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STABILITY AND FINITE SUM OF WILSON FRAMES

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Abstract. Wilson frames have been defined in [10]. In this paper we discuss about finite sum of Wilson frames in case of similar window functions. A necessary and sufficient condition is obtained under which finite sum of Wilson frames is a Wilson frame. Finally, a sufficient condition under which finite sum of Wilson Bessel sequences is a Bessel sequence for $L^2(\mathbb{R})$ in terms of their frame bounds is obtained.

1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer[5] in 1952. Daubechies, Meyer and Grossmann [3],in 1986, reintroduced frames and observed that frames can be used to find series expansions of functions in $L^2(\mathbb{R})$. Time-frequency representations play a fundamental role in many practical applications as they provide localized information of signals in time and frequency domains. Series representations in terms of frame capture such information at prescribed discrete points in time-frequency plane. Gabor frames find wide applications in time-frequency analysis, signals processing, etc [2,7,8,9].

However, in view of the Balian-Low Theorem, Gabor frames which are Reisz basis for $L^2(\mathbb{R})$ do not possess good localization properties in time and frequency simultaneously. This drawback was overcome by a Wilson system defined by Daubechies, Jaffard and Journe in [4]. K. Bittner in [1] generalised the system defined by Daubechies, Jaffard and Journe which also overcame the drawbacks of the Balian-Low Theorem. In this paper we discuss stability of Wilson frames and finite sum of Wilson frames in case of same window functions.

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2. Preliminaries

Throughout the paper $\mathbb{H}$ will denote a Hilbert space and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ denotes the set of Natural numbers.

Definition 2.1. Let $\mathbb{H}$ be a Hilbert space and I be an index set. A family of vectors $\{f_i\}_{i\in I} \subset \mathbb{H}$ is called a *frame* for $\mathbb{H}$ if there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in \mathbb{H} \tag{2.1}$$

The positive constants A and B are called *lower* and *upper frame bound* for the frame $\{f_i\}_{i\in I}$, respectively. The inequality (2.1) is called the *frame inequality.* If in (2.1) only the upper inequality holds, then $\{f_i\}_{i\in I}$ is called a *Bessel sequence.*

For a frame $\{f_i\}_{i\in I} \subset \mathbb{H}$ the *frame operator* $S : \mathbb{H} \to \mathbb{H}$ defined as $Sx = \sum\limits_{n\in I} \langle x, f_n\rangle f_n$, $x \in \mathbb{H}$.

For $a \in \mathbb{R}$ and $g \in L^2(\mathbb{R})$, the *translation operator* T_a on $L^2(\mathbb{R})$ is defined as $T_a g(x) = g(x - a)$, $x \in \mathbb{R}$ and the *modulation operator* E_a on $L^2(\mathbb{R})$ is defined as $E_a g(x) = \exp(2\pi iax)g(x)$,$x \in \mathbb{R}$.

Definition 2.2. [2] Let $g \in L^2(\mathbb{R})$ and a, b are positive constants. Then the sequence $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is called a *Gabor system* for $L^2(\mathbb{R})$.
If the Gabor system $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, then it is called a *Gabor frame* and if $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence for $L^2(\mathbb{R})$, then it is called a *Gabor Bessel sequence*. For literature related to Gabor frames, one may refer to [2,5,6,7,13].

Definition 2.3. [1] Let $w_0, w_{-1} \in L^2(\mathbb{R})$. Then the *Wilson system* associated with $w_0, w_{-1} \in L^2(\mathbb{R})$ is a sequence of functions $\{\psi_j^k : w_0, w_{-1} \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ in $L^2(\mathbb{R})$ defined by

$$\psi_j^k(x) = \begin{cases} \varepsilon_k \cos(2k\pi x) w_0\left(x - \dfrac{j}{2}\right), & \text{if} j \in 2\mathbb{Z} \\ 2\sin(2(k+1)\pi x) w_{-1}\left(x - \dfrac{j+1}{2}\right) & \text{if } j \in 2\mathbb{Z}+1 \end{cases}$$

and

$$\varepsilon_k = \begin{cases} \sqrt{2}, & \text{if } k = 0 \\ 2, & \text{if } k \in \mathbb{N}. \end{cases}$$

If $w_0 = w_{-1}$, then the Wilson system in $L^2(\mathbb{R})$ is represented as a sequence of functions $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$,$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
If the Wilson system is a frame for $L^2(\mathbb{R})$, it is called a *Wilson frame*. If the Wilson system is a Bessel sequence $L^2(\mathbb{R})$, it is called a *Wilson Bessel sequence*. For literature on Wilson systems, one may refer to [1,4,6,10]

Remark 2.4. [10] If we choose $w_0 = w_{-1}$ in the Wilson frame $\{\psi_j^k : w_0, w_{-1} \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ in $L^2(\mathbb{R})$, then

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 = 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_k T_j w_0\rangle|^2 \quad \text{for all } f \in L^2(\mathbb{R})$$

3. Stability of Wilson frames with $w_0 = w_{-1}$

In [10], we have seen that Wilson frames are closely related to Gabor frames in case $w_0 = w_{-1}$. Stability of Gabor frames is discussed in [12]. In the following result, we discuss stability of Wilson frames with $w_0 = w_{-1}$.

Let $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ be a Wilson frame associated with $w_0 \in L^2(\mathbb{R})$. Let $h \in L^2(\mathbb{R})$ be such that $\{\psi_j^{'k} : w_0 + h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a Wilson Bessel sequence. Then, the Wilson system associated with $h \in L^2(\mathbb{R})$ may or may not be a Wilson frame for $L^2(\mathbb{R})$.

Example 3.1. Let $w_0 = h = \chi_{[0,1)}$. Then $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a Wilson frame for $L^2(\mathbb{R})$ and $\{\psi_j^k : w_0 + h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a Wilson Bessel sequence for $L^2(\mathbb{R})$. Also, $\{\psi_j^k : h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a Wilson frame for $L^2(\mathbb{R})$.

Example 3.2. Let $w_0 = \chi_{[0,1)}$ and $h = \chi_{[0,\frac{1}{2})}$. Then $\{\psi_j^k : w_0, w_{-1} \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a Wilson frame for $L^2(\mathbb{R})$ and $\{\psi_j^k : w_0 + h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a Wilson Bessel sequence for $L^2(\mathbb{R})$. But $\{\psi_j^k : h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is not a Wilson frame for $L^2(\mathbb{R})$.

In the following result, we give sufficient conditions for the stability of Wilson frames with $w_0 = w_{-1}$.

Theorem 3.3. *Let $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ be a Wilson frame for $L^2(\mathbb{R})$ having A and B as its lower and upper frame bound, respectively. Let S be its frame operator and let $h \in L^2(\mathbb{R})$ be any function such that the Wilson system $\{\psi_j^k : w_0 + h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a Wilson Bessel Sequence with Wilson Bessel bound M. Then $\{\psi_j^k; h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a Wilson Bessel sequence for $L^2(\mathbb{R})$. Also ,if $(A^2\|S\|^{-1} - 2M) > 0$, then $\{\psi_j^{''k} : h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a Wilson frame for $L^2(\mathbb{R})$.*

Proof. Let $f \in L^2(\mathbb{R})$. Consider the Wilson system $\{\psi_j^k : h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$. Then, by Remark 2.4 , we have

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_j^{''k}\rangle|^2 = 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_k T_j h\rangle|^2$$

Therefore

$$\sum_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}} |\langle f, \psi_j^{''k}\rangle|^2 = 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_j(w_0+h)\rangle - \langle f, E_kT_jw_0\rangle|^2$$
$$\leq 4\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_j(w_0+h)\rangle|^2 + 4\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2.$$

Since $\{\psi_j^k : w_0+h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}}$ is a Wilson Bessel sequence with Bessel bound M, we have

$$\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_j(w_0+h)\rangle|^2 \leq \left(\frac{M}{2}\right)\|f\|^2.$$

Since $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}}$ is a Wilson frame for $L^2(\mathbb{R})$ with upper frame bound B, we have

$$\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2 \leq \left(\frac{B}{2}\right)\|f\|^2.$$

Hence, we obtain

$$\sum_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 \leq 2\{M+B\}\|f\|^2, \quad \text{for all } f\in L^2(\mathbb{R}).$$

Also, for $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned}(A^2\|S\|^{-1} - 2M)\|f\|^2 &\leq (A^2\frac{1}{A}\|f\|^2 - 2M\|f\|^2)\\ &= A\|f\|^2 - 2M\|f\|^2\\ &\leq \sum_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}} |\langle f, \psi_j^k\rangle|^2 - 2\sum_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}} |\langle f, \psi_j^{'k}\rangle|^2\\ &= 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2 - 4\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_j(w_0+h)\rangle|^2.\end{aligned}$$

Therefore, we get

$$(A^2\|S\|^{-1} - 2M)\|f\|^2 \leq 2(\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2 - 2\sum_{j,w\in\mathbb{Z}} |\langle f, E_kT_j(w_0+h)\rangle|^2).$$

Thus

$$(A^2\|S\|^{-1} - 2M)\|f\|^2 \leq 4\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jh\rangle|^2.$$

Hence $\{\psi_j^{''k} : h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}}$ is a Wilson frame with frame bounds $\frac{1}{4}(A^2\|S\|^{-1} - 2M)$ and $2(M+B)$. $\square$

Remark 3.4. The condition that $(A^2\|S\|^{-1} - 2M) > 0$ is not necessary as seen in Example 3.1, where $A = B = 1$ and $M = 4$.

Finally, we give necessary and sufficient condition for the Wilson system $\{\psi_j^k : h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ to be a Wilson Bessel sequence in terms of a Wilson frame.

Theorem 3.5. *Let $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ be a Wilson frame for $L^2(\mathbb{R})$ having A and B as its lower and upper frame bound respectively. Let $h \in L^2(\mathbb{R})$ be any function. Then, the Wilson system $\{\psi_j^k : h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a Wilson Bessel sequence in $L^2(\mathbb{R})$ if and only if there exist $C > 0$ such that*

$$\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_j(w_0 - h)\rangle|^2 \leq C \sum_{j,k\in\mathbb{Z}} |\langle f, \psi_j^k\rangle|^2, \textit{for all} \ \ f \in L^2(\mathbb{R}).$$

Proof. Let $f \in L^2(\mathbb{R})$. Then

$$\begin{aligned}\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_j(w_0 - h)\rangle|^2 &= \sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0 - E_kT_jh\rangle|^2 \\ &= \sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle - \langle f, E_kT_jh\rangle|^2 \\ &\leq 2 \sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2 + 2 \sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jh\rangle|^2.\end{aligned}$$

Suppose $\{\psi_j^k : h \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a Bessel sequence with Bessel bound M. Then

$$\frac{1}{M} \sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jh\rangle|^2 \leq \|f\|^2.$$

As $\{\psi_j^k : w_0 \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a Wilson frame for $L^2(\mathbb{R})$ having A as its lower frame bound , we obtain

$$\|f\|^2 \leq \frac{1}{A} \sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2.$$

Thus

$$\frac{1}{M} \sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jh\rangle|^2 \leq \frac{1}{A} \sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2.$$

Hence

$$\begin{aligned}\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_j(w_0 - h)\rangle|^2 \quad &\leq \quad 2(\frac{M}{A} + 1) \sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2 \\ &\leq \quad (\frac{M}{A} + 1) \sum_{j,k\in\mathbb{Z}} |\langle f, \psi_j^k\rangle|^2,\end{aligned}$$

where $C = \frac{M}{A} + 1$.

Conversely, for $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned}\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jh\rangle|^2 =& \sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_j(h-w_0)\rangle + \langle f, E_kT_jw_0\rangle|^2\\ \leq& 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_j(h-w_0)\rangle|^2 + 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2\\ \leq& 2C\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2 + 2\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{j,k\in\mathbb{Z}} (|\langle f, E_kT_jh\rangle|^2 &\leq 2(C+1)\sum_{j,k\in\mathbb{Z}} |\langle f, E_kT_jw_0\rangle|^2\\ &= (C+1)\sum_{j,k\in\mathbb{Z}} |\langle f, \psi_j^k\rangle|^2\\ &\leq (C+1)B\|f\|^2.\end{aligned}$$

□

4. Finite Sum of Wilson Frames with $w_0 = w_{-1}$

In this section, we study finite sum of Wilson frames and obtain conditions under which finite sum of Wilson frames for $L^2(\mathbb{R})$ is a Wilson frame for $L^2(\mathbb{R})$.

Let $\Lambda_r = \{1, 2, 3.....r\}$ be a finite set and $\{\psi_{j,i}^k : w_i \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$, $i \in \Lambda_k$, $w_i \in L^2(\mathbb{R})$ be Wilson frames for $L^2(\mathbb{R})$ associated with $w_i \in L^2(\mathbb{R})$. Consider the Wilson system $\{\psi_j^k : \sum_{i=1}^r w_i \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ associated with $\sum_{i=1}^r w_i \in L^2(\mathbb{R})$. Then, $\{\psi_j^k : \sum_{i=1}^r w_i \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ may not be a frame for $L^2(\mathbb{R})$, where

$$\sum_{i=1}^r w_i(x) = w_1(x) + w_2(x) + ... + w_r(x).$$

Example 4.1. Let $g = \chi_{[0,1]}$, $h = e^{-\frac{x^2}{4}}$. Then

(1) If $w_i = g$ for all $i \in \Lambda_r$, then $\{\psi_j^k : \sum_{i=1}^r w_i \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a frame for $L^2(\mathbb{R})$.

Indeed, since $\{E_kT_j(\sum_{i=1}^r w_i)\}_{j,k\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, the Wilson system $\{\psi_j^k : \sum_{i=1}^r w_i \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a frame for $L^2(\mathbb{R})$.

(2) If $w_i = g$ for $1 \leq i \leq r-1$ and $w_r = h$, then $\{\psi_j^k : \sum_{i=1}^{r} w_i \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a frame for $L^2(\mathbb{R})$ but $\{\psi_{j,r}^k : w_r \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is not a frame for $L^2(\mathbb{R})$.

(3) If $w_i = g$ for all $i = 1, 3, ...r-1$ and $w_i = -g$ for all $i = 2, 4, ...r$, where r is even, then $\{\psi_j^k : \sum_{i=1}^{r} w_i \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is not a frame for $L^2(\mathbb{R})$ but $\{\psi_{j,i}^k : w_i \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a frame for $L^2(\mathbb{R})$ for $i \in \Lambda_r$.

Now, we give necessary and sufficient condition under which finite sum of Wilson frames is a frame for $L^2(\mathbb{R})$.

Theorem 4.2. *Let $\{\psi_{j,i}^k : w_i \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$, $i \in \Lambda_r$, $w_i \in L^2(\mathbb{R})$ be Wilson frames for $L^2(\mathbb{R})$. Let $\{\alpha_i\}_{i=1}^r$ be any scalars. Then $\{\psi_j^k : \sum_{i=1}^{r} \alpha_i w_i \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a frame for $L^2(\mathbb{R})$ if and only if there exists $M > 0$ and some $p \in \{1, 2, ...r\}$ such that*

$$M \sum_{j,k\in\mathbb{Z}} |\langle f, E_k T_j w_p\rangle|^2 \leq \sum_{j,k\in\mathbb{Z}} |\langle f, \sum_{i=1}^{r} \alpha_i E_k T_j w_i\rangle|^2, f \in L^2(\mathbb{R}).$$

Proof. For each $1 \leq i \leq r$, let A_i and B_i be bounds of the frame $\{\psi_{j,i}^k : w_i \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ respectively. Then, for any $f \in L^2(\mathbb{R})$

$$MA_p\|f\|^2 \leq \quad M \sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle f, \psi_{j,p}^k\rangle|^2, f \in L^2(\mathbb{R}).$$

Using Remark 2.4

$$\begin{aligned} MA_p\|f\|^2 \leq& 2M \sum_{j,k\in\mathbb{Z}} |\langle f, E_k T_j w_p\rangle|^2 \\ \leq& 2 \sum_{j,k\in\mathbb{Z}} |\langle f, \sum_{i=1}^{r} \alpha_i E_k T_j w_i\rangle|^2, f \in L^2(\mathbb{R}). \end{aligned}$$

Thus

$$MA_p\|f\|^2 \leq \quad \sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} \left| \left\langle \sum_{i=1}^{r} \alpha_i \psi_{j,i}^k, f \right\rangle \right|^2, \; f \in L^2(\mathbb{R}).$$

Also

$$\begin{aligned}\sum_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}}\left|\left\langle\sum_{i=1}^{r}\alpha_i\psi_{j,i}^{k},f\right\rangle\right|^2 =&2\sum_{j,k\in\mathbb{Z}}|\langle f,\sum_{i=1}^{r}\alpha_iE_kT_jw_i\rangle|^2\\ =&2\sum_{j,k\in\mathbb{Z}}|\sum_{i=1}^{r}\alpha_i\langle f,E_kT_jw_i\rangle|^2\\ \leq&2r\sum_{i=1}^{r}\left(|\alpha_i|^2\sum_{j,k\in\mathbb{Z}}|\langle f,E_kT_jw_i\rangle|^2\right)\\ \leq&2r(\max|\alpha_i|^2)\left(\sum_{i=1}^{r}B_i\right)\|f\|^2, f\in L^2(\mathbb{R}).\end{aligned}$$

Hence $\{\psi_j^k:\sum\limits_{i=1}^{r}\alpha_iw_i\in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}}$ is a frame for $L^2(\mathbb{R})$.

Conversely, let $\{\psi_j^k:\sum\limits_{i=1}^{r}\alpha_iw_i\in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}}$ be a frame for $L^2(\mathbb{R})$ with bounds A and B. For each $p\in\lambda_r$, let $\{\psi_j^k:w_p\in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}}$ be a frame with frame bounds A_p and B_p respectively. Then

$$\frac{2}{B_p}\leq\|f\|^2,\quad\text{for all}\quad f\in L^2(\mathbb{R}).$$

Also, we have

$$\|f\|^2\leq\frac{2}{A}\sum_{j,k\in\mathbb{Z}}|\sum_{i=1}^{r}\alpha_i\langle f,E_kT_jw_i\rangle|^2,\quad f\in L^2(\mathbb{R}).$$

Thus

$$M\sum_{j,k\in\mathbb{Z}}|\langle f,E_kT_jw_p\rangle|^2\leq\frac{2}{A}\sum_{j,k\in\mathbb{Z}}|\sum_{i=1}^{r}\alpha_i\langle f,E_kT_jw_i\rangle|^2,\ f\in L^2(\mathbb{R}),$$

where $M=\dfrac{A}{B_p}$. □

Finally, we prove a sufficient condition under which finite sum of Wilson Bessel sequences is a Bessel sequence for $L^2(\mathbb{R})$ in terms of their frame bounds.

Theorem 4.3. *Let* $\{\psi_{j,i}^k:w_i\in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}}, i\in\Lambda_r$, $w_i\in L^2(\mathbb{R})$ *be Wilson Bessel sequences for* $L^2(\mathbb{R})$ *with bounds* B_i *. Let* $\{\alpha_i\}_{i=1}^{r}$ *be any scalars. Then,* $\{\psi_j^k:\sum\limits_{i=1}^{k}\alpha_iw_i\in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}_0}}$ *is a Bessel sequence for* $L^2(\mathbb{R})$.

Proof. Let $f \in L^2(\mathbb{R})$ be any function. Then, by Remark 2.4, we have

$$\begin{aligned}
\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle \psi_j^k, f\rangle|^2 =& 2\sum_{j,k\in\mathbb{Z}} \left|\langle \sum_{i=1}^{r} \alpha_i E_k T_j w_i, f\rangle\right|^2 \\
=& 2\sum_{j,k\in\mathbb{Z}} \left|\sum_{i=1}^{r} \langle \alpha_i E_k T_j w_i, f\rangle\right|^2 \\
\leq& 2r\sum_{j,k\in\mathbb{Z}} \left|\alpha_1 \langle E_k T_j w_1, f\rangle\right|^2 + ... + 2r\sum_{j,k\in\mathbb{Z}} \left|\alpha_r \langle E_k T_j w_r, f\rangle\right|^2 \\
=& 2r|\alpha_1|^2 \sum_{j,k\in\mathbb{Z}} \left|\langle E_k T_j w_1, f\rangle\right|^2 + ...2r|\alpha_r|^2 \sum_{j,k\in\mathbb{Z}} \left|\langle E_k T_j w_r, f\rangle\right|^2 \\
=& r|\alpha_1|^2 \sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} \left|\langle \psi_{j,1}^k, f\rangle\right|^2 + ...r|\alpha_r|^2 \sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} \left|\langle \psi_{j,r}^k, f\rangle\right|^2
\end{aligned}$$

So

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}} |\langle \psi_j^k, f\rangle|^2 \leq r|\alpha_1|^2 B_1 \|f\|^2 + ...r|\alpha_r|^2 B_r \|f\|^2 \leq r\left(\sum_{i=1}^{r} |\alpha_i|^2 B_i\right)\|f\|^2, \;\; f \in L^2(\mathbb{R}).$$

Hence $\{\psi_j^k : \sum_{i=1}^{k} \alpha_i w_i \in L^2(\mathbb{R})\}_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{N}_0}}$ is a Bessel sequence for $L^2(\mathbb{R})$ with Bessel bound $r\left(\sum_{i=1}^{r} |\alpha_i|^2 B_i\right)$. □

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