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A FEW RESULTS ON RELATIVE RITT TYPE AND RELATIVE RITT WEAK TYPE OF ENTIRE FUNCTIONS REPRESENTED BY VECTOR VALUED DIRICHLET SERIES

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Abstract. In this paper we wish to study some growth properties of entire functions represented by a vector valued Dirichlet series on the basis of relative Ritt type and relative Ritt weak type.

1. Introduction, Definitions and Notations

Let $f(s)$ be an entire function of the complex variable $s = \sigma + it$ (σ and t are real variables) defined by everywhere absolutely convergent vector valued Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} \tag{1.1}$$

where a_n's belong to a Banach space $(E, \|.\|)$ and λ_n's are non-negative real numbers such that $0 < \lambda_n < \lambda_{n+1}\,(n \geq 1)$, $\lambda_n \to \infty$ as $n \to \infty$ and satisfy the conditions

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = D < \infty$$

and

$$\limsup_{n\to\infty} \frac{\log \|a_n\|}{\lambda_n} = -\infty \ .$$

If σ_c and σ_a denote respectively the abscissa of convergence and absolute convergence of (1.1), then in this case clearly $\sigma_a = \sigma_c = \infty$.

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The function $M_f(\sigma)$ known as maximum modulus function corresponding to an entire function $f(s)$ defined by (1.1), is written as follows

$$M_f(\sigma) = \underset{-\infty<t<\infty}{l.u.b.} \|f(\sigma + it)\| .$$

However, the Ritt order [3] of $f(s)$, denoted by ρ_f which is generally used in computational purpose is defined in terms of the growth of $f(s)$ with respect to the $\exp\exp z$ function as

$$\rho_f = \limsup_{\sigma\to\infty} \frac{\log\log M_f(\sigma)}{\log\log M_{\exp\exp z}(\sigma)} = \limsup_{\sigma\to\infty} \frac{\log\log M_f(\sigma)}{\sigma} .$$

Similarly, one can define the Ritt lower order of $f(s)$, denoted by λ_f in the following manner:

$$\lambda_f = \liminf_{\sigma\to\infty} \frac{\log\log M_f(\sigma)}{\sigma} .$$

An entire function represented by vector valued Dirichlet series for which order and lower order are the same is said to be of regular growth. Functions which are not of regular growth are said to be of irregular growth.

To compare the relative growth of two entire functions having same non zero finite Ritt order, one may introduce the definition of Ritt type and Ritt lower type in the following manner:

Definition 1.1. The Ritt type and Ritt lower type respectively denoted by Δ_f and $\overline{\Delta}_f$ of an entire function f are defined as follows:

$$\Delta_f = \limsup_{\sigma\to\infty} \frac{\log M_f(\sigma)}{\exp\left[\rho_f \cdot \sigma\right]} \quad \text{and} \quad \overline{\Delta}_f = \liminf_{\sigma\to\infty} \frac{\log M_f(\sigma)}{\exp\left[\rho_f \cdot \sigma\right]},\ 0 < \rho_f < \infty .$$

Analogously to determine the relative growth of two entire functions represented by vector valued Dirichlet series having same non zero finite Ritt-lower order, one may introduce the definition of Ritt-weak type in the following way:

Definition 1.2. The Ritt-weak type denoted by τ_f of an entire function f represented by vector valued Dirichlet series is defined as follows:

$$\tau_f = \liminf_{\sigma\to\infty} \frac{\log M_f(\sigma)}{\exp\left[\lambda_f \cdot \sigma\right]} ,\ 0 < \lambda_f < \infty .$$

Also one may define the growth indicator $\overline{\tau}_f$ of an entire function f represented by vector valued Dirichlet series in the following manner :

$$\overline{\tau}_f = \limsup_{\sigma\to\infty} \frac{\log M_f(\sigma)}{\exp\left[\lambda_f \cdot \sigma\right]} ,\ 0 < \lambda_f < \infty .$$

During the past decades, several authors {cf. [3],[4],[5],[7],[9]} made closed investigations on the properties of entire Dirichlet series related to Ritt order. Further, B. L. Srivastava [8] defined different growth parameters such as order and lower order of entire functions represented by vector valued Dirichlet series. He also obtained the results for coefficient characterization of order.

G. S. Srivastava [6]} introduced the relative Ritt order between two entire functions represented by vector valued Dirichlet series to avoid comparing growth just with $\exp\exp z$ which is as follows:

$$\begin{aligned}\rho_g(f) &= \inf\left\{\mu > 0 : M_f(\sigma) < M_g(\sigma\mu) \text{ for all } \sigma > \sigma_0(\mu)\right\}\\ &= \limsup_{\sigma\to\infty} \frac{M_g^{-1}M_f(\sigma)}{\sigma}\ .\end{aligned}$$

Similarly, one can define the relative Ritt lower order of $f(s)$ with respect to $g(s)$, denoted by $\lambda_g(f)$ in the following manner:

$$\lambda_g(f) = \liminf_{\sigma\to\infty} \frac{M_g^{-1}M_f(\sigma)}{\sigma}\ .$$

Further to compare the relative growth of two entire functions represented by vector valued Dirichlet series having same non zero finite relative Ritt order with respect to another entire function represented by vector valued Dirichlet series, Datta et al. [1] introduced the definitions of relative Ritt-type and relative Ritt-lower type in the following manner:

Definition 1.3. [1] The relative Ritt-type and relative Ritt-lower type denoted respectively by $\Delta_g(f)$ and $\overline{\Delta}_g(f)$ of an entire function f with respect to another entire function g both represented by vector valued Dirichlet series are respectively defined as follows:

$$\begin{aligned}\Delta_g(f) &= \limsup_{\sigma\to\infty} \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\rho_g(f)\cdot\sigma\right]} \text{ and}\\ \overline{\Delta}_g(f) &= \liminf_{\sigma\to\infty} \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\rho_g(f)\cdot\sigma\right]},\ 0 < \rho_g(f) < \infty\ .\end{aligned}$$

Further to determine the relative growth of two entire functions represented by vector valued Dirichlet series having same non zero finite relative Ritt-lower order with respect to another entire function represented by vector valued Dirichlet series Datta et al. [1] also introduced the definition of relative Ritt-weak type in the following way:

Definition 1.4. [1] The relative Ritt-weak type denoted by $\tau_g(f)$ of an entire function f with respect to another entire function g both represented by vector valued Dirichlet series is defined as follows:

$$\tau_g(f) = \liminf_{\sigma\to\infty} \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\lambda_g(f)\cdot\sigma\right]}\ ,\ 0 < \lambda_g(f) < \infty\ .$$

Also one may define the growth indicator $\overline{\tau}_g(f)$ of an entire function f with respect to another entire function g both represented by vector valued Dirichlet series in the following manner :

$$\overline{\tau}_g(f) = \limsup_{\sigma\to\infty} \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\lambda_g(f)\cdot\sigma\right]}\ ,\ 0 < \lambda_g(f) < \infty\ .$$

In the paper we study some relative growth properties of entire functions represented by vector valued Dirichlet series using relative Ritt-order, relative Ritt-type and relative Ritt-weak type.

2. Lemmas

In this section we present a lemma due to Datta et al. [2]:

Lemma 2.1. [2] Let f and g be any two entire functions represented by vector valued Dirichlet series such that $0 < \lambda_f \le \rho_f < \infty$ and $0 < \lambda_g \le \rho_g < \infty$. Then

$$\frac{\lambda_f}{\rho_g} \le \lambda_g(f) \le \min\left\{\frac{\lambda_f}{\lambda_g}, \frac{\rho_f}{\rho_g}\right\} \le \max\left\{\frac{\lambda_f}{\lambda_g}, \frac{\rho_f}{\rho_g}\right\} \le \rho_g(f) \le \frac{\rho_f}{\lambda_g} .$$

3. Main Results

In this section we state the main results of the paper.

Theorem 3.1. Let f and g be any two entire functions represented by vector valued Dirichlet series such that $0 < \lambda_f \le \rho_f < \infty$ and $0 < \lambda_g \le \rho_g < \infty$. Then

$$\max\left\{\left[\frac{\overline{\Delta}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\Delta_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\right\} \le \Delta_g(f) \le \min\left\{\left[\frac{\overline{\tau}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\Delta_f}{\overline{\Delta}_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\overline{\tau}_f}{\overline{\Delta}_g}\right]^{\frac{1}{\rho_g}}\right\}$$

and

$$\left[\frac{\overline{\Delta}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} \le \overline{\Delta}_g(f) \le \min\left\{\left[\frac{\overline{\Delta}_f}{\overline{\Delta}_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\Delta_f}{\Delta_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\tau_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\overline{\Delta}_f}{\overline{\Delta}_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\overline{\tau}_f}{\Delta_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\tau_f}{\overline{\Delta}_g}\right]^{\frac{1}{\rho_g}}\right\} .$$

Proof. From the definitions of Δ_f and $\overline{\Delta}_f$, we have for all sufficiently large values of σ that

$$M_f(\sigma) \le \exp\left[\left(\Delta_f + \varepsilon\right)\exp\left[\rho_f \cdot \sigma\right]\right], \tag{3.2}$$

$$M_f(\sigma) \ge \exp\left[\left(\overline{\Delta}_f - \varepsilon\right)\exp\left[\rho_f \cdot \sigma\right]\right] \tag{3.3}$$

and also for a sequence of values of σ tending to infinity, we get that

$$M_f(\sigma) \ge \exp\left[\left(\Delta_f - \varepsilon\right)\exp\left[\rho_f \cdot \sigma\right]\right], \tag{3.4}$$

$$M_f(\sigma) \le \exp\left[\left(\overline{\Delta}_f + \varepsilon\right)\exp\left[\rho_f \cdot \sigma\right]\right] . \tag{3.5}$$

Similarly from the definitions of Δ_g and $\overline{\Delta}_g$, it follows for all sufficiently large values of σ that

$$M_g(\sigma) \le \exp\left[\left(\Delta_g + \varepsilon\right)\exp\left[\rho_g \cdot \sigma\right]\right]$$

$$\textit{i.e.,} \qquad \sigma \le M_g^{-1}\left[\exp\left[\left(\Delta_g + \varepsilon\right)\exp\left[\rho_g \cdot \sigma\right]\right]\right]$$

$$\textit{i.e.,} \qquad M_g^{-1}(\sigma) \ge \left[\frac{\log\left(\frac{\log \sigma}{(\Delta_g + \varepsilon)}\right)}{\rho_g}\right], \tag{3.6}$$

$$
\begin{aligned}
& M_g(\sigma) \geq \exp\left[\left(\overline{\Delta}_g - \varepsilon\right)\exp\left[\rho_g \cdot \sigma\right]\right] \\
\textit{i.e.,} \quad & \sigma \geq M_g^{-1}\left[\exp\left[\left(\overline{\Delta}_g - \varepsilon\right)\exp\left[\rho_g \cdot \sigma\right]\right]\right] \\
\textit{i.e.,} \quad & M_g^{-1}(\sigma) \leq \left[\frac{\log\left(\frac{\log \sigma}{(\overline{\Delta}_g - \varepsilon)}\right)}{\rho_g}\right]
\end{aligned}
\tag{3.7}
$$

and for a sequence of values of σ tending to infinity, we obtain that

$$
\begin{aligned}
& M_g(\sigma) \geq \exp\left[\left(\Delta_g - \varepsilon\right)\exp\left[\rho_g \cdot \sigma\right]\right] \\
\textit{i.e.,} \quad & \sigma \geq M_g^{-1}\left[\exp\left[\left(\Delta_g - \varepsilon\right)\exp\left[\rho_g \cdot \sigma\right]\right]\right] \\
\textit{i.e.,} \quad & M_g^{-1}(\sigma) \leq \left[\frac{\log\left(\frac{\log \sigma}{(\Delta_g - \varepsilon)}\right)}{\rho_g}\right],
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
& M_g(\sigma) \leq \exp\left[\left(\overline{\Delta}_g + \varepsilon\right)\exp\left[\rho_g \cdot \sigma\right]\right] \\
\textit{i.e.,} \quad & \sigma \leq M_g^{-1}\left[\exp\left[\left(\overline{\Delta}_g + \varepsilon\right)\exp\left[\rho_g \cdot \sigma\right]\right]\right] \\
\textit{i.e.,} \quad & M_g^{-1}(\sigma) \geq \left[\frac{\log\left(\frac{\log \sigma}{(\overline{\Delta}_g - \varepsilon)}\right)}{\rho_g}\right].
\end{aligned}
\tag{3.9}
$$

Further from the definitions of $\overline{\tau}_f$ and τ_f, we have for all sufficiently large values of σ that

$$
M_f(\sigma) \leq \exp\left[\left(\overline{\tau}_f + \varepsilon\right)\exp\left[\lambda_f \cdot \sigma\right]\right], \tag{3.10}
$$

$$
M_f(\sigma) \geq \exp\left[\left(\tau_f - \varepsilon\right)\exp\left[\lambda_f \cdot \sigma\right]\right] \tag{3.11}
$$

and also for a sequence of values of σ tending to infinity, we get that

$$
M_f(\sigma) \geq \exp\left[\left(\overline{\tau}_f - \varepsilon\right)\exp\left[\lambda_f \cdot \sigma\right]\right], \tag{3.12}
$$

$$
M_f(\sigma) \leq \exp\left[\left(\tau_f + \varepsilon\right)\exp\left[\lambda_f \cdot \sigma\right]\right]. \tag{3.13}
$$

Similarly from the definitions of $\overline{\tau}_g$ and τ_g, it follows for all sufficiently large values of σ that

$$
\begin{aligned}
& M_g(\sigma) \leq \exp\left[\left(\overline{\tau}_g + \varepsilon\right)\exp\left[\lambda_g \cdot \sigma\right]\right] \\
& \textit{i.e.,}\ \sigma \leq M_g^{-1}\left[\exp\left[\left(\overline{\tau}_g + \varepsilon\right)\exp\left[\lambda_g \cdot \sigma\right]\right]\right] \\
\textit{i.e.,} \quad & M_g^{-1}(\sigma) \geq \left[\frac{\log\left(\frac{\log \sigma}{(\overline{\tau}_g + \varepsilon)}\right)}{\lambda_g}\right],
\end{aligned}
\tag{3.14}
$$

$$\begin{aligned}
& M_g(\sigma) \geq \exp\left[\left(\tau_g - \varepsilon\right)\exp\left[\lambda_g \cdot \sigma\right]\right] \\
i.e., \quad & \sigma \geq M_g^{-1}\left[\exp\left[\left(\tau_g - \varepsilon\right)\exp\left[\lambda_g \cdot \sigma\right]\right]\right] \\
i.e., \quad & M_g^{-1}(\sigma) \leq \left[\frac{\log\left(\frac{\log \sigma}{(\tau_g - \varepsilon)}\right)}{\lambda_g}\right] \qquad (3.15)
\end{aligned}$$

and for a sequence of values of σ tending to infinity, we obtain that

$$\begin{aligned}
& M_g(\sigma) \geq \exp\left[\left(\overline{\tau}_g - \varepsilon\right)\exp\left[\lambda_g \cdot \sigma\right]\right] \\
i.e., \quad & \sigma \geq M_g^{-1}\left[\exp\left[\left(\overline{\tau}_g - \varepsilon\right)\exp\left[\lambda_g \cdot \sigma\right]\right]\right] \\
i.e., \quad & M_g^{-1}(\sigma) \leq \left[\frac{\log\left(\frac{\log \sigma}{(\overline{\tau}_g - \varepsilon)}\right)}{\lambda_g}\right], \qquad (3.16) \\
& M_g(\sigma) \leq \exp\left[\left(\tau_g + \varepsilon\right)\exp\left[\lambda_g \cdot \sigma\right]\right] \\
i.e., \quad & \sigma \leq M_g^{-1}\left[\exp\left[\left(\tau_g + \varepsilon\right)\exp\left[\lambda_g \cdot \sigma\right]\right]\right] \\
i.e., \quad & M_g^{-1}(\sigma) \geq \left[\frac{\log\left(\frac{\log \sigma}{(\tau_g - \varepsilon)}\right)}{\lambda_g}\right]. \qquad (3.17)
\end{aligned}$$

Now from (3.4) and in view of (3.14), we get for a sequence of values of σ tending to infinity that

$$\begin{aligned}
& M_g^{-1}M_f(\sigma) \geq M_g^{-1}\left[\exp\left[\left(\Delta_f - \varepsilon\right)\exp\left[\rho_f \cdot \sigma\right]\right]\right] \\
i.e., \quad & M_g^{-1}M_f(\sigma) \geq \left[\frac{\log\left(\frac{\log\exp\left[\left(\Delta_f - \varepsilon\right)\exp\left[\rho_f \cdot \sigma\right]\right]}{(\overline{\tau}_g + \varepsilon)}\right)}{\lambda_g}\right] \\
i.e., \quad & M_g^{-1}M_f(\sigma) \geq \log\left[\left[\frac{\left(\Delta_f - \varepsilon\right)}{\left(\overline{\tau}_g + \varepsilon\right)}\right]^{\frac{1}{\lambda_g}} \cdot \exp\left[\frac{\rho_f}{\lambda_g} \cdot \sigma\right]\right] \\
i.e., \quad & \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\frac{\rho_f}{\lambda_g} \cdot \sigma\right]} \geq \left[\frac{\left(\Delta_f - \varepsilon\right)}{\left(\overline{\tau}_g + \varepsilon\right)}\right]^{\frac{1}{\lambda_g}}.
\end{aligned}$$

Since in view of Lemma 2.1, $\frac{\rho_f}{\lambda_g} \geq \rho_g(f)$ and as $\varepsilon(>0)$ is arbitrary, therefore it follows from above that

$$\limsup_{\sigma\to\infty} \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\rho_g(f) \cdot \sigma\right]} \geq \left[\frac{\Delta_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}$$

$$i.e., \ \Delta_g(f) \geq \left[\frac{\Delta_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} . \tag{3.18}$$

Similarly from (3.3) and in view of (3.17), it follows for a sequence of values of σ tending to infinity that

$$\begin{aligned}
& M_g^{-1} M_f(\sigma) \geq M_g^{-1}\left[\exp\left[\left(\overline{\Delta}_f - \varepsilon\right)\exp\left[\rho_f \cdot \sigma\right]\right]\right] \\
i.e., \quad & M_g^{-1} M_f(\sigma) \geq \left[\frac{\log\left(\frac{\log\exp\left[\left(\overline{\Delta}_f - \varepsilon\right)\exp\left[\rho_f \cdot \sigma\right]\right]}{\left(\tau_g - \varepsilon\right)}\right)}{\lambda_g}\right] \\
i.e., \quad & M_g^{-1} M_f(\sigma) \geq \log\left[\left[\frac{\left(\overline{\Delta}_f - \varepsilon\right)}{\left(\tau_g + \varepsilon\right)}\right]^{\frac{1}{\lambda_g}} \cdot \exp\left[\frac{\rho_f}{\lambda_g} \cdot \sigma\right]\right] \\
i.e., \quad & \frac{\exp M_g^{-1} M_f(\sigma)}{\exp\left[\frac{\rho_f}{\lambda_g} \cdot \sigma\right]} \geq \left[\frac{\left(\overline{\Delta}_f - \varepsilon\right)}{\left(\tau_g + \varepsilon\right)}\right]^{\frac{1}{\lambda_g}} .
\end{aligned}$$

Since in view of Lemma 2.1, it follows that $\frac{\rho_f}{\lambda_g} \geq \rho_g(f)$. Also $\varepsilon(>0)$ is arbitrary, so we get from above that

$$\begin{aligned}
& \limsup_{\sigma\to\infty} \frac{\exp M_g^{-1} M_f(\sigma)}{\exp\left[\rho_g(f) \cdot \sigma\right]} \geq \left[\frac{\overline{\Delta}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}} \\
i.e., \quad & \Delta_g(f) \geq \left[\frac{\overline{\Delta}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}} .
\end{aligned} \tag{3.19}$$

Again in view of (3.15), we have from (3.10) for all sufficiently large values of σ that

$$\begin{aligned}
& M_g^{-1} M_f(\sigma) \leq M_g^{-1}\left[\exp\left[\left(\overline{\tau}_f + \varepsilon\right)\exp\left[\lambda_f \cdot \sigma\right]\right]\right] \\
i.e., \quad & M_g^{-1} M_f(\sigma) \leq \left[\frac{\log\left(\frac{\log\exp\left[\left(\overline{\tau}_f + \varepsilon\right)\exp\left[\lambda_f \cdot \sigma\right]\right]}{\left(\tau_g - \varepsilon\right)}\right)}{\lambda_g}\right] \\
i.e., \quad & M_g^{-1} M_f(\sigma) \leq \log\left[\left[\frac{\left(\overline{\tau}_f + \varepsilon\right)}{\left(\tau_g - \varepsilon\right)}\right]^{\frac{1}{\lambda_g}} \cdot \exp\left[\frac{\lambda_f}{\lambda_g} \cdot \sigma\right]\right] \\
i.e., \quad & \frac{\exp M_g^{-1} M_f(\sigma)}{\exp\left[\frac{\lambda_f}{\lambda_g} \cdot \sigma\right]} \leq \left[\frac{\left(\overline{\tau}_f + \varepsilon\right)}{\left(\tau_g - \varepsilon\right)}\right]^{\frac{1}{\lambda_g}} .
\end{aligned}$$

Since in view of Lemma 2.1, we get that $\frac{\lambda_f}{\lambda_g} \leq \rho_g(f)$ and as $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\limsup_{\sigma\to\infty} \frac{\exp M_g^{-1} M_f(\sigma)}{\exp\left[\rho_g(f)\cdot\sigma\right]} \leq \left[\frac{\overline{\tau}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}$$

$$\text{i.e.,} \qquad \Delta_g(f) \leq \left[\frac{\overline{\tau}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}} . \tag{3.20}$$

Again in view of (3.7), we have from (3.2) for all sufficiently large values of σ that

$$M_g^{-1} M_f(\sigma) \leq M_g^{-1}\left[\exp\left[\left(\Delta_f+\varepsilon\right)\exp\left[\rho_f\cdot\sigma\right]\right]\right]$$

$$\text{i.e.,} M_g^{-1} M_f(\sigma) \leq \left[\frac{\log\left(\frac{\log\exp\left[\left(\Delta_f+\varepsilon\right)\exp\left[\rho_f\cdot\sigma\right]\right]}{\left(\overline{\Delta}_g-\varepsilon\right)}\right)}{\rho_g}\right]$$

$$\text{i.e.,} \; M_g^{-1} M_f(\sigma) \leq \log\left[\left[\frac{\left(\Delta_f+\varepsilon\right)}{\left(\overline{\Delta}_g-\varepsilon\right)}\right]^{\frac{1}{\rho_g}}\cdot\exp\left[\frac{\rho_f}{\rho_g}\cdot\sigma\right]\right]$$

$$\text{i.e.,} \; \frac{\exp M_g^{-1} M_f(\sigma)}{\exp\left[\frac{\rho_f}{\rho_g}\cdot\sigma\right]} \leq \left[\frac{\left(\Delta_f+\varepsilon\right)}{\left(\overline{\Delta}_g-\varepsilon\right)}\right]^{\frac{1}{\rho_g}} . \tag{3.21}$$

As in view of Lemma 2.1, it follows that $\frac{\rho_f}{\rho_g} \leq \rho_g(f)$ and since $\varepsilon(>0)$ is arbitrary, we get from (3.21) that

$$\limsup_{\sigma\to\infty} \frac{\exp M_g^{-1} M_f(\sigma)}{\exp\left[\rho_g(f)\cdot\sigma\right]} \leq \left[\frac{\Delta_f}{\overline{\Delta}_g}\right]^{\frac{1}{\rho_g}}$$

$$\text{i.e.,} \; \Delta_g(f) \leq \left[\frac{\Delta_f}{\overline{\Delta}_g}\right]^{\frac{1}{\rho_g}} . \tag{3.22}$$

Further in view of (3.7), we have from (3.10) for all sufficiently large values of σ that

$$M_g^{-1} M_f(\sigma) \qquad \leq M_g^{-1}\left[\exp\left[\left(\overline{\tau}_f+\varepsilon\right)\exp\left[\lambda_f\cdot\sigma\right]\right]\right] \tag{3.23}$$

$$\text{i.e.,} \qquad M_g^{-1} M_f(\sigma) \leq \left[\frac{\log\left(\frac{\log\exp\left[\left(\overline{\tau}_f+\varepsilon\right)\exp\left[\lambda_f\cdot\sigma\right]\right]}{\left(\overline{\Delta}_g-\varepsilon\right)}\right)}{\rho_g}\right]$$

$$i.e., \quad M_g^{-1}M_f(\sigma) \le \log\left[\left[\frac{\left(\overline{\tau}_f+\varepsilon\right)}{\left(\overline{\Delta}_g-\varepsilon\right)}\right]^{\frac{1}{\rho_g}}\cdot\exp\left[\frac{\lambda_f}{\rho_g}\cdot\sigma\right]\right]$$

$$i.e., \quad \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\frac{\lambda_f}{\rho_g}\cdot\sigma\right]} \le \left[\frac{\left(\overline{\tau}_f+\varepsilon\right)}{\left(\overline{\Delta}_g-\varepsilon\right)}\right]^{\frac{1}{\rho_g}} .$$

Since in view of Lemma 2.1, we get that $\frac{\lambda_f}{\rho_g} \le \rho_g(f)$ and as $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\limsup_{\sigma\to\infty}\frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\rho_g(f)\cdot\sigma\right]} \le \left[\frac{\overline{\tau}_f}{\overline{\Delta}_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e.,\ \Delta_g(f) \le \left[\frac{\overline{\tau}_f}{\overline{\Delta}_g}\right]^{\frac{1}{\rho_g}} . \tag{3.24}$$

Thus the first part of the theorem follows from (3.18), (3.19), (3.20), (3.22) and (3.24).

Further from (3.3) and in view of (3.14), we get for all sufficiently large values of σ that

$$M_g^{-1}M_f(\sigma) \ge M_g^{-1}\left[\exp\left[\left(\overline{\Delta}_f-\varepsilon\right)\exp\left[\rho_f\cdot\sigma\right]\right]\right]$$

$$i.e., \quad M_g^{-1}M_f(\sigma) \ge \left[\frac{\log\left(\frac{\log\exp\left[\left(\overline{\Delta}_f-\varepsilon\right)\exp\left[\rho_f\cdot\sigma\right]\right]}{\left(\overline{\tau}_g+\varepsilon\right)}\right)}{\lambda_g}\right]$$

$$i.e., \quad M_g^{-1}M_f(\sigma) \ge \log\left[\left[\frac{\left(\overline{\Delta}_f-\varepsilon\right)}{\left(\overline{\tau}_g+\varepsilon\right)}\right]^{\frac{1}{\lambda_g}}\cdot\exp\left[\frac{\rho_f}{\lambda_g}\cdot\sigma\right]\right]$$

$$i.e. \quad , \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\frac{\rho_f}{\lambda_g}\cdot\sigma\right]} \ge \left[\frac{\left(\overline{\Delta}_f-\varepsilon\right)}{\left(\overline{\tau}_g+\varepsilon\right)}\right]^{\frac{1}{\lambda_g}} .$$

Now in view of Lemma 2.1, it follows that $\frac{\rho_f}{\lambda_g} \ge \rho_g(f)$. Since $\varepsilon(>0)$ is arbitrary, we get from above that

$$\liminf_{\sigma\to\infty}\frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\rho_g(f)\cdot\sigma\right]} \ge \left[\frac{\overline{\Delta}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}$$

$$i.e.,\ \overline{\Delta}_g(f) \ge \left[\frac{\overline{\Delta}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} . \tag{3.25}$$

Also in view of (3.8), we get from (3.2) for a sequence of values of σ tending to infinity that

$$M_g^{-1}M_f(\sigma) \leq M_g^{-1}\left[\exp\left[\left(\Delta_f+\varepsilon\right)\exp\left[\rho_f\cdot\sigma\right]\right]\right]$$

$$i.e., \qquad M_g^{-1}M_f(\sigma) \leq \left[\frac{\log\left(\frac{\log\exp\left[\left(\Delta_f+\varepsilon\right)\exp\left[\rho_f\cdot\sigma\right]\right]}{\left(\Delta_g-\varepsilon\right)}\right)}{\rho_g}\right]$$

$$i.e., \qquad M_g^{-1}M_f(\sigma) \leq \log\left[\left[\frac{\left(\Delta_f+\varepsilon\right)}{\left(\Delta_g-\varepsilon\right)}\right]^{\frac{1}{\rho_g}}\cdot\exp\left[\frac{\rho_f}{\rho_g}\cdot\sigma\right]\right]$$

$$i.e., \qquad \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\frac{\rho_f}{\rho_g}\cdot\sigma\right]} \leq \left[\frac{\left(\Delta_f+\varepsilon\right)}{\left(\Delta_g-\varepsilon\right)}\right]^{\frac{1}{\rho_{g(m,p)}}}. \tag{3.26}$$

Again in view of Lemma 2.1, $\frac{\rho_f}{\rho_g} \leq \rho_g(f)$ and $\varepsilon(>0)$ is arbitrary, so we get from (3.26) that

$$\liminf_{\sigma\to\infty}\frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\rho_g(f)\cdot\sigma\right]} \leq \left[\frac{\Delta_f}{\Delta_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e., \ \overline{\Delta}_g(f) \leq \left[\frac{\Delta_f}{\Delta_g}\right]^{\frac{1}{\rho_g}}. \tag{3.27}$$

Likewise from (3.5) and in view of (3.7), it follows for a sequence of values of σ tending to infinity that

$$M_g^{-1}M_f(\sigma) \leq M_g^{-1}\left[\exp\left[\left(\overline{\Delta}_f+\varepsilon\right)\exp\left[\rho_f\cdot\sigma\right]\right]\right]$$

$$i.e., \qquad M_g^{-1}M_f(\sigma) \leq \left[\frac{\log\left(\frac{\log\exp\left[\left(\overline{\Delta}_f+\varepsilon\right)\exp\left[\rho_f\cdot\sigma\right]\right]}{\left(\overline{\Delta}_g-\varepsilon\right)}\right)}{\rho_g}\right]$$

$$i.e., \qquad M_g^{-1}M_f(\sigma) \leq \log\left[\left[\frac{\left(\overline{\Delta}_f+\varepsilon\right)}{\left(\overline{\Delta}_g-\varepsilon\right)}\right]^{\frac{1}{\rho_g}}\cdot\exp\left[\frac{\rho_f}{\rho_g}\cdot\sigma\right]\right]$$

$$i.e., \qquad \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\frac{\rho_f}{\rho_g}\cdot\sigma\right]} \leq \left[\frac{\left(\overline{\Delta}_f+\varepsilon\right)}{\left(\overline{\Delta}_g-\varepsilon\right)}\right]^{\frac{1}{\rho_g}}. \tag{3.28}$$

Analogously, we get from (3.28) that

$$\liminf_{\sigma\to\infty}\frac{M_g^{-1}M_f(\sigma)}{\exp\left[\rho_g(f)\cdot\sigma\right]}\le\left[\frac{\overline{\Delta}_f}{\overline{\Delta}_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e.,\ \overline{\sigma}_g(f)\le\left[\frac{\overline{\Delta}_f}{\overline{\Delta}_g}\right]^{\frac{1}{\rho_g}}, \tag{3.29}$$

since in view of Lemma 2.1, $\frac{\rho_f}{\rho_g}\le\rho_g(f)$ and $\varepsilon(>0)$ is arbitrary.

Further in view of (3.16), we get from (3.10) for a sequence of values of σ tending to infinity that

$$M_g^{-1}M_f(\sigma)\le M_g^{-1}\left[\exp\left[\left(\overline{\tau}_f+\varepsilon\right)\exp\left[\lambda_f\cdot\sigma\right]\right]\right]$$

$$i.e.,\qquad M_g^{-1}M_f(\sigma)\le\left[\frac{\log\left(\frac{\log\exp\left[\left(\overline{\tau}_f+\varepsilon\right)\exp\left[\lambda_f\cdot\sigma\right]\right]}{\left(\overline{\tau}_g-\varepsilon\right)}\right)}{\lambda_g}\right]$$

$$i.e.,\qquad M_g^{-1}M_f(\sigma)\le\log\left[\left[\frac{\left(\overline{\tau}_f+\varepsilon\right)}{\left(\overline{\tau}_g-\varepsilon\right)}\right]^{\frac{1}{\lambda_g}}\cdot\exp\left[\frac{\lambda_f}{\lambda_g}\cdot\sigma\right]\right]$$

$$i.e.,\qquad \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\frac{\lambda_f}{\lambda_g}\cdot\sigma\right]}\le\left[\frac{\left(\overline{\tau}_f+\varepsilon\right)}{\left(\overline{\tau}_g-\varepsilon\right)}\right]^{\frac{1}{\lambda_g}}.$$

As in view of Lemma 2.1, we get that $\frac{\lambda_f}{\lambda_g}\le\rho_g(f)$ and as $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\liminf_{\sigma\to\infty}\frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\rho_g(f)\cdot\sigma\right]}\le\left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}$$

$$i.e.,\ \overline{\Delta}_g(f)\le\left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}. \tag{3.30}$$

Similarly from (3.13) and in view of (3.15), it follows for a sequence of values of σ tending to infinity that

$$M_g^{-1}M_f(\sigma)\le M_g^{-1}\left[\exp\left[\left(\tau_f+\varepsilon\right)\exp\left[\lambda_f\cdot\sigma\right]\right]\right]$$

$$i.e., \quad M_g^{-1}M_f(\sigma) \le \left[\frac{\log\left(\frac{\log\exp\left[(\tau_f+\varepsilon)\exp[\lambda_f\cdot\sigma]\right]}{(\tau_g-\varepsilon)}\right)}{\lambda_g}\right]$$

$$i.e., \quad M_g^{-1}M_f(\sigma) \le \log\left[\left[\frac{(\tau_f+\varepsilon)}{(\tau_g-\varepsilon)}\right]^{\frac{1}{\lambda_g}}\cdot\exp\left[\frac{\lambda_f}{\lambda_g}\cdot\sigma\right]\right]$$

$$i.e., \quad \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\frac{\lambda_f}{\lambda_g}\cdot\sigma\right]} \le \left[\frac{(\tau_f+\varepsilon)}{(\tau_g-\varepsilon)}\right]^{\frac{1}{\lambda_g}}.$$

Also in view of Lemma 2.1, we get that $\frac{\lambda_f}{\lambda_g} \le \rho_g(f)$ and as $\varepsilon(>0)$ is arbitrary, therefore it follows from above that

$$\liminf_{\sigma\to\infty}\frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\rho_g(f)\cdot\sigma\right]} \le \left[\frac{\tau_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}$$

$$i.e., \ \overline{\Delta}_g(f) \le \left[\frac{\tau_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}. \tag{3.31}$$

Again in view of (3.8), we get from (3.10) for a sequence of values of σ tending to infinity that

$$M_g^{-1}M_f(\sigma) \le M_g^{-1}\left[\exp\left[\left(\overline{\tau}_f+\varepsilon\right)\exp\left[\lambda_f\cdot\sigma\right]\right]\right]$$

$$i.e., \quad M_g^{-1}M_f(\sigma) \le \left[\frac{\log\left(\frac{\log\exp\left[(\overline{\tau}_f+\varepsilon)\exp[\lambda_f\cdot\sigma]\right]}{(\Delta_g-\varepsilon)}\right)}{\rho_g}\right]$$

$$i.e., \quad M_g^{-1}M_f(\sigma) \le \log\left[\left[\frac{(\overline{\tau}_f+\varepsilon)}{(\Delta_g-\varepsilon)}\right]^{\frac{1}{\rho_g}}\cdot\exp\left[\frac{\lambda_f}{\rho_g}\cdot\sigma\right]\right]$$

$$i.e., \quad \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\frac{\lambda_f}{\rho_g}\cdot\sigma\right]} \le \left[\frac{(\overline{\tau}_f+\varepsilon)}{(\Delta_g-\varepsilon)}\right]^{\frac{1}{\rho_g}}.$$

Since in view of Lemma 2.1, we get that $\frac{\lambda_f}{\rho_g} \le \rho_g(f)$ and as $\varepsilon(>0)$ is arbitrary, so it follows from above that

$$\liminf_{\sigma\to\infty}\frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\rho_g(f)\cdot\sigma\right]} \le \left[\frac{\overline{\tau}_f}{\Delta_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e., \ \overline{\Delta}_g(f) \le \left[\frac{\overline{\tau}_f}{\Delta_g}\right]^{\frac{1}{\rho_g}}. \tag{3.32}$$

Similarly from (3.13) and in view of (3.7), it follows for a sequence of values of σ tending to infinity that

$$M_g^{-1}M_f(\sigma) \le M_g^{-1}\left[\exp\left[\left(\tau_f+\varepsilon\right)\exp\left[\lambda_f\cdot\sigma\right]\right]\right]$$

$$i.e., \quad M_g^{-1}M_f(\sigma) \le \left[\frac{\log\left(\frac{\log\exp\left[\left(\tau_f+\varepsilon\right)\exp\left[\lambda_f\cdot\sigma\right]\right]}{\left(\overline{\Delta}_g-\varepsilon\right)}\right)}{\rho_g}\right]$$

$$i.e., \quad M_g^{-1}M_f(\sigma) \le \log\left[\left[\frac{\left(\tau_f+\varepsilon\right)}{\left(\overline{\Delta}_g-\varepsilon\right)}\right]^{\frac{1}{\rho_g}}\cdot\exp\left[\frac{\lambda_f}{\rho_g}\cdot\sigma\right]\right]$$

$$i.e., \quad \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\frac{\lambda_f}{\rho_g}\cdot\sigma\right]} \le \left[\frac{\left(\tau_f+\varepsilon\right)}{\left(\overline{\Delta}_g-\varepsilon\right)}\right]^{\frac{1}{\rho_g}} .$$

As in view of Lemma 2.1, we get that $\frac{\lambda_f}{\rho_g} \le \rho_g(f)$ and as $\varepsilon(>0)$ is arbitrary, therefore it follows from above that

$$\liminf_{\sigma\to\infty}\frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\rho_g(f)\cdot\sigma\right]} \le \left[\frac{\tau_f}{\overline{\Delta}_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e., \ \overline{\Delta}_g(f) \le \left[\frac{\tau_f}{\overline{\Delta}_g}\right]^{\frac{1}{\rho_g}} . \tag{3.33}$$

Hence the second part of the theorem follows from (3.25), (3.27), (3.29), (3.30), (3.31), (3.32) and (3.33). □

Theorem 3.2. Let f and g be any two entire functions represented by vector valued Dirichlet series such that $0<\lambda_f\le\rho_f<\infty$ and $0<\lambda_g\le\rho_g<\infty$. Then

$$\max\left\{\left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}},\left[\frac{\tau_f}{\tau_g}\right]^{\frac{1}{\lambda_g}},\left[\frac{\overline{\Delta}_f}{\overline{\Delta}_g}\right]^{\frac{1}{\rho_g}},\left[\frac{\Delta_f}{\Delta_g}\right]^{\frac{1}{\rho_g}},\left[\frac{\Delta_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}},\left[\frac{\overline{\Delta}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}\right\} \le \overline{\tau}_g(f) \le \left[\frac{\overline{\tau}_f}{\Delta_g}\right]^{\frac{1}{\rho_g}}$$

and

$$\max\left\{\left[\frac{\overline{\Delta}_f}{\Delta_g}\right]^{\frac{1}{\rho_g}},\left[\frac{\tau_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}},\left[\frac{\overline{\Delta}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\right\} \le \tau_g(f) \le \min\left\{\left[\frac{\tau_f}{\overline{\Delta}_g}\right]^{\frac{1}{\rho_g}},\left[\frac{\overline{\tau}_f}{\Delta_g}\right]^{\frac{1}{\rho_g}}\right\} .$$

Proof. We obtain from (3.12) and (3.14), for a sequence of values of σ tending to infinity that

$$M_g^{-1}M_f(\sigma) \ge M_g^{-1}\left[\exp\left[\left(\overline{\tau}_f-\varepsilon\right)\exp\left[\lambda_f\cdot\sigma\right]\right]\right]$$

$$i.e., \quad M_g^{-1}M_f(\sigma) \ge \left[\frac{\log\left(\frac{\log\exp\left[\left(\overline{\tau}_f-\varepsilon\right)\exp\left[\lambda_f\cdot\sigma\right]\right]}{\left(\overline{\tau}_g+\varepsilon\right)}\right)}{\lambda_g}\right]$$

$$i.e., \quad M_g^{-1}M_f(\sigma) \geq \log\left[\left[\frac{\left(\overline{\tau}_f - \varepsilon\right)}{\left(\overline{\tau}_g + \varepsilon\right)}\right]^{\frac{1}{\lambda_g}} \cdot \exp\left[\frac{\lambda_f}{\lambda_g} \cdot \sigma\right]\right]$$

$$i.e., \quad \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\frac{\lambda_f}{\lambda_g} \cdot \sigma\right]} \geq \left[\frac{\left(\overline{\tau}_f - \varepsilon\right)}{\left(\overline{\tau}_g + \varepsilon\right)}\right]^{\frac{1}{\lambda_g}} .$$

Since in view of Lemma 2.1, we get that $\frac{\lambda_f}{\lambda_g} \geq \lambda_g(f)$ and as $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\limsup_{\sigma\to\infty} \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\lambda_g(f) \cdot \sigma\right]} \geq \left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}$$

$$i.e., \ \overline{\tau}_g(f) \geq \left[\frac{\overline{\tau}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} . \tag{3.34}$$

Further we obtain from (3.11) and (3.17), for a sequence of values of σ tending to infinity that

$$M_g^{-1}M_f(\sigma) \geq M_g^{-1}\left[\exp\left[\left(\tau_f - \varepsilon\right)\exp\left[\lambda_f \cdot \sigma\right]\right]\right]$$

$$i.e., \quad M_g^{-1}M_f(\sigma) \geq \left[\frac{\log\left(\frac{\log\exp\left[\left(\tau_f - \varepsilon\right)\exp\left[\lambda_f \cdot \sigma\right]\right]}{\left(\tau_g - \varepsilon\right)}\right)}{\lambda_g}\right]$$

$$i.e., \quad M_g^{-1}M_f(\sigma) \geq \log\left[\left[\frac{\left(\tau_f - \varepsilon\right)}{\left(\tau_g - \varepsilon\right)}\right]^{\frac{1}{\lambda_g}} \cdot \exp\left[\frac{\lambda_f}{\lambda_g} \cdot \sigma\right]\right]$$

$$i.e., \quad \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\frac{\lambda_f}{\lambda_g} \cdot \sigma\right]} \geq \left[\frac{\left(\tau_f - \varepsilon\right)}{\left(\tau_g - \varepsilon\right)}\right]^{\frac{1}{\lambda_g}} .$$

As in view of Lemma 2.1, we get that $\frac{\lambda_f}{\lambda_g} \geq \lambda_g(f)$ and as $\varepsilon\,(> 0)$ is arbitrary, therefore it follows from above that

$$\limsup_{\sigma\to\infty} \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\lambda_g(f) \cdot \sigma\right]} \geq \left[\frac{\tau_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}$$

$$i.e., \ \overline{\tau}_g(f) \geq \left[\frac{\tau_f}{\tau_g}\right]^{\frac{1}{\lambda_g}} . \tag{3.35}$$

Now from (3.4) and in view of (3.6), we get for a sequence of values of σ tending to infinity that

$$M_g^{-1}M_f(\sigma) \geq M_g^{-1}\left[\exp\left[\left(\Delta_f - \varepsilon\right)\exp\left[\rho_f \cdot \sigma\right]\right]\right]$$

$$\text{i.e.,}\qquad M_g^{-1}M_f(\sigma) \geq \left[\frac{\log\left(\frac{\log\exp\left[\left(\Delta_f-\varepsilon\right)\exp\left[\rho_f\cdot\sigma\right]\right]}{\left(\Delta_g+\varepsilon\right)}\right)}{\rho_g}\right]$$

$$\text{i.e.,}\qquad M_g^{-1}M_f(\sigma) \geq \log\left[\left[\frac{\left(\Delta_f-\varepsilon\right)}{\left(\Delta_g+\varepsilon\right)}\right]^{\frac{1}{\rho_g}}\cdot\exp\left[\frac{\rho_f}{\rho_g}\cdot\sigma\right]\right]$$

$$\text{i.e.,}\qquad \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\frac{\rho_f}{\rho_g}\cdot\sigma\right]} \geq \left[\frac{\left(\Delta_f-\varepsilon\right)}{\left(\Delta_g+\varepsilon\right)}\right]^{\frac{1}{\rho_g}}. \tag{3.36}$$

Analogously from (3.3) and in view of (3.9), it follows for a sequence of values of σ tending to infinity that

$$M_g^{-1}M_f(\sigma) \geq M_g^{-1}\left[\exp\left[\left(\overline{\Delta}_f-\varepsilon\right)\exp\left[\rho_f\cdot\sigma\right]\right]\right]$$

$$\text{i.e.,}\qquad M_g^{-1}M_f(\sigma) \geq \left[\frac{\log\left(\frac{\log\exp\left[\left(\overline{\Delta}_f-\varepsilon\right)\exp\left[\rho_f\cdot\sigma\right]\right]}{\left(\overline{\Delta}_g-\varepsilon\right)}\right)}{\rho_g}\right]$$

$$\text{i.e.,}\qquad M_g^{-1}M_f(\sigma) \geq \log\left[\left[\frac{\left(\overline{\Delta}_f-\varepsilon\right)}{\left(\overline{\Delta}_g+\varepsilon\right)}\right]^{\frac{1}{\rho_g}}\cdot\exp\left[\frac{\rho_f}{\rho_g}\cdot\sigma\right]\right]$$

$$\text{i.e.,}\qquad \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\frac{\rho_f}{\rho_g}\cdot\sigma\right]} \geq \left[\frac{\left(\overline{\Delta}_f-\varepsilon\right)}{\left(\overline{\Delta}_g+\varepsilon\right)}\right]^{\frac{1}{\rho_g}}. \tag{3.37}$$

As in view of Lemma 2.1, $\frac{\rho_f}{\rho_g} \geq \lambda_g(f)$ and $\varepsilon(>0)$ is arbitrary, we get from (3.36) that

$$\limsup_{\sigma\to\infty}\frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\lambda_g(f)\cdot\sigma\right]} \geq \left[\frac{\Delta_f}{\Delta_g}\right]^{\frac{1}{\rho_g}}$$

$$\text{i.e.,}\quad \overline{\tau}_g(f) \geq \left[\frac{\Delta_f}{\Delta_g}\right]^{\frac{1}{\rho_g}}. \tag{3.38}$$

Similarly, we get from (3.37) that

$$\limsup_{\sigma\to\infty}\frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\lambda_g(f)\cdot\sigma\right]} \geq \left[\frac{\overline{\Delta}_f}{\overline{\Delta}_g}\right]^{\frac{1}{\rho_g}}$$

$$\text{i.e.,}\quad \overline{\tau}_g(f) \geq \left[\frac{\overline{\Delta}_f}{\overline{\Delta}_g}\right]^{\frac{1}{\rho_g}}, \tag{3.39}$$

since in view of Lemma 2.1, $\frac{\rho_f}{\rho_g} \leq \lambda_g(f)$ and $\varepsilon(>0)$ is arbitrary.

Likewise from (3.4) and in view of (3.14), we get for a sequence of values of σ tending to infinity that

$$\begin{aligned}
& M_g^{-1} M_f(\sigma) \geq M_g^{-1}\left[\exp\left[\left(\Delta_f - \varepsilon\right)\exp\left[\rho_f \cdot \sigma\right]\right]\right] \\
i.e., \quad & M_g^{-1} M_f(\sigma) \geq \left[\frac{\log\left(\frac{\log\exp\left[\left(\Delta_f - \varepsilon\right)\exp\left[\rho_f \cdot \sigma\right]\right]}{\left(\overline{\tau}_g + \varepsilon\right)}\right)}{\lambda_g}\right] \\
i.e., \quad & M_g^{-1} M_f(\sigma) \geq \log\left[\left[\frac{\left(\Delta_f - \varepsilon\right)}{\left(\overline{\tau}_g + \varepsilon\right)}\right]^{\frac{1}{\lambda_g}} \cdot \exp\left[\frac{\rho_f}{\lambda_g} \cdot \sigma\right]\right] \\
i.e., \quad & \frac{\exp M_g^{-1} M_f(\sigma)}{\exp\left[\frac{\rho_f}{\lambda_g} \cdot \sigma\right]} \geq \left[\frac{\left(\Delta_f - \varepsilon\right)}{\left(\overline{\tau}_g + \varepsilon\right)}\right]^{\frac{1}{\lambda_g}} .
\end{aligned}$$

Since in view of Lemma 2.1, we get that $\frac{\rho_f}{\lambda_g} \geq \lambda_g(f)$ and as $\varepsilon (> 0)$ is arbitrary, it follows from above that

$$\begin{aligned}
\limsup_{\sigma \to \infty} \frac{\exp M_g^{-1} M_f(\sigma)}{\exp\left[\lambda_g(f) \cdot \sigma\right]} & \geq \left[\frac{\Delta_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} \\
i.e., \ \overline{\tau}_g(f) & \geq \left[\frac{\Delta_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} . \qquad (3.40)
\end{aligned}$$

Further from (3.3) and in view of (3.17), it follows for a sequence of values of σ tending to infinity that

$$\begin{aligned}
& M_g^{-1} M_f(\sigma) \geq M_g^{-1}\left[\exp\left[\left(\overline{\Delta}_f - \varepsilon\right)\exp\left[\rho_f \cdot \sigma\right]\right]\right] \\
i.e., \quad & M_g^{-1} M_f(\sigma) \geq \left[\frac{\log\left(\frac{\log\exp\left[\left(\overline{\Delta}_f - \varepsilon\right)\exp\left[\rho_f \cdot \sigma\right]\right]}{\left(\tau_g - \varepsilon\right)}\right)}{\lambda_g}\right] \\
i.e., \quad & M_g^{-1} M_f(\sigma) \geq \log\left[\left[\frac{\left(\overline{\Delta}_f - \varepsilon\right)}{\left(\tau_g - \varepsilon\right)}\right]^{\frac{1}{\lambda_g}} \cdot \exp\left[\frac{\rho_f}{\lambda_g} \cdot \sigma\right]\right] \\
i.e., \quad & \frac{\exp M_g^{-1} M_f(\sigma)}{\exp\left[\frac{\rho_f}{\lambda_g} \cdot \sigma\right]} \geq \left[\frac{\left(\overline{\Delta}_f - \varepsilon\right)}{\left(\tau_g - \varepsilon\right)}\right]^{\frac{1}{\lambda_g}} .
\end{aligned}$$

As in view of Lemma 2.1, we get that $\frac{\rho_f}{\lambda_g} \geq \lambda_g(f)$ and as $\varepsilon(>0)$ is arbitrary, therefore it follows from above that

$$\limsup_{\sigma\to\infty} \frac{\exp M_g^{-1} M_f(\sigma)}{\exp\left[\lambda_g(f)\cdot\sigma\right]} \geq \left[\frac{\overline{\Delta}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}$$

$$i.e.,\ \overline{\tau}_g(f) \geq \left[\frac{\overline{\Delta}_f}{\tau_g}\right]^{\frac{1}{\lambda_g}}. \tag{3.41}$$

Again from (3.7) and (3.10), we have for all sufficiently large values of σ that

$$M_g^{-1} M_f(\sigma) \leq M_g^{-1}\left[\exp\left[\left(\overline{\tau}_f+\varepsilon\right)\exp\left[\lambda_f\cdot\sigma\right]\right]\right]$$

$$i.e., \qquad M_g^{-1} M_f(\sigma) \leq \left[\frac{\log\left(\frac{\log\exp\left[\left(\overline{\tau}_f+\varepsilon\right)\exp\left[\lambda_f\cdot\sigma\right]\right]}{\left(\overline{\Delta}_g-\varepsilon\right)}\right)}{\rho_g}\right]$$

$$i.e., \qquad M_g^{-1} M_f(\sigma) \leq \log\left[\left[\frac{\left(\overline{\tau}_f+\varepsilon\right)}{\left(\overline{\Delta}_g-\varepsilon\right)}\right]^{\frac{1}{\rho_g}}\cdot\exp\left[\frac{\lambda_f}{\rho_g}\cdot\sigma\right]\right]$$

$$i.e., \qquad \frac{\exp M_g^{-1} M_f(\sigma)}{\exp\left[\frac{\lambda_f}{\rho_g}\cdot\sigma\right]} \leq \left[\frac{\left(\overline{\tau}_f+\varepsilon\right)}{\left(\overline{\Delta}_g-\varepsilon\right)}\right]^{\frac{1}{\rho_g}}.$$

Since in view of Lemma 2.1, we get that $\frac{\lambda_f}{\rho_g} \leq \lambda_g(f)$ and as $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\limsup_{\sigma\to\infty} \frac{\exp M_g^{-1} M_f(\sigma)}{\exp\left[\lambda_g(f)\cdot\sigma\right]} \leq \left[\frac{\overline{\tau}_f}{\overline{\Delta}_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e.,\ \overline{\tau}_g(f) \leq \left[\frac{\overline{\tau}_f}{\overline{\Delta}_g}\right]^{\frac{1}{\rho_g}}. \tag{3.42}$$

Thus the first part of the theorem follows from (3.34), (3.35), (3.38), (3.39), (3.40), (3.41) and (3.42).

Further from (3.11) and in view of (3.14), we get for all sufficiently large values of σ that

$$M_g^{-1} M_f(\sigma) \geq M_g^{-1}\left[\exp\left[\left(\tau_f-\varepsilon\right)\exp\left[\lambda_f\cdot\sigma\right]\right]\right]$$

$$i.e., \qquad M_g^{-1} M_f(\sigma) \geq \left[\frac{\log\left(\frac{\log\exp\left[\left(\tau_f-\varepsilon\right)\exp\left[\lambda_f\cdot\sigma\right]\right]}{\left(\overline{\tau}_g+\varepsilon\right)}\right)}{\lambda_g}\right]$$

$$i.e., \quad M_g^{-1}M_f(\sigma) \geq \log\left[\left[\frac{(\tau_f-\varepsilon)}{(\overline{\tau}_g+\varepsilon)}\right]^{\frac{1}{\lambda_g}} \cdot \exp\left[\frac{\lambda_f}{\lambda_g}\cdot\sigma\right]\right]$$

$$i.e., \quad \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\frac{\lambda_f}{\lambda_g}\cdot\sigma\right]} \geq \left[\frac{(\tau_f-\varepsilon)}{(\overline{\tau}_g+\varepsilon)}\right]^{\frac{1}{\lambda_g}} .$$

Since in view of Lemma 2.1, we get that $\frac{\lambda_f}{\lambda_g} \geq \lambda_g(f)$ and as $\varepsilon(>0)$ is arbitrary,so it follows from above that

$$\liminf_{\sigma\to\infty}\frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\lambda_g(f)\cdot\sigma\right]} \geq \left[\frac{\tau_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}$$

$$i.e., \ \tau_g(f) \geq \left[\frac{\tau_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} . \tag{3.43}$$

Again from (3.3) and in view of (3.6), we get for all sufficiently large values of σ that

$$M_g^{-1}M_f(\sigma) \geq M_g^{-1}\left[\exp\left[\left(\overline{\Delta}_f-\varepsilon\right)\exp\left[\rho_f\cdot\sigma\right]\right]\right]$$

$$i.e., \quad M_g^{-1}M_f(\sigma) \geq \left[\frac{\log\left(\frac{\log\exp\left[\left(\overline{\Delta}_f-\varepsilon\right)\exp\left[\rho_f\cdot\sigma\right]\right]}{(\Delta_g+\varepsilon)}\right)}{\rho_g}\right]$$

$$i.e., \quad M_g^{-1}M_f(\sigma) \geq \log\left[\left[\frac{(\overline{\Delta}_f-\varepsilon)}{(\Delta_g+\varepsilon)}\right]^{\frac{1}{\rho_g}} \cdot \exp\left[\frac{\rho_f}{\rho_g}\cdot\sigma\right]\right]$$

$$i.e., \quad \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\frac{\rho_f}{\rho_g}\cdot\sigma\right]} \geq \left[\frac{(\overline{\Delta}_f-\varepsilon)}{(\Delta_g+\varepsilon)}\right]^{\frac{1}{\rho_g}} . \tag{3.44}$$

As in view of Lemma 2.1, $\frac{\rho_f}{\rho_g} \geq \lambda_g(f)$ and $\varepsilon(>0)$ is arbitrary, we get from (3.44) that

$$\liminf_{\sigma\to\infty}\frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\lambda_g(f)\cdot\sigma\right]} \geq \left[\frac{\overline{\Delta}_f}{\Delta_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e., \ \tau_g(f) \geq \left[\frac{\overline{\Delta}_f}{\Delta_g}\right]^{\frac{1}{\rho_g}} . \tag{3.45}$$

Again from (3.3) and in view of (3.14), we get for all sufficiently large values of σ that

$$M_g^{-1}M_f(\sigma) \geq M_g^{-1}\left[\exp\left[\left(\overline{\Delta}_f-\varepsilon\right)\exp\left[\rho_f\cdot\sigma\right]\right]\right]$$

$$i.e., \quad M_g^{-1}M_f(\sigma) \geq \left[\frac{\log\left(\frac{\log\exp\left[\left(\overline{\Delta}_f-\varepsilon\right)\exp\left[\rho_f\cdot\sigma\right]\right]}{(\overline{\tau}_g+\varepsilon)}\right)}{\lambda_g}\right]$$

$$i.e., \qquad M_g^{-1}M_f(\sigma) \geq \log\left[\left[\frac{\left(\overline{\Delta}_f-\varepsilon\right)}{\left(\overline{\tau}_g+\varepsilon\right)}\right]^{\frac{1}{\lambda_g}} \cdot \exp\left[\frac{\rho_f}{\lambda_g}\cdot\sigma\right]\right]$$

$$i.e., \qquad \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\frac{\rho_f}{\lambda_g}\cdot\sigma\right]} \geq \left[\frac{\left(\overline{\Delta}_f-\varepsilon\right)}{\left(\overline{\tau}_g+\varepsilon\right)}\right]^{\frac{1}{\lambda_g}} .$$

Since in view of Lemma 2.1, we get that $\frac{\rho_f}{\lambda_g} \geq \lambda_g(f)$ and as $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{\sigma\to\infty}\frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\lambda_g(f)\cdot\sigma\right]} \geq \left[\frac{\overline{\Delta}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}$$

$$i.e., \ \tau_g(f) \geq \left[\frac{\overline{\Delta}_f}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} . \tag{3.46}$$

Moreover, we get from (3.8) and (3.10) for a sequence of values of σ tending to infinity that

$$M_g^{-1}M_f(\sigma) \leq M_g^{-1}\left[\exp\left[\left(\overline{\tau}_f+\varepsilon\right)\exp\left[\lambda_f\cdot\sigma\right]\right]\right]$$

$$i.e., \qquad M_g^{-1}M_f(\sigma) \leq \left[\frac{\log\left(\frac{\log\exp\left[\left(\overline{\tau}_f+\varepsilon\right)\exp\left[\lambda_f\cdot\sigma\right]\right]}{\left(\Delta_g-\varepsilon\right)}\right)}{\rho_g}\right]$$

$$i.e., \qquad M_g^{-1}M_f(\sigma) \leq \log\left[\left[\frac{\left(\overline{\tau}_f+\varepsilon\right)}{\left(\Delta_g-\varepsilon\right)}\right]^{\frac{1}{\rho_g}} \cdot \exp\left[\frac{\lambda_f}{\rho_g}\cdot\sigma\right]\right]$$

$$i.e., \qquad \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\frac{\lambda_f}{\rho_g}\cdot\sigma\right]} \leq \left[\frac{\left(\overline{\tau}_f+\varepsilon\right)}{\left(\Delta_g-\varepsilon\right)}\right]^{\frac{1}{\rho_g}} .$$

As in view of Lemma 2.1, we get that $\frac{\lambda_f}{\rho_g} \leq \lambda_g(f)$ and as $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{\sigma\to\infty}\frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\lambda_g(f)\cdot\sigma\right]} \leq \left[\frac{\overline{\tau}_f}{\Delta_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e., \ \tau_g(f) \leq \left[\frac{\overline{\tau}_f}{\Delta_g}\right]^{\frac{1}{\rho_g}} . \tag{3.47}$$

Similarly, from (3.13) and in view of (3.7), it follows for a sequence of values of σ tending to infinity that

$$M_g^{-1}M_f(\sigma) \leq M_g^{-1}\left[\exp\left[\left(\tau_f+\varepsilon\right)\exp\left[\lambda_f\cdot\sigma\right]\right]\right]$$

$$i.e., \quad M_g^{-1}M_f(\sigma) \le \left[\frac{\log\left(\frac{\log\exp[(\tau_f+\varepsilon)\exp[\lambda_f\cdot\sigma]]}{(\overline{\Delta}_g-\varepsilon)}\right)}{\rho_g}\right]$$

$$i.e., \quad M_g^{-1}M_f(\sigma) \le \log\left[\left[\frac{(\tau_f+\varepsilon)}{(\overline{\Delta}_g-\varepsilon)}\right]^{\frac{1}{\rho_g}} \cdot \exp\left[\frac{\lambda_f}{\rho_g}\cdot\sigma\right]\right]$$

$$i.e., \quad \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\frac{\lambda_f}{\rho_g}\cdot\sigma\right]} \le \left[\frac{(\tau_f+\varepsilon)}{(\overline{\Delta}_g-\varepsilon)}\right]^{\frac{1}{\rho_g}} .$$

Since in view of Lemma 2.1, we get that $\frac{\lambda_f}{\rho_g} \le \lambda_g(f)$ and as $\varepsilon (> 0)$ is arbitrary, therefore it follows from above that

$$\liminf_{\sigma\to\infty} \frac{\exp M_g^{-1}M_f(\sigma)}{\exp\left[\lambda_g(f)\cdot\sigma\right]} \le \left[\frac{\tau_f}{\overline{\Delta}_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e., \ \tau_g(f) \le \left[\frac{\tau_f}{\overline{\Delta}_g}\right]^{\frac{1}{\rho_g}} . \tag{3.48}$$

Hence the second part of the theorem follows from (3.43), (3.45), (3.46), (3.47) and (3.48). □

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