. In a similar way we get that $yx^{-1}$ is invertible in $(E, p_\alpha)$. If $z = (yx^{-1})^{-1}$, then $y(x^{-1}z) = e$, that is, $y$ is invertible and hence $(E, p_\alpha)$ is a $Q$-algebra. Therefore $(E, \{p_\alpha : \alpha \in \Lambda\})$ is a $Q$-algebra. $\square$

Since by [1, p. 17, Corollary 2], every complete locally $m$-pseudoconvex algebra is an invertive algebra, we conclude that if $E$ is a complete $m$-($\rho$-convex) algebra, then 1), 2,) 3), 4), 5) and 6) in the previous Theorem are equivalent.

In [2] has been shown that in the case of locally pseudoconvex (not necessarily unital) algebras, 1) and 6) are equivalent.

**Remark 3.3.** The condition $\rho_\alpha = \rho$ for every $\alpha \in \Lambda$ that appears in the hypothesis of the previous Theorem, is only used in 4) $\Rightarrow$ 3), we note (see the proof of 4) $\Rightarrow$ 3)) that under the hypothesis $(E, \{p_\alpha : \alpha \in \Lambda\})$ is a complex locally $m$-pseudoconvex algebra with unit $e$, $\rho = \inf\{\rho_\alpha : \alpha \in \Lambda\} > 0$ and if $r(x)^\rho < 1$ then 1), 2), 3), 4), 5) and 6) are equivalent.

In the non-unital case, Theorem 3.1 remains valid dropping conditions 2) and 3).

Additional properties concerning $Q$-algebras can be found in [2], [4] and [6].

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