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ON RADICALS AND REPRESENTATIONS OF TOPOLOGICAL ALGEBRAS

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Abstract. In this paper we generalize some results about Jacobson radical, topological radical and irreducible representations known for Banach algebras to the case of quite general topological algebras.

1. Introduction

The present paper is a continuation of the work started in [1] and [2]. Our aim is to generalize some results about the radicals and irreducible representations from the case of Banach algebras, studied in [6], to the case of general topological algebras with very few extra conditions on the algebra or its topology.

Let $\mathbb{K}$ denote either the field $\mathbb{R}$ of all real numbers or the field $\mathbb{C}$ of all complex numbers and let A be an algebra over $\mathbb{K}$. As usual, we will denote by $\mathrm{Rad}(A)$ the Jacobson radical of A and by $\mathrm{Inv}(A)$ the set of invertible elements of a unital algebra A.

By a topological algebra over the field $\mathbb{K}$ we will mean a topological vector space over the field $\mathbb{K}$, in which there is also defined the multiplication, which turns the space into an algebra and which is separately continuous in the topology of the topological vector space.

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2. Jacobson radical

It is well-known fact that $\mathrm{Rad}(A)$ can be represented as the intersection of all maximal regular left ideals of A or as the intersection of all maximal regular right ideals of A (see [7], Theorem 1.2.2, p. 11). An algebra A is called semisimple, if $\mathrm{Rad}(A) = \{\theta_A\}$, where θ_A stands for the zero element of algebra A.

We will start with the generalization of Corollary 3.2.2 from [6], p. 35.

Proposition 2.1. *Let A be a unital topological algebra in which there exists a neighbourhood O of zero such that*

$$\sum_{k=0}^{\infty} y^k \in A$$

for every $y \in O$. Then

(a) $(e_A - O) \cap I = \emptyset$ *for every left, right or two-sided ideal I of A;*

(b) *every maximal left, right or two-sided ideal of A is closed;*

(c) $\mathrm{Rad}(A)$ *is closed in A;*

(d) $A/\mathrm{Rad}(A)$ *is a semisimple topological algebra.*

Proof. (a) It is known that an ideal (left, right or two-sided) does not contain invertible elements of A. By Proposition 2.1 from [2] it is known that $e_A - y \in \mathrm{Inv}(A)$ for every $y \in O$. Hence, $e_A - O \subseteq \mathrm{Inv}(A)$. Thus, the condition (a) follows.

(b) Let I be a left, right or two-sided ideal of A. Every neighbourhood contains an open neighbourhood. Thus, there exists an open neighbourhood $U \subseteq O$ such that $e_A - U$ is open. Obviously, $(e_A - U) \cap I \subseteq (e_A - O) \cap I = \emptyset$.

Suppose that there exists $i \in \mathrm{cl}_A(I) \cap (e_A - U)$. Then there exists a net $(i_\lambda)_{\lambda\in\Lambda} \in I$ such that $(i_\lambda)_{\lambda\in\Lambda} \underset{\lambda}{\to} i$. Since $i \in e_A - U$, then $e_A - U$ is a neighbourhood of i. Thus, there ekists $\lambda_0 \in \Lambda$ such that $i_\lambda \in e_A - U$ for every $\lambda > \lambda_0$. But then there exists $\lambda_1 > \lambda_0$ such that $i_{\lambda_1} \in I \cap (e_A - U) = \emptyset$. Since this leads to a contradiction, then $\mathrm{cl}_A(I) \cap (e_A - U) = \emptyset$, as well. This means that $I \subseteq \mathrm{cl}_A(I) \neq A$. Hence, $\mathrm{cl}_A(I)$ is a left, right or two-sided ideal in A. If I is a maximal ideal, then $I = \mathrm{cl}_A(I)$.

(c) Since $\mathrm{Rad}(A)$ is an intersection of all maximal left ideals of A and all maximal left ideals of A are closed, then $\mathrm{Rad}(A)$ is also closed in A.

(d) The claim follows by Theorem 3.1.5 from [6], p. 35. □

3. Representations and antirepresentations

Let X be a topological linear space over $\mathbb{K}$ and $\mathcal{L}(X)$ the set of all continuous linear maps on X. Then $\mathcal{L}(X)$ becomes an algebra over $\mathbb{K}$ considering pointwise addition and the composite as multiplication. Moreover, $\mathcal{L}(X)$ becomes a topological algebra, when we endow it with the topology of simple convergence. A subbase of neighbourhoods of zero of this topology consists of sets in the form

$$T(S,O) = \{L \in \mathcal{L}(X) : L(S) \subset O\},$$

where S is a finite subset of X and O is a neighbourhood of zero in X (for more details, see [4], p. 26).

Let A be a topological algebra over $\mathbb{K}$ and X a nontrivial (i. e., $X \neq \{\theta_X\}$) topological linear space over $\mathbb{K}$. A map $\pi : A \to L(X)$ is called a *representation of A on X*, if π is an algebra homomorphism, and an *antirepresentation of A on X*, if π is an algebra antihomomorphism. The difference between algebra homomorphism and algebra antihomomorphism is that for an algebra homomorphism we have $\pi(ab) = \pi(a)\pi(b)$ but for an algebra antihomomorphism we have $\pi(ab) = \pi(b)\pi(a)$ for every $a, b \in A$. In case A is unital, its representation on X has to be a unital algebra homomorphism. The kernel of a representation (or an antirepresentation) π is a set $\ker(\pi) = \{a \in A : \pi(a) = \theta_{L(X)}\}$, where $\theta_{L(X)}(x) = \theta_X$ for every $x \in X$.

A linear subspace Y of X is said to be *invariant under the* (*anti-*)*representation* π, if $(\pi(a))(Y) \subseteq Y$ for every $a \in A$. A(n) (anti-)representation $\pi : A \to L(X)$ is called *irreducible*, if the only invariant subspaces under π are $\{\theta_X\}$ and X.

Remark 3.1. Take a unital topological algebra A and consider the quotient space $X = A/L$ (respectively, $X = A/R$), where L is a closed maximal left ideal of A (respectively, R is a closed maximal right ideal of A) with the quotient map $q : A \to X$. The topology τ_X on the quotient space X is defined through the topology τ_A of A as follows:

$$\tau_X = \{Y \subset X : q^{-1}(Y) \in \tau_A\}.$$

Consider the map $\pi : A \to \mathcal{L}(X)$, defined by $(\pi(b))([a]) := [ba]$ (respectively, $(\pi(b))([a]) := [ab]$) for every $b \in A$ and every $[a] = q(a) \in X$. This (anti)representation is irreducible because of the maximality of the ideal L (respectively, R). It is also a continuous representation, because the ideal L (respectively, R) is closed.

Let now A be a unital topological algebra. The left (right) topological radical $\mathrm{rad}_l(A)$ ($\mathrm{rad}_r(A)$) of a topological algebra A is sometimes defined as the intersection of the kernels of all continuous irreducible (anti)representations of A on topological linear spaces (see [4], p. 27). Another way is to define the left topological radical as the intersection of all closed maximal left ideals of A and the right topological radical as an intersection of all closed maximal right ideals of A (see [3] or [5]).

For a unital algebra A and $a \in A$ we denote by $\sigma(a)$ the spectrum of a in A, i.e., the set

$$\sigma(a) = \{\lambda \in \mathbb{K} : a - \lambda e_A \notin \text{ Inv } (A)\}.$$

Now we are ready to generalise Theorem 4.2.1 from [6], p. 81, for arbitrary topological algebras instead of Banach algebras.

Theorem 3.2. *Let A be a unital topological algebra over $\mathbb{K}$. Then*

(i) *for every continuous irreducible* (*anti*)*representation π of A on a Hausdorff topological linear space there exists a closed maximal left* (*right*) *ideal M of A such that* $\ker(\pi) = \{a \in A : aA \subseteq M\}$ (*respectively,* $\ker(\pi) = \{a \in A : Aa \subseteq M\}$)*;*

(ii) *for every closed maximal left* (*right*) *ideal M of A there exists a continuous irreducible* (*anti*)*representation π of A on a Hausdorff topological linear space such that* $\ker(\pi) = \{a \in A : aA \in M\}$ (*respectively,* $\ker(\pi) = \{a \in A : Aa \in M\}$)*;*

(iii) *the left* (*right*) *topological radical of A is the intersection of the kernels of all continuous irreducible* (*anti*)*representations of A on Hausdorff topological linear spaces;*

(iv) *for every* $a \in A$*, the union of spectrums* $\sigma_{\mathcal{L}(X)}(\pi(a))$ *of elements* $\pi(a)$ *in the corresponding algebras* $\pi(A)$*, where* $\pi : A \to \mathcal{L}(X)$ *runs over the set of all continuous irreducible* (*anti*)*representations of* A *on Hausdorff topological linear spaces* X*, is a subset of the spectrum* $\sigma(a)$ *of* a *in* A*.*

Moreover, if

a) *for every element* $a \in A$*, which has no left* (*respectively, right*) *inverse, there exists a closed maximal left* (*respectively, right*) *ideal* M *of* A *such that* $a \in M$*,*

and

b) *for every element* a *in the left* (*respectively, right*) *topological radical of* A *the element* $e_A - a$ *is right* (*respectively, left*) *invertible in* A*,*

then

(v) *for every* $a \in A$*, the spectrum* $\sigma(a)$ *of* a *in* A *is equal to the union of spectrums* $\sigma_{\mathcal{L}(X)}(\pi(a))$ *of elements* $\pi(a)$ *in the corresponding algebras* $\pi(A)$*, where* $\pi : A \to \mathcal{L}(X)$ *runs over the set of all continuous irreducible* (*anti*)*representations of* A *on Hausdorff topological linear spaces* X*.*

Proof. (i) We will present the proof for representations. The proof for antirepresentations is similar. Let X be a nontrivial Hausdorff topological linear space over $\mathbb{K}$, $\pi : A \to \mathcal{L}(X)$ a continuous irreducible representation of A on X and $x \in X \backslash \{\theta_X\}$. Consider the set $F_x = \{(\pi(a))(x) : a \in A\}$. Since A is unital, then $\pi(e_A)$ is the identity map, hence, $x = (\pi(e_A))(x) \in F_x$. Take any $b \in A$. Then

$$(\pi(b))(F_x) = \{(\pi(b))(\pi(a)(x)) : a \in A\} = \{(\pi(ba))(x) : a \in A\} \subseteq F_x.$$

Hence, F_x is invariant under the representation π. Since π was irreducible and $F_x \neq \{\theta_X\}$, then $F_x = X$.

Let $M = \{a \in A : (\pi(a))(x) = \theta_X\}$. Then $e_A \notin M$, because $(\pi(e_A))(x) = x \neq \theta_X$. Thus, $M \neq A$. Hence, M is a left (right) ideal of A.

Suppose, that there exists a left (right) ideal $N \notin \{A, M\}$ of A such that $M \subseteq N$. Let $W = \{(\pi(n))(x) : n \in N\} \neq \{\theta_X\}$. Take any $a \in A$. Then $an \in N$ (respectively, $na \in N$) and

$$(\pi(a))(W) = \{(\pi(a))((\pi(n))(x)) : n \in N\} = \{(\pi(an))(x) : n \in N\} \subseteq W.$$

Since π is an irreducible representation and $W \neq \{\theta_X\}$, then $W = X$. Hence, there exists $e \in N$ such that $(\pi(e))(x) = x$.

Take any $a \in A$. Then

$$(\pi(ae - a))(x) = (\pi(a))((\pi(e))(x)) - (\pi(a))(x) = (\pi(a))(x) - (\pi(a))(x) = \theta_X$$

(respectively, $\pi(ea - a)(x))\theta_X$). Hence, $ae - a \in M$ (respectively, $ea - a \in M$) for every $a \in A$. But now, $a = ae - (ae - a) \in N + M \subset N$ (respectively, $a = ea - (ea - a) \in N$) for every $a \in A$. Hence, $N = A$, which is a contradiction. Therefore, M is a maximal left (right) ideal of A.

Since M is maximal, then $\mathrm{cl}_A(M) = M$ or $\mathrm{cl}_A(M) = A$, because $\mathrm{cl}_A(M)$ is also a left (respectively, right) ideal of A. If $\mathrm{cl}_A(M) = A$, then there exists a net $(a_\lambda)_{\lambda \in \Lambda}$ of elements of M converging to $e_A \in A \backslash M$. Since π is continuous, then $(\pi(a_\lambda))_{\lambda \in \Lambda}$ converges to $\pi(e_A)$.

Take any neighbourhood Y of zero in X. Then there exists a $\lambda_Y \in \Lambda$ such that

$$\pi(a_\lambda) \in T(\pi(e_A), Y) = \pi(e_A) - T(\{x\}, Y) = \{h \in \mathcal{L}(X) : (\pi(e_A) - h)(x) \in Y\}$$

for every $\lambda > \lambda_Y$, where $T(\pi(e_A), Y)$ is a neighbourhood of $\pi(e_A)$ in $\mathcal{L}(X)$, defined by Y. Thus, for every neighbourhood Y of zero in X there exists $\lambda_Y \in \Lambda$ such that

$$x = (\pi(e_A))(x) = (\pi(e_A))(x) - (\pi(a_\lambda))(x) = (\pi(e_A) - \pi(a_\lambda))(x) \in Y$$

for $\lambda > \lambda_Y$. Therefore, $x \in Y$ for every neighbourhood Y of zero in X, which means that $x = \theta_X$, because X is a Hausdorff space. But this is a contradiction with the choice of x. Hence, $\mathrm{cl}_A(M) = M$ and M is closed maximal left (right) ideal of A.

Take any $a \in \ker(\pi)$. Then $(\pi(a))(y) = \theta_X$ for every $y \in X$ and we obtain

$$(\pi(ab))(x) = (\pi(a))((\pi(b))(x)) = \theta_X$$

(similarily, $(\pi(ba))(x) = (\pi(b))(\pi(a)(x)) = (\pi(b))(\theta_X) = \theta_X$) for every $b \in A$. Hence,

$$\ker(\pi) \subseteq \{a \in A : aA \subseteq M\}$$

(respectively, $\ker(\pi) \subseteq \{a \in A : Aa \subseteq M\}$).

Let $b \in \{a \in A : aA \subseteq M\}$ (respectively, $b \in \{a \in A : Aa \subseteq M\}$). Then $b = be_A \in M$ (respectively, $b = e_A b \in M$) and

$$\theta_X = (\pi(be_A))(x) = (\pi(b))(\pi(e_A)(x)) = (\pi(b))(x)$$

(similarily, $\theta_X = (\pi(e_A b))(x) = (\pi(e_A))(\pi(b))(x)) = (\pi(b))(x))$ for every $x \in X$. Hence, $b \in \ker(\pi)$. Thus, for every continuous irreducible representation π of A there exists a closed maximal left (right) ideal M of A such that $\ker(\pi) = \{a \in A : aA \subseteq M\}$ (respectively, $\ker(\pi) = \{a \in A : Aa \subseteq M\}$).

(ii) Let M be a closed maximal left (right) ideal of A. Take $X = A/M$. Then X is a Hausdorff topological linear space, because M was closed. Then one can consider a continuous irreducible representation (respectively, an antirepresentation) $\pi : A \to \mathcal{L}(X)$ as in Remark 3.1.

Now,

$$\ker(\pi) = \{a \in A : \pi(a) = \theta_{\mathcal{L}(X)}\} =$$
$$= \{a \in A : ab \in M \quad \text{for all} \quad b \in A\} = \{a \in A : aA \subseteq M\}$$

(respectively,

$$\ker(\pi) = \{a \in A : ba \in M \quad \text{for all} \quad b \in A\} = \{a \in A : Aa \subseteq M\}).$$

(iii) See the proof of statements a) and b) of Theorem 1 from [4], p. 27. There it is shown that the intersection of continuous irreducible representations of a topological algebra is equal to the intersection of closed maximal regular left ideals of A. In case A is a unital algebra, every ideal of A is regular. Replacing representations by antirepresentations and left ideals by right ideals, the rest of this claim follows identically (see [5]).

(iv) Take $a \in A$ and an element λ from the union of spectrums $\sigma_{\mathcal{L}(X)}(\pi(a))$ of elements $\pi(a)$ in the corresponding algebras $\pi(A)$, where $\pi : A \to \mathcal{L}(X)$ runs over the set of all continuous irreducible (anti)representations of A on Hausdorff topological linear spaces. Then there exists a Hausdorff topological linear space X and a continuous irreducible (anti)representation $\pi : A \to \mathcal{L}(X)$ such that $\lambda \in \sigma_{\mathcal{L}(X)}(\pi(a))$.

This means that $\pi(a)-\lambda e_{\mathcal{L}(X)}=\pi(a-\lambda e_A)$ is not invertible in $\mathcal{L}(X)$. Suppose that $a-\lambda e_A\in \mathrm{Inv}(A)$. Then there exists $b\in A$ such that

$$(a-\lambda e_A)b=e_A=b(a-\lambda e_A).$$

But then

$$\pi(a-\lambda e_A)\pi(b)=\pi((a-\lambda e_A)b)=\pi(e_A)=e_{\mathcal{L}(X)}=$$
$$=\pi(e_A)=\pi(b(a-\lambda e_A))=\pi(b)\pi(a-\lambda e_A),$$

which means that $\pi(a-\lambda e_A)$ has an inverse $\pi(b)$ and is invertible. Since this gives us a contradiction, then $a-\lambda e_A\notin \mathrm{Inv}(A)$ and we obtain that the union of spectrums $\sigma_{\mathcal{L}(X)}(\pi(a))$ of elements $\pi(a)$ in the corresponding algebras $\pi(A)$, where $\pi: A\to\mathcal{L}(X)$ runs over the set of all continuous irreducible (anti)representations of A, is a subset of $\sigma(a)$.

(v) Take $a\in A$ and $\lambda\in\sigma(a)$. Then $a-\lambda e_A\notin \mathrm{Inv}(A)$.

Suppose that $a-\lambda e_A$ is not invertible from left (respectively, from right) in A. By assumption a), there exists a closed maximal left (respectively, right) ideal M of A such that $a-\lambda e_A\in M$. Take $X=A/M$ and consider a map $\pi: A\to\mathcal{L}(X)$, defined in Remark 1. Then we will obtain a representation (respectively, an antirepresentation) π of A on a Hausdorff topological linear space X, as noticed in Remark 1. Since $a-\lambda e_A\in M$, we have

$$(\pi(a-\lambda e_A))([e_A])=[(a-\lambda e_A)e_A]=[a-\lambda e_A]=\theta_X$$

(respectively,

$$(\pi(a-\lambda e_A))([e_A])=[e_A(a-\lambda e_A)]=[a-\lambda e_A]=\theta_X.)$$

If $\pi(a-\lambda e_A)$ was invertible in $\mathcal{L}(X)$, then there should exist $f\in\mathcal{L}(X)$ such that $(f\pi(a-\lambda e_A))([x])=[x]=(\pi(a-\lambda(e_A)f)([x])$ for every $x\in A$. Since $e_A\notin M$, then

$$[e_A]\neq\theta_X=f(\theta_X)=f((\pi(a-\lambda e_A))([e_A]))=(f\pi(a-\lambda e_A))([e_A]).$$

Hence, $\pi(a)-\lambda e_{\mathcal{L}(X)}=\pi(a-\lambda e_A)$ is not invertible in $\mathcal{L}(X)$, which means that $\lambda\in\sigma_{\mathcal{L}(X)}(\pi(a))$.

Suppose now, that $a-\lambda e_A$ is invertible from left (respectively, from right) in A and let $i\in A$ be the left (respectively, right) inverse of $a-\lambda e_A$ in A. Take

$$j=e_A-(a-\lambda e_A)i=e_A-ai+\lambda i$$

(respectively, $j=e_A-i(a-\lambda e_A)=e_A-ia+\lambda i$). Then

$$j(a-\lambda e_A)=(e_A-ai+\lambda i)(a-\lambda e_A)=a-\lambda e_A-a+\lambda e_A=\theta_A$$

(respectively, $(a-\lambda e_A)j=\theta_A$).

Suppose that j does not belong to the left (respectively, right) topological radical of A. Then there exist a topological linear space X and a continuous representation (respectively, an antirepresentation) $\pi: A\to\mathcal{L}(X)$ such that $\pi(j)\neq\theta_{\mathcal{L}(X)}$. Then

$$\pi(j)\pi(a-\lambda e_A)=\pi(j(a-\lambda e_A))=\pi(\theta_A)=\theta_{\mathcal{L}(X)}$$

(respectively, $\pi(a-\lambda e_A)\pi(j)=\theta_{\mathcal{L}(X)}$). Suppose that $\pi(a-\lambda e_A)$ is invertible in $\mathcal{L}(X)$ and denote by $g\in\mathcal{L}(X)$ the inverse of $\pi(a-\lambda e_A)$. Then

$$\pi(j)=\pi(j)\pi(a-\lambda e_A)g=\theta_{\mathcal{L}(X)}g=\theta_{\mathcal{L}(X)}$$

(similarily in the case of antirepresentations). But it is a contradiction with the condition that $\pi(j) \neq \theta_{\mathcal{L}(X)}$. This shows that

$$\pi(a) - \lambda e_{\mathcal{L}(X)} = \pi(a - \lambda e_A)$$

is not invertible in $\mathcal{L}(X)$ and $\lambda \in \sigma_{\mathcal{L}(X)}(\pi(a))$.

Suppose that j belongs to the left (respectively, right) topological radical of A. By assumption b), $e_A - j = (a - \lambda e_A)i$ is invertible from right (respectively, $e_A - j = i(a-\lambda e_A)$ is invertible from left). Hence, $a-\lambda e_A$ is also invertible from right (respectively, left), thus invertible, which is a contradiction with the assumption that $\lambda \in \sigma(a)$. Thus, j does not belong to the left (respectively, right) topological radical of A and the spectrum $\sigma(a)$ is a subset of the union of spectrums $\sigma_{\mathcal{L}(X)}(\pi(a))$ of elements $\pi(a)$ in the corresponding algebras $\pi(A)$, where $\pi : A \to \mathcal{L}(X)$ runs over the set of all continuous irreducible (anti)representations of A.

Hence, those sets coincide and the claim is proved. □

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