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EXPONENTIAL STABILITY FOR A STRUCTURE WITH INTERFACIAL SLIP AND MEMORY

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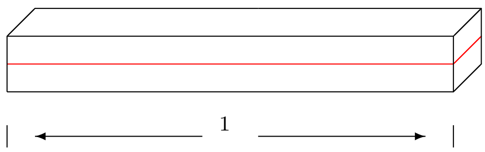
Abstract. In this work we consider a structure given by a laminated beam consisting of two identical layers uniform of length 1, taking into account that an adhesive of the small thickness is bonding the two layers and produce the interfacial slip.



The memory effect with a dissipative relaxation function together with other stronger dissipative effects, for example the frictional damping, can not produce any rate of decay if the relaxation function does not decay uniformly. In this work we prove that the effect of the memory together with the frictional damping produces stabilization for this structure. We applied the Energy Method, that consists in the use of appropriated multiplies to build a functional of Lyapunov for the system.

1. Introduction

There are few papers that deal with systems of interfacial slip, we cite [1], [2], [3], [5], [6], [7] and references therein. We mention Hansen [3], Hansen and Spies [2] where the mathematical model (1.1)-(1.3) for two-layered beams with structural damping due to

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the interfacial slip was derived.

$$\rho u_{tt} + G(\psi - u_x)_x = 0,\ x \in (0,1),\ t \geq 0, \tag{1.1}$$
$$I_\rho(3S_{tt} - \psi_{tt}) - G(\psi - u_x) - D(3S_{xx} - \psi_{xx}) = 0,\ x \in (0,1),\ t \geq 0, \tag{1.2}$$
$$3I_\rho S_{tt} + 3G(\psi - u_x) + 4\delta_0 S + 4\gamma_0 S_t - 3DS_{xx} = 0,\ x \in (0,1),\ t \geq 0. \tag{1.3}$$

In this model, $u(x,t)$ denotes the transverse displacement, $\psi(x,t)$ represents the rotation angle and $S(x,t)$ is proportional to the amount of slip along the interface at time t and longitudinal spatial variable x. The coefficients $\rho, G, I_\rho, D, \delta_0, \gamma_0$ are the density, the shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness and adhesive damping of the beams. The equation (1.14) describes the dynamics of the slip. In [5], Wang et all, proved that the frictional damping created by the interfacial slip alone is not enough to stabilize the system (1.1)-(1.3) exponentially to its equilibrium state.

Now consider $s(x,t) = 3S(x,t)$ and $\rho_1 = \rho,\ \rho_2 = I_\rho,\ k = G,\ b = D,\ 3\delta = \delta_0,\ 3\gamma = \gamma_0$, then we deduce from (1.1)-(1.3) the following system

$$\rho_1 u_{tt} + k(\psi - u_x)_x = 0, \tag{1.4}$$
$$\rho_2(s - \psi)_{tt} - b(s - \psi)_{xx} - k(\psi - u_x) = 0, \tag{1.5}$$
$$\rho_2 s_{tt} - bs_{xx} + 3k(\psi - u_x) + 4\delta s + 4\gamma s_t = 0. \tag{1.6}$$

Combining (1.5) and (1.6) we have

$$\rho_1 u_{tt} + k(\psi - u_x)_x = 0, \tag{1.7}$$
$$\rho_2 \psi_{tt} - b\psi_{xx} + 4k(\psi - u_x) + \delta s + \beta s_t = 0. \tag{1.8}$$

Note that for $s = 0$ in (1.8) we obtain the following system

$$\rho_1 u_{tt} + k(\psi - u_x)_x = 0,$$
$$\rho_2 \psi_{tt} - b\psi_{xx} + 4k(\psi - u_x) = 0,$$

that is a conservative system closely related with Timoshenko's beams.

For frictional damping, if we introduce the damping αu_t on the transverse displacement and $\beta\psi_t$ on the rotation angle in the Timoshenko's system, we obtain the dissipative system

$$\rho_1 u_{tt} + k(\psi - u_x)_x + \alpha u_t = 0, \tag{1.9}$$
$$\rho_2 \psi_{tt} - b\psi_{xx} + k(\psi - u_x) + \beta\psi_t = 0, \tag{1.10}$$

that is exponentially stable, see for example [9] and reference therein.

Motivated by the exponential stability of (1.9)-(1.10) Raposo et all in the work [1] investigated the action of the frictional damping and proved the exponential stability for the following system of laminated beams

$$\rho_1 u_{tt} + k(\psi - u_x)_x + \alpha u_t = 0,$$
$$\rho_2(s - \psi)_{tt} - b(s - \psi)_{xx} - k(\psi - u_x) + \beta(s - \psi)_t = 0,$$
$$\rho_2 s_{tt} - bs_{xx} + 3k(\psi - u_x) + 4\delta s + 4\gamma s_t = 0.$$

Then, the frictional damping has the same influence on both models.

Now we consider the usual convolution term

$$(g * u_{xx})(t) := \int_0^t g(t-s)u_{xx}(s)\,ds.$$

If g is a function of exponential type such that

$$\left.\begin{aligned} g>0,\ \exists\, c, c_0, c_1, c_2 > 0 : -c_0 g(t) \le g'(t) \le -c_1 g(t),\ |g''(t)| \le c_2 g(t), \\ c - \textstyle\int_0^\infty g\,ds > 0. \end{aligned}\right\} \tag{1.11}$$

is satisfied, then, Ammar-Khodja et all, see [11], proved that the viscoelastic system of Timoshenko with memory

$$\rho_1\phi_{tt} - k(\phi_x + \psi)_x = 0,$$
$$\rho_2\psi_{tt} - b\psi_{xx} + g * \psi_{xx} + k(\phi_x + \psi) = 0,$$

decays uniformly if and only if the coefficients satisfy

$$\frac{\rho_1}{\rho_2} = \frac{k}{b}.$$

In recent paper, Lo and Tatar, see [7], showed that for viscoelastic material there isn't need any kind of internal or boundary control to stabilize exponentially the system of laminated beams with interfacial slip. They consider the following system

$$\rho u_{tt} + G(\psi - u_x)_x = 0, \tag{1.12}$$

$$I_\rho(3S_{tt} - \psi_{tt}) - G(\psi - u_x) - D(3S - \psi)_{xx} + \int_0^t h(t-r)(3S-\psi)_{xx}(r)dt = 0, \tag{1.13}$$

$$I_\rho S_{tt} + G(\psi - u_x) + \frac{4}{3}\gamma S + \frac{4}{3}\alpha S_t - S_{xx} = 0. \tag{1.14}$$

Instead of the frictional damping, the viscoelastic damping may have different influence when acts in the stabilization of Timoshenko's system and when is effective on the structure with interfacial slip. In this direction, this work deal with the following system

$$\rho_1 u_{tt} + k(\psi - u_x)_x + \alpha u_t + g * u_{xx} - u_{xx} = 0, \tag{1.15}$$

$$\rho_2(s-\psi)_{tt} - b(s-\psi)_{xx} - k(\psi - u_x) + \beta(s-\psi)_t = 0, \tag{1.16}$$

$$\rho_2 s_{tt} - b s_{xx} + 3k(\psi - u_x) + 4\delta s + 4\gamma s_t = 0, \tag{1.17}$$

where $(x,t) \in (0,1) \times (0,\infty)$ with cantilever boundary conditions

$$u(0,t) = \psi(0,t) = s(0,t) = 0, \tag{1.18}$$

$$s_x(1,t) = \psi_x(1,t) = 0,\ \ u_x(1,t) = \psi(1,t), \tag{1.19}$$

and initial data

$$(u(x,0), \psi(x,0), s(x,0)) = (u_0(x), \psi_0(x), s_0(x)), \tag{1.20}$$

$$(u_t(x,0), \psi_t(x,0), s_t(x,0)) = (u_1(x), \psi_1(x), s_1(x)). \tag{1.21}$$

with the memory kernel g of type (1.11) satisfying

$$1 - \int_0^\infty g\,ds > 0.$$

We remark that the memory effect is a subtle damping mechanism, because, the simple memory term $g * u_{xx}$ can imposes a destabilization of the system, as even more reason, another important observation is that when the memory effect with a dissipative relaxation function together with other stronger dissipative effects, then the resulting dissipation does not produce any rate of decay if the relaxation function does not decay uniformly, see the work of Fabrizio and Polidoro [12].

The well-posedness of the system has been addressed in [3], We have weak solutions in $(\mathcal{V} \times L^2)^3$ and strong solutions in $(\mathcal{V} \times H^1)^3$ where

$$\mathcal{V} = \{v \,:\, v \in H^k(0,1), v(0) = 0\}, \;\; k = 1, 2.$$

In this work, we have interested in the asymptotic behaviour, in fact, we need to know if the effect of the memory together with the frictional damping produces stabilization of the system (1.15)- (1.17). We will use Sobolev spaces with its proprieties as in [4] and for $f, g \in \mathcal{V} \times H^1$ we denote

$$\int_0^1 f(r)g(r)\, dr = \langle f, g\rangle.$$

Now let us briefly outline the contents of this paper. We applied the Energy Method, that consists in the use of appropriated multiplies to build a functional of Lyapunov for the system. In Section 2 we prove that the energy is dissipative. In Section 3 we give some technical lemmas and finally in section 4 we show the exponential stability.

2. Energy of the system

First consider the following simple lemma (cf. Lemma 3.2 from [10]).

Lemma 2.1. *For $f, h \in C^1([0,\infty), \mathbb{R})$ we have*

$$2\,[f * h]\, h' = f'\Box h - f(t)|h|^2 - \frac{d}{dt}\left\{ f\Box h - \left(\int_0^t f\, ds\right)|h|^2 \right\},$$

where the symbol $\Box$ denotes the following convolution

$$(g\Box h)(t) := \int_0^t g(t-s)|h(t) - h(s)|^2\, ds.$$

In this section we prove the dissipative proprieties of the energy of the system. To achieve this we will use the multiplicative technique.

Proposition 2.2. The energy of the system given by

$$E(t) = \frac{1}{2}\left[3\rho_1||u_t||^2 + 3k||\psi - u_x||^2 + \rho_2||s_t||^2 + b||s_x||^2 + 4\delta||s||^2\right]$$
$$+\frac{1}{2}\left[3\rho_2||(s-\psi)_t||^2 + 3b||(s-\psi)_x||^2 + 3\int_0^1 g\Box u_x\, dx + \left(1 - \int_0^t g\, ds\right)3||u_x||^2\right]$$

satisfies

$$\frac{d}{dt}E(t) = -3\alpha||u_t|| - 4\gamma||s_t||^2 - 3\beta||(s-\psi)_t||^2 + \frac{3}{2}\int_0^1 g'\Box u_x\, dx - \frac{3}{2}g(t)||u_x||^2.$$

Proof. First we multiply (1.15) by $3u_t$ and we obtain

$$\langle \rho_1 u_{tt}, 3u_t\rangle + \langle k(\psi - u_x)_x, 3u_t\rangle + \langle \alpha u_t, 3u_t\rangle + \langle g * u_{xx}, 3u_t\rangle - \langle u_{xx}, 3u_t\rangle = 0.$$

Using integration by parts and boundary conditions we have

$$\frac{1}{2}\frac{d}{dt}3\rho_1||u_t||^2 - \langle 3k(\psi - u_x), u_{xt}\rangle + 3\alpha||u_t||^2 - 3\langle g * u_x, u_{xt}\rangle + 3\langle u_x, u_{xt}\rangle = 0.$$

The last question can be written as

$$\frac{1}{2}\frac{d}{dt}3\rho_1||u_t||^2 - \langle 3k(\psi - u_x), (u_x - \psi + \psi)_t\rangle + 3\alpha||u_t||^2$$
$$-3\langle g * u_x, u_{xt}\rangle + 3\frac{1}{2}\frac{d}{dt}||u_x||^2 = 0,$$

that is,

$$\frac{1}{2}\frac{d}{dt}3\rho_1||u_t||^2 + \langle 3k(\psi - u_x), (\psi - u_x)_t\rangle - \langle 3k(\psi - u_x), \psi_t\rangle + 3\alpha||u_t||^2$$
$$-3\langle g * u_x, u_{xt}\rangle + 3\frac{1}{2}\frac{d}{dt}||u_x||^2 = 0,$$

Now using Lemma 1 with $f = g$ and $h = u_x$ we have

$$2\left[g * u_x\right] u_{xt} = g'\Box u_x - g(t)|u_x| - \frac{d}{dt}\left\{g\Box u_x - \left(\int_0^t g\, ds\right)|u_x|^2\right\} \tag{2.22}$$

so,

$$2\langle g * u_x, u_{xt}\rangle = \int_0^1 g'\Box u_x\, dx - \int_0^1 g(t)|u_x|^2\, dx$$
$$-\frac{d}{dt}\int_0^1 \left\{g\Box u_x - \left(\int_0^t g\, ds\right)|u_x|^2\right\} dx \tag{2.23}$$

and then

$$\frac{1}{2}\frac{d}{dt}\left[3\rho_1||u_t||^2 + 3k||\psi - u_x||^2\right] - \langle 3k(\psi - u_x), \psi_t\rangle + 3\frac{1}{2}\frac{d}{dt}||u_x||^2$$
$$+3\frac{1}{2}\frac{d}{dt}\int_0^1 \left\{g\Box u_x - \left(\int_0^t g\, ds\right)|u_x|\right\} dx =$$
$$= -3\alpha||u_t||^2 + 3\frac{1}{2}\int_0^1 g'\Box u_x\, dx - 3\frac{1}{2}\int_0^1 g(t)|u_x|^2\, dx$$

that can be write as

$$\frac{1}{2}\frac{d}{dt}\left[3\rho_1||u_t||^2 + 3k||\psi - u_x||^2\right] - \langle 3k(\psi - u_x), \psi_t\rangle$$
$$+3\frac{1}{2}\frac{d}{dt}\int_0^1 \left\{g\Box u_x + \left(1 - \int_0^t g\, ds\right)|u_x|^2\right\} dx =$$
$$= -3\alpha||u_t||^2 + 3\frac{1}{2}\int_0^1 g'\Box u_x\, dx - 3\frac{1}{2}\int_0^1 g(t)|u_x|^2\, dx$$

that is,

$$\frac{1}{2}\frac{d}{dt}\left[3\rho_1||u_t||^2+3k||\psi-u_x||^2+3\int_0^1 g\Box u_x\,dx+\left(1-\int_0^t g\,ds\right)3\int_0^1 |u_x|^2\,dx\right]$$
$$-\langle 3k(\psi-u_x),\psi_t\rangle=-3\alpha||u_t||^2+3\frac{1}{2}\int_0^1 g'\Box u_x\,dx-3\frac{1}{2}\int_0^1 g(t)|u_x|^2\,dx$$

an then

$$\frac{1}{2}\frac{d}{dt}\left[3\rho_1||u_t||^2+3k||\psi-u_x||^2+3\int_0^1 g\Box u_x\,dx+\left(1-\int_0^t g\,ds\right)3||u_x||^2\right]$$
$$-\langle 3k(\psi-u_x),\psi_t\rangle=-3\alpha||u_t||^2+3\frac{1}{2}\int_0^1 g'\Box u_x\,dx-3\frac{1}{2}g(t)||u_x||^2 \tag{2.24}$$

Now multiplying (1.17) by s_t we have

$$\langle\rho_2 s_{tt},s_t\rangle-\langle bs_{xx},s_t\rangle+\langle 3k(\psi-u_x),s_t\rangle+\langle 4\delta s,s_t\rangle+\langle 4\gamma s_t,s_t\rangle=0.$$

Performing integration by parts we get

$$\frac{1}{2}\frac{d}{dt}\left[\rho_2||s_t||^2+b||s_x||^2+4\delta||s||^2\right]+\langle 3k(\psi-u_x),s_t\rangle=-4\gamma||s_t||^2. \tag{2.25}$$

In this moment we multiply (1.16) by $3(s-\psi)_t$ and we obtain

$$\langle\rho_2(s-\psi)_{tt},3(s-\psi)_t\rangle-\langle b(s-\psi)_{xx},3(s-\psi)_t\rangle$$
$$-\langle k(\psi-u_x),3(s-\psi)_t\rangle+\langle\beta(s-\psi)_t,3(s-\psi)_t\rangle=0.$$

Performing integration by parts and using boundary conditions we get

$$\frac{1}{2}\frac{d}{dt}\left[3\rho_2||(s-\psi)_t||^2+3b||(s-\psi)_x||^2\right]$$
$$-\langle 3k(\psi-u_x),(s-\psi)_t\rangle=-3\beta||(s-\psi)_t||^2. \tag{2.26}$$

Finally, adding (2.24), (2.25), (2.26) our conclusion old.

3. Technical lemmas

In the section previous we observe that the functional of energy restores some terms of energy with a negative sign. We are interested in building the functional of that restores the full energy of the system with negative sign and for this goal consider the following lemmas.

Lemma 3.1. *Defining*

$$L_1(t)=\langle\rho_2 s_t,s\rangle+2\gamma||s||^2,$$

we have

$$\frac{d}{dt}L_1(t)\le -b||s_x||^2-\delta||s||^2+C_{\epsilon_0}||\psi-u_x||^2+\rho_2||s_t||^2.$$

Proof. Using (1.17) we obtain

$$\begin{aligned}\frac{d}{dt}\langle\rho_2 s_t, s\rangle &= \langle\rho_2 s_{tt}, s\rangle + \langle\rho_2 s_t, s_t\rangle\\ &= \langle b s_{xx}, s\rangle - \langle 3k(\psi-u_x), s\rangle - \langle 4\delta s, s\rangle - \langle 4\gamma s_t, s\rangle + \langle\rho_2 s_t, s_t\rangle\\ &= -b||s_x||^2 - \langle 3k(\psi-u_x), s\rangle - 4\delta||s||^2 - 4\gamma\frac{1}{2}\frac{d}{dt}||s||^2 + \rho_2||s_t||^2\end{aligned}$$

Using Young's inequality, we obtain

$$\frac{d}{dt}\langle\rho_2 s_t, s\rangle \le -b||s_x||^2 - 4\delta||s||^2 + \epsilon_0||s||^2 + C_{\epsilon_0}||\psi-u_x||^2 - 2\gamma\frac{d}{dt}||s||^2 + \rho_2||s_t||^2.$$

So, for $\epsilon_0 = 3\delta$ we have

$$\frac{d}{dt}L_1(t) \le -b||s_x||^2 - \delta||s||^2 + C_{\epsilon_0}||\psi-u_x||^2 + \rho_2||s_t||^2.$$

Lemma 3.2. *Defining*

$$L_2(t) = \langle\rho_2(s-\psi)_t, (s-\psi)\rangle + \beta||(s-\psi)||^2,$$

we have

$$\frac{d}{dt}L_2(t) \le -\frac{b}{4}||(s-\psi)_x||^2 - \frac{b}{8}||\psi_x||^2 + \frac{b}{2}||s_x||^2 + C_{\epsilon_1}||\psi-u_x||^2 + \rho_2||(s-\psi)_t||^2.$$

Proof. Using (1.16) we obtain

$$\begin{aligned}\frac{d}{dt}\langle\rho_2(s-\psi)_t, (s-\psi)\rangle &= \langle\rho_2(s-\psi)_{tt}, (s-\psi)\rangle + \langle\rho_2(s-\psi)_t, (s-\psi)_t\rangle\\ &= \langle b(s-\psi)_{xx}, (s-\psi)\rangle + \langle k(\psi-u_x), (s-\psi)\rangle\\ &- \langle\beta(s-\psi)_t, (s-\psi)\rangle + \langle\rho_2(s-\psi)_t, (s-\psi)_t\rangle\\ &= -\langle b(s-\psi)_x, (s-\psi)_x\rangle + \langle k(\psi-u_x), (s-\psi)\rangle\\ &- \langle\beta(s-\psi)_t, (s-\psi)\rangle + \langle\rho_2(s-\psi)_t, (s-\psi)_t\rangle\\ &= -b||(s-\psi)_x||^2 + \langle k(\psi-u_x), (s-\psi)\rangle - \frac{1}{2}\frac{d}{dt}\beta||s-\psi||^2 + \rho_2||(s-\psi)_t||^2.\end{aligned}$$

Using Young's inequality we obtain

$$\frac{d}{dt}\langle\rho_2(s-\psi)_t, (s-\psi)\rangle \le -b||(s-\psi)_x||^2 + \epsilon_1||(s-\psi)||^2 + C_{\epsilon_1}||\psi-u_x||^2 - \frac{1}{2}\frac{d}{dt}\beta||(s-\psi)||^2 + \rho_2||(s-\psi)_t||^2,$$

Now we applied Poincare's inequality and we obtain

$$\frac{d}{dt}\langle\rho_2(s-\psi)_t, (s-\psi)\rangle \le -b||(s-\psi)_x||^2 + \epsilon_1 C_p||(s-\psi)_x||^2 + C_{\epsilon_1}||\psi-u_x||^2 - \frac{1}{2}\frac{d}{dt}\beta||(s-\psi)||^2 + \rho_2||(s-\psi)_t||^2,$$

and then

$$\frac{d}{dt}\langle\rho_2(s-\psi)_t, (s-\psi)\rangle \le -(b-\epsilon_1 C_p)||(s-\psi)_x||^2 + C_{\epsilon_1}||\psi-u_x||^2 - \frac{1}{2}\frac{d}{dt}\beta||(s-\psi)||^2 + \rho_2||(s-\psi)_t||^2,$$

So, for $\epsilon_1 = \frac{b}{2Cp}$ we have

$$\frac{d}{dt}L_2(t) \le -\frac{b}{2}||(s-\psi)_x||^2 + C_{\epsilon_1}||\psi-u_x||^2 + \rho_2||(s-\psi)_t||^2.$$

We write

$$\frac{d}{dt}L_2(t) \le -\frac{b}{4}||(s-\psi)_x||^2 - \frac{b}{4}||(s-\psi)_x||^2 + C_{\epsilon_1}||\psi-u_x||^2 + \rho_2||(s-\psi)_t||^2,$$

and using

$$-||(s-\psi)_x||^2 \leq -\frac{1}{2}||\psi_x||^2 + 2||s_x||^2,$$

we obtain

$$\frac{d}{dt}L_2(t) \leq -\frac{b}{4}||(s-\psi)_x||^2 - \frac{b}{8}||\psi_x||^2 + \frac{b}{2}||s_x||^2 + C_{\epsilon_1}||\psi - u_x||^2 + \rho_2||(s-\psi)_t||^2.$$

Lemma 3.3. *Defining*

$$L_3(t) = \langle \rho_1 u_t, u\rangle + \frac{1}{2}\alpha||u||^2,$$

$$\frac{d}{dt}L_3(t) \leq -\frac{k}{2}||\psi - u_x||^2 + \frac{k}{2}C_p||\psi_x||^2 + \epsilon||g * u_x||^2 + (C_\epsilon + 1)||u_x||^2 + \rho_1||u_t||^2.$$

Proof. Using (1.15) we obtain

$$\begin{aligned}
\frac{d}{dt}\langle \rho_1 u_t, u\rangle &= \langle \rho_1 u_{tt}, u\rangle + \langle \rho_1 u_t, u_t\rangle \\
&= -\langle k(\psi - u_x)_x, u\rangle - \langle \alpha u_t, u\rangle - \langle \alpha g * u_{xx}, u\rangle - \langle u_{xx}, u\rangle + \langle \rho_1 u_t, u_t\rangle \\
&= \langle k(\psi - u_x), u_x\rangle - \frac{1}{2}\frac{d}{dt}\alpha||u||^2 + \langle \alpha g * u_x, u_x\rangle + \langle u_x, u_x\rangle + \rho_1||u_t||^2 \\
&= \langle k(\psi - u_x), u_x - \psi + \psi\rangle - \frac{1}{2}\frac{d}{dt}\alpha||u||^2 + \langle \alpha g * u_x, u_x\rangle + ||u_x||^2 + \rho_1||u_t||^2 \\
&= -k||\psi - u_x||^2 + \langle k(\psi - u_x), \psi\rangle - \frac{1}{2}\frac{d}{dt}\alpha||u||^2 + \langle \alpha g * u_x, u_x\rangle + ||u_x||^2 + \rho_1||u_t||^2
\end{aligned}$$

Using Young's inequality, we obtain

$$\frac{d}{dt}\langle \rho_1 u_t, u\rangle \leq -k||\psi - u_x||^2 + \frac{k}{2}||\psi - u_x||^2 + \frac{k}{2}||\psi||^2 - \frac{1}{2}\frac{d}{dt}\alpha||u||^2 + \epsilon||g * u_x||^2 + C_\epsilon||u_x||^2 + ||u_x||^2 + \rho_1||u_t||^2,$$

and using Poicare's inequality we have

$$\frac{d}{dt}\langle \rho_1 u_t, u\rangle \leq -k||\psi - u_x||^2 + \frac{k}{2}||\psi - u_x||^2 + \frac{k}{2}C_p||\psi_x||^2 - \frac{1}{2}\frac{d}{dt}\alpha||u||^2 + \epsilon||g * u_x||^2 + C_\epsilon||u_x||^2 + ||u_x||^2 + \rho_1||u_t||^2,$$

from where follows that

$$\frac{d}{dt}L_3(t) \leq -\frac{k}{2}||\psi - u_x||^2 + \frac{k}{2}C_p||\psi_x||^2 + \epsilon||g * u_x||^2 + (C_\epsilon + 1)||u_x||^2 + \rho_1||u_t||^2.$$

4. Exponential stability

Now we are in position to prove our principal result. We consider the kernel of type (1.11) and we look for the exponential decay of the full energy $E(t)$ using the lemmas previous to building a Lyapunov functional. So we have the following theorem.

Theorem 4.1. *The problem (1.15)-(1.21) is exponentially stable, that is,*

$$E(t) \leq C\, E(0)\, e^{-\,\omega\, t}, \text{ for some } C > 0,\, \omega > 0.$$

Proof. We have

$$\frac{d}{dt}E(t) = -3\alpha||u_t|| - 4\gamma||s_t||^2 - 3\beta||(s-\psi)_t||^2 + \frac{3}{2}\int_0^1 g'\Box u_x\,dx - \frac{3}{2}g(t)||u_x||^2.$$

$$\frac{d}{dt}L_1(t) \leq -b||s_x||^2 - \delta||s||^2 + C_{\epsilon_0}||\psi - u_x||^2 + \rho_2||s_t||^2.$$

$$\frac{d}{dt}L_2(t) \leq -\frac{b}{4}||(s-\psi)_x||^2 - \frac{b}{8}||\psi_x||^2 + \frac{b}{2}||s_x||^2 + C_{\epsilon_1}||\psi - u_x||^2 + \rho_2||(s-\psi)_t||^2.$$

$$\frac{d}{dt}L_3(t) \leq -\frac{k}{2}||\psi - u_x||^2 + \frac{k}{2}C_p||\psi_x||^2 + \epsilon||g * u_x||^2 + (C_\epsilon + 1)||u_x||^2 + \rho_1||u_t||^2.$$

Let $N_1 > N_2 > N_3 > 0$ be positive constants. We define the functional of Lyapunov by

$$\mathcal{L}(t) = N_1\,E(t) + L_1(t) + N_2\,L_2(t) + N_3\,L_3(t),$$

and we have for ϵ absolutely small and N_3 big enough that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\alpha_0\,E(t).$$

Moreover, there are positive constants α_1, α_2 such that for $t \geq 0$

$$\alpha_1\,E(t) \leq \mathcal{L}(t) \leq \alpha_2\,E(t)$$

whence

$$\frac{d}{dt}\mathcal{L}(t) \leq -\omega\,\mathcal{L}(t)$$

for $\omega = \alpha_0/\alpha_2$, and hence our conclusion follows.

Conclusion

From the point of view of the asymptotic behavior, we have an interesting difference between Timoshenko's beam and the models of structure with interfacial slip. The frictional damping has the same influence on both models while the viscosity introduced by the memory effect is different in each system. The action of Memory stabilizes the Timoshenko's system only when the speed of the waves are equal, while, for structure with interfacial slip no condition is necessary on the coefficients for to obtain exponential decay.

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