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DUAL FRAMES ON FINITE DIMENSIONAL QUATERNIONIC HILBERT SPACE

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Abstract. Khokulan et al. [15] introduced frames for finite dimensional quaternionic Hilbert spaces. In this paper, we will study frames for quaternonic Hilbert spaces and discuss different types of dual frames of a given frame in a quaternionic Hilbert space.

1. Introduction

While working on some deep problems in non-harmonic Fourier series, Duffin and Schaeffer [12] indroduced *frames for Hilbert spaces.* According to the Parseval's identity

" If $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal bases in a Hilbert space $\mathcal{H}$, then

$$\sum_{n=1}^{\infty}|\langle x,e_n\rangle|^2=\|x\|^2,\quad x\in\mathcal{H}."$$

Thus, the idea of frames emerged in order to provide the relaxation to the Parseval's identity into an inequality. This leads us to the following definition:

"A sequence $\{x_n\}_{n\in\mathbb{N}}\subset\mathcal{H}$ is said to be a *frame* for a Hilbert space $\mathcal{H}$ if there exist positive constants A and B such that

$$A\|x\|^2\leq\sum_{n=1}^{\infty}|\langle x,x_n\rangle|^2\leq B\|x\|^2,\quad \text{for all } x\in\mathcal{H}." \tag{1}$$

The positive constants A and B, respectively, are called lower and upper frame bounds for the frame $\{x_n\}_{n\in\mathbb{N}}$. The inequality (1) is called the *frame inequality* for the frame $\{x_n\}_{n\in\mathbb{N}}$. A frame $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ is said to be

- *tight* if it is possible to choose $A=B$.
- *Parseval* if it is a tight frame with $A=B=1$.

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Another important class of sequences to study frame theory is the class of Bessel sequences. *Bessel sequences* are the sequences which only satisfy upper frame inequality. These sequences are in general need not to be bases but have some properties which are similar to orthonormal bases.

Frame theory began to spread among researchers when in 1986, Daubechies, Grossmann and Meyer published a fundamental paper [11] and observed that frames can be also used to provide the series expansions of functions in $L^2(\mathbb{R})$. In some sense, frames can be considered as a generalization to orthonormal bases in Hilbert spaces. The main property of frames which makes them so useful is their redundancy, due to which, representation of signals using frames is advantageous over basis expansions in a variety of practical applications. Frame works as an important tool in the study of signal and image processing [4], filter bank theory [6], wireless communications [14] and sigma-delta quantization [5]. For more literature on frame theory, one may refer to [7, 10].

Recently, Khokulan et al. [15] introduced and studied frames for finite dimensional quaternionic Hilbert spaces. In this paper, we will discuss some different types of dual frames of a given frame in a left quaternionic Hilbert space and give various types of reconstructions with the help of dual frame.

Throughout this paper we will denote $\mathbb{H}$ as a non-commutative field of quaternion, $V_L(\mathbb{H})$ as a left quaternionic Hilbert space and ran T will denote the range of operator T.

2. Quaternionic Hilbert space

Let $\mathbb{H}$ be the non-commutative field of quaternion, i.e.,

$$\mathbb{H} = \{x_0 + x_1 i + x_2 j + x_3 k \ : \ x_0,\ x_1,\ x_2,\ x_3 \in \mathbb{R}\}$$

where $i^2 = j^2 = k^2 = -1$; $ij = -ji = k$; $jk = -kj = i$ and $ki = -ik = j$. For each quaternion $q = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$, define conjugate of q denoted by $\overline{q}$ as

$$\overline{q} = x_0 - x_1 i - x_2 j - x_3 k \in \mathbb{H}.$$

If $q = x_0 + x_1 i + x_2 j + x_3 k$ is a quaternion then x_0 is called the real part of q and $x_1 i + x_2 j + x_3 k$ is called the imaginary part of q. The modulus of quaternion $q = x_0 + x_1 i + x_2 j + x_3 k$ is defined as

$$|q| = (\overline{q}q)^{1/2} = (q\overline{q})^{1/2} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

For every non-zero quaternion $q = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$, there exists a unique inverse q^{-1} as

$$q^{-1} = \frac{\overline{q}}{|q|^2} = \frac{x_0 - x_1 i - x_2 j - x_3 k}{\sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}}.$$

Definition 2.1. A left quaternionic Hilbert space $V_L(\mathbb{H})$ is a vector space under left multiplication by quaternionic scalars together with the binary mapping $\langle .|. \rangle$: $V_L(\mathbb{H}) \times V_L(\mathbb{H}) \to \mathbb{H}$ (called the scalar (quaternion) product) which satisfies following properties:

(a) $\overline{\langle v_1|v_2 \rangle} = \langle v_2|v_1 \rangle$ for all $v_1, v_2 \in V_L(\mathbb{H})$.
(b) $\langle v|v \rangle > 0$ if $v \neq 0$.

(c) $\langle v|v_1 + v_2\rangle = \langle v|v_1\rangle + \langle v|v_2\rangle$ for all $v, v_1, v_2 \in V_L(\mathbb{H})$
(d) $\langle qv|u\rangle = q\langle v|u\rangle$ for all $v, u \in V_L(\mathbb{H})$ and $q \in \mathbb{H}$.

In view of Definition 2.1, we have following properties of left quaternionic Hilbert space:
(i) $\langle v|qu\rangle = \langle v|u\rangle\overline{q}$ for all $v, u \in V_L(\mathbb{H})$ and $q \in \mathbb{H}$
(ii) $pv_1 + qv_2 \in V_L(\mathbb{H})$, for all $v_1, v_2 \in V_L(\mathbb{H})$ and $p, q \in \mathbb{H}$

One may observe that $\mathbb{H}(= \mathbb{H}(\mathbb{H}))$ is a left quaternionic Hilbert space with respect to the quaternion product

$$\langle p|q\rangle = p\overline{q},\ p, q \in \mathbb{H}.$$

Also, if we take the vector space

$$\mathbb{H}^m = \{q = (q_1, q_2, \cdots, q_m) : q_i \in \mathbb{H}\}$$

under left multiplication by quaternionic scalars together with the quaternion product on $\mathbb{H}^m$ as

$$\langle p|q\rangle_{\mathbb{H}^m} = \sum_{n=1}^{m} p_i\overline{q_i},\quad p = (p_1, \cdots, p_m) \text{ and } q = (q_1, \cdots, q_m) \in V_L(\mathbb{H}).$$

Then $\mathbb{H}^m$ is a left quaternionic Hilbert space with above defined quaternion product.

Definition 2.2 ([1])**.** Let $V_L(\mathbb{H})$ be a left quaternionic Hilbert space and $T : V_L(\mathbb{H}) \to V_L(\mathbb{H})$ be an operator. Then T is said to be

- *left linear* if $T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$ for all $v_1, v_2 \in V_L(\mathbb{H})$ and $\alpha, \beta \in \mathbb{H}$.
- *bounded* if there exist $K \geq 0$ such that $\|T(v)\| \leq K\|v\|$ for all $v \in V_L(\mathbb{H})$.

Theorem 2.3 ([20])**.** *Let $\mathbb{H}$ be a left quaternionic Hilbert space. Then the left dual space of $\mathbb{H}$ is congruent to the space $\mathbb{H}$.*

Definition 2.4 ([1])**.** Let $V_L(\mathbb{H})$ be a left quaternionic Hilbert space and $T : V_L(\mathbb{H}) \to V_L(\mathbb{H})$ be an operator. Then the adjoint operator T^* of T is defined by

$$\langle v|Tu\rangle = \langle T^*v|u\rangle, \text{ for all } u, v \in V_L(\mathbb{H}).$$

Further, T is said to be *self-adjoint* if $T = T^*$.

Theorem 2.5 ([1])**.** *Let S and T be two bounded operator on $V_L(\mathbb{H})$. Then*
(a) $\langle Tv|u\rangle = \langle v|S^*u\rangle$.
(b) $(S + T)^* = S^* + T^*$.
(c) $(ST)^* = T^*S^*$.
(d) $(S^*)^* = S$.
(e) $I^* = I$, *where I is an identity operator on $V_L(\mathbb{H})$.*
(f) If S is an invertible operator then $(S^{-1})^* = (S^*)^{-1}$.

Definition 2.6. Let $V_L(\mathbb{H})$ be a finite dimensional left quaternionic Hilbert space. Then a sequence $\{v_n\}_{n=1}^m$ is said to be left basis for $V_L(\mathbb{H})$ if
(1) $V_L(\mathbb{H}) =$ left span $\{v_n\}_{n=1}^m =$ span $[v_n]_{n=1}^m$.
(2) $\{v_n\}_{n=1}^m$ is a linearly independent set.

Definition 2.7. A left basis $\{e_n\}_{n=1}^m$ for $V_L(\mathbb{H})$ is said to be left orthonormal basis if

$$\langle e_i|e_j\rangle = \delta_{ij}, \text{ for all } i, j \in \{1, 2, \cdots, m\}.$$

Khokulan et al. [15] introduced frames for finite dimensional quaternionic Hilbert spaces and gave the following definition:

Definition 2.8. A sequence $\{v_n\}_{n=1}^{m} \subset V_L(\mathbb{H})$ is said to be a *frame* for a left quaternionic Hilbert space $V_L(\mathbb{H})$ if there exist positive constants A and B such that

$$A\|v\|^2 \leq \sum_{n=1}^{m} |\langle v|v_n\rangle|^2 \leq B\|v\|^2, \quad \text{for all } v \in V_L(\mathbb{H}). \tag{2}$$

These positive constants A and B, respectively, are called lower and upper frame bounds for the frame $\{v_n\}_{n=1}^{m}$. The inequality (2) is called the *frame inequality* for the frame $\{v_n\}_{n\in\mathbb{N}}$. A frame $\{v_n\}_{n=1}^{m}$ in $V_L(\mathbb{H})$ is said to be

- *tight* if it is possible to choose $A,\ B$ satisfying inequality (2) with $A = B$.
- *Parseval* if it is tight with $A = B = 1$.
- *normalized* if $\|v_n\| = 1$, for all $n = 1, \cdots, m$.

If $\{v_n\}_{n=1}^{m}$ is a frame for $V_L(\mathbb{H})$, then the bounded linear operator $T : \mathbb{H}^m \to V_L(\mathbb{H})$ given by

$$T(\{q_n\}_{n=1}^{m}) = \sum_{n=1}^{m} q_n v_n, \text{ for all } \{q_n\}_{n=1}^{m} \in \mathbb{H}^m \tag{3}$$

is called the pre-frame operator or synthesis operator. The adjoint operator $T^* : V_L(\mathbb{H}) \to \mathbb{H}^m$ of T is given by

$$T^*(v) = \{\langle v|v_n\rangle\}_{n=1}^{m}, \text{ for all } v \in V_L(\mathbb{H}) \tag{4}$$

is called the analysis operator. By composing T and T^*, we obtain the frame operator $S_L : V_L(\mathbb{H}) \to V_L(\mathbb{H})$ defined as

$$\begin{aligned} S_L(v) &= TT^*(v) \\ &= \sum_{n=1}^{m} \langle v|v_n\rangle v_n, \text{ for all } v \in V_L(\mathbb{H}). \end{aligned}$$

3. Dual of frame for Finite Dimensional Quaternionic Hilbert Space

Let $\{v_n\}_{n=1}^{m}$ be a frame for a left quaternionic Hilbert space $V_L(\mathbb{H})$ and let T & T^* be as defined (3) & (4), respectively. Then T is injective if and only if T^* is surjective. We list this observation in the form of following result and omit the proof as it is straight forward.

Theorem 3.1. *Let $V_L(\mathbb{H})$ be a left quaternionic Hilbert space, $\{v_n\}_{n=1}^{m}$ be a sequence in $V_L(\mathbb{H})$ and T and T^* are the bounded operators as defined in (3) and (4). Then following statements are equivalent:*

(a) $\{v_n\}_{n=1}^{m}$ is frame for $V_L(\mathbb{H})$.
(b) T is injective.
(c) T^* is surjective.

If $V_L(\mathbb{H})$ be a left quaternionic Hilbert space, $\{v_n\}_{n=1}^{m}$ be a frame for $V_L(\mathbb{H})$ with the frame operator S_L and Φ be an invertible operator on $V_L(\mathbb{H})$. Then $\{\Phi v_n\}_{n=1}^{m}$ is also a frame for $V_L(\mathbb{H})$. In fact, if T is the analysis operator of $\{v_n\}_{n=1}^m$ and V is the analysis operator of $\{\Phi v_n\}_{n=1}^m$. Then, for each $v \in V_L(\mathbb{H})$, we have

$$\begin{aligned} V(v) &= \{\langle v|\Phi v_n\rangle\}_{n=1}^m \\ &= \{\langle \Phi^* v|v_n\rangle\}_{n=1}^m \\ &= T\Phi^*(v). \end{aligned}$$

This gives

$$V = T\Phi^*.$$

As $(T\Phi^*)^* = \Phi T^*$ is surjective, therefore by Theorem 3.1, $\{\Phi v_n\}_{n=1}^m$ is also a frame for $V_L(\mathbb{H})$. Moreover, if U is the frame operator of the frame $\{\Phi v_n\}_{n=1}^m$. Then, for each $v \in V_L(\mathbb{H})$, we have

$$\begin{aligned} U(v) &= \sum_{n=1}^m \langle v|\Phi v_n\rangle \Phi v_n \\ &= \Phi\left(\sum_{n=1}^m \langle v|\Phi v_n\rangle v_n\right) \\ &= \Phi\left(\sum_{n=1}^m \langle \Phi^* v|v_n\rangle v_n\right) \\ &= \Phi S_L \Phi^*(v). \end{aligned}$$

Thus, the frame operator for the frame $\{\Phi v_n\}_{n=1}^m$ is $U = \Phi S_L \Phi^*$. Further, it is easy to observe that if T is a self-adjoint on $V_L(\mathbb{H})$, then

$$\begin{aligned} \langle Tv|v\rangle &= \langle v|Tv\rangle \\ &= \overline{\langle Tv|v\rangle}, \quad v \in V_L(\mathbb{H}). \end{aligned}$$

Thus, for each $v \in V_L(\mathbb{H})$, $\langle Tv|v\rangle \in \mathbb{R}$.

In view of the above discussion, we have gave the following definition

Definition 3.2. Let T and U be self-adjoint operators on $V_L(\mathbb{H})$. Then we write $T \leq U$ if $\langle Tv|v\rangle \leq \langle Uv|v\rangle$, for all $v \in V_L(\mathbb{H})$.

Next, we prove the following result

Theorem 3.3. *Let $V_L(\mathbb{H})$ be a quaternionic Hilbert space, $\{v_n\}_{n=1}^m$ be a frame for $V_L(\mathbb{H})$ with frame bounds A and B and frame operator S_L. Then $\{S_L^{-1} v_n\}_{n=1}^m$ is also a frame for $V_L(\mathbb{H})$ with frame bounds B^{-1}, A^{-1} and frame operator S_L^{-1}.*

Proof. As $\{v_n\}_{n=1}^m$ is a frame for $V_L(\mathbb{H})$ and S_L^{-1} is an invertible operator. Therefore $\{S_L^{-1} v_n\}_{n=1}^m$ is also a frame for $V_L(\mathbb{H})$. Further, the frame operator of the frame $\{S_L^{-1} v_n\}_{n=1}^m$ is given by

$$S_L^{-1} S_L (S_L^{-1})^* = S_L^{-1}.$$

Furthermore, as $\{v_n\}_{n=1}^m$ is a frame for $V_L(\mathbb{H})$ with frame bounds A and B and frame operator S_L, we have

$$A\langle v|v\rangle \leq \langle S_L v|v\rangle \leq B\langle v|v\rangle, \text{ for all } v \in V_L(\mathbb{H}).$$

Thus in view of Definition 3.2, we have $AI \leq S_L \leq BI$. As the operator S_L^{-1} commutes with both S_L and I. Therefore, $B^{-1}I \leq S_L^{-1} \leq A^{-1}I$. □

The new frame obtained in the Theorem 3.3 is called the canonical dual frame and due to this we have the following definition:

Definition 3.4. Let $V_L(\mathbb{H})$ be a quaternionic Hilbert space and $\{v_n\}_{n=1}^m$ a frame for $V_L(\mathbb{H})$ with frame operator S_L. Then, the frame $\{S_L^{-1}v_n\}_{n=1}^m$ is called canonical dual for the frame $\{v_n\}_{n=1}^m$.

In the next result, we show that the canonical dual of a Parseval frame is the frame itself.

Theorem 3.5. *Let $V_L(\mathbb{H})$ be a left quaternionic Hilbert space and $\{v_n\}_{n=1}^m$ be a Parseval frame for $V_L(\mathbb{H})$. Then, canonical dual of $\{v_n\}_{n=1}^m$ is $\{v_n\}_{n=1}^m$.*

Proof. Let $\{v_n\}_{n=1}^m$ is a Parseval frame for $V_L(\mathbb{H})$ with the frame operator S_L. Then, we have

$$\|v\|^2 = \sum_{n=1}^{m} |\langle v|v_n\rangle^2, \ v \in V_L(\mathbb{H}).$$

This gives

$$\begin{aligned}\langle v|v\rangle &= \sum_{n=1}^{m} |\langle v|v_n\rangle^2 \\ &= \langle S_L v|v\rangle, \quad v \in V_L(\mathbb{H}).\end{aligned}$$

Thus, $S_L : V_L(\mathbb{H}) \to V_L(\mathbb{H})$ is the identity operator on $V_L(\mathbb{H})$. Hence, canonical dual of $\{v_n\}_{n=1}^m$ is $\{S_L^{-1}v_n = v_n\}_{n=1}^m$. □

In view of Theorem 3.5, we have a following remark

Remark 3.6. Let $V_L(\mathbb{H})$ be a left quaternionic Hilbert space and $\{v_n\}_{n=1}^m$ be a Parseval frame for $V_L(\mathbb{H})$. Then, the reconstruction formula takes the following form:

$$v = \sum_{n=1}^{m} \langle v|v_n\rangle v_n, \ v \in V_L(\mathbb{H}).$$

In general, dual frames for a given frame other than the canonical dual frame in a left quaternionic Hilbert space also exist. In fact we have a following definition

Definition 3.7. Let $V_L(\mathbb{H})$ be a left quaternionic Hilbert space and $\{v_n\}_{n=1}^m$ be a frame for $V_L(\mathbb{H})$. Then, a frame $\{u_n\}_{n=1}^m \subset V_L(\mathbb{H})$ is said to be a *alternate dual frame* for $\{v_n\}_{n=1}^m$ if

$$v = \sum_{n=1}^{m} \langle v|v_n\rangle u_n, \ v \in V_L(\mathbb{H}).$$

In view of above definition, one may observe that canonical dual frame for $V_L(\mathbb{H})$ is also a alternate dual frame for $V_L(\mathbb{H})$ and alternate dual for a frame for $V_L(\mathbb{H})$ is not unique. In order to show their existence we give the following example

Example 3.8. Let $\{e_j\}_{j=1}^{n}$ be a sequence of standard orthonormal basis in $V_L(\mathbb{H})$. Then, define a sequence $\{v_i\}_{i=1}^{2n}$ in $V_L(\mathbb{H})$ by

$$v_{2j-1} = v_{2j} = e_j, \ j = 1, 2, \cdots, n.$$

Then $\{v_i\}_{i=1}^{2n}$ is a frame for $V_L(\mathbb{H})$. Moreover, canonical dual $\{\widetilde{v}_n\}_{i=1}^{2n}$ of $\{v_n\}_{i=1}^{2n}$ is given by

$$\widetilde{v}_{2j-1} = \widetilde{v}_{2j} = \frac{e_j}{2}, \quad j = 1, 2, \cdots, n.$$

Next, define sequences $\{u_n\}_{i=1}^{2n}$ and $\{z_n\}_{i=1}^{2n}$ in $V_L(\mathbb{H})$ by

$$u_{2j-1} = e_j \ \text{ and } \ u_{2j} = 0, \quad j = 1, 2, \cdots, n.$$

and

$$z_{2j-1} = 0 \ \text{ and } \ z_{2j} = e_j, \quad j = 1, 2, \cdots, n.$$

Then $\{u_i\}_{i=1}^{2n}$ and $\{z_i\}_{i=1}^{2n}$ are alternate dual frames for $\{v_i\}_{i=1}^{2n}$ in $V_L(\mathbb{H})$. □

Next, we give various types of reconstruction using a given dual frame.

Theorem 3.9. *Let $\{v_n\}_{n=1}^{m}$ and $\{u_n\}_{n=1}^{m}$ be a frames for a quaternionic Hilbert space $V_L(\mathbb{H})$ with analysis operators T and U, respectively. Then the following statements are equivalent:*

(i) $v = \sum_{n=1}^{m} \langle v|v_n\rangle u_n$, *for all* $v \in V_L(\mathbb{H})$.

(ii) $v = \sum_{n=1}^{m} \langle v|u_n\rangle v_n$, *for all* $v \in V_L(\mathbb{H})$.

(iii) $\langle v|u\rangle = \sum_{n=1}^{m} \langle v|v_n\rangle\langle u_n|u\rangle$, *for all* $v \in V_L(\mathbb{H})$.

(iv) $T^*U = \mathcal{I}$ *and* $U^*T = \mathcal{I}$.

Proof. $(i) \Rightarrow (ii)$ For each $v, u \in V_L(\mathbb{H})$, we have

$$\begin{aligned} \langle v|u\rangle &= \left\langle \sum_{n=1}^{m} \langle v|v_n\rangle u_n|u \right\rangle \\ &= \sum_{n=1}^{m} \langle v|v_n\rangle\langle u_n|u\rangle \\ &= \left\langle v \middle| \sum_{n=1}^{m} \overline{\langle u_n|u\rangle} v_n \right\rangle \\ &= \left\langle v \middle| \sum_{n=1}^{m} \langle u|u_n\rangle v_n \right\rangle. \end{aligned}$$

This gives

$$v = \sum_{n=1}^{m} \langle v|u_n\rangle v_n, \quad v \in V_L(\mathbb{H}).$$

$(ii) \Rightarrow (iii)$ For each $v, u \in V_L(\mathbb{H})$, we have

$$\begin{aligned}
\langle v|u\rangle &= \left\langle v \middle| \sum_{n=1}^{m} \langle u|u_n\rangle v_n \right\rangle \\
&= \sum_{n=1}^{m} \langle v|\langle u|u_n\rangle v_n\rangle \\
&= \sum_{n=1}^{m} \langle v|v_n\rangle \overline{\langle u|u_n\rangle} \\
&= \sum_{n=1}^{m} \langle v|v_n\rangle \langle u_n|u\rangle.
\end{aligned}$$

$(iii) \Rightarrow (ii)$ Straight forward.
$(ii) \Rightarrow (iv)$ For each $x \in V_L(\mathbb{H})$, we have

$$\begin{aligned}
T^*U(v) &= T^*\{\langle v|u_n\rangle\} \\
&= \sum_{n=1}^{m} \langle v|u_n\rangle v_n \\
&= v.
\end{aligned}$$

This gives $T^*U = \mathcal{I}.$. Also, $U^*T = (T^*U)^* = \mathcal{I}$.
$(iv) \Rightarrow (i)$ Straight forward. □

Now, we show that for a given frame for a left quaternionic Hilbert space, the coefficient sequence with respect to the canonical dual frame has the minimal $\mathbb{H}^m$ norm among all the alternate dual frames.

Theorem 3.10. *Let $V_L(\mathbb{H})$ be a left quaternionic Hilbert space and $\{v_n\}_{n=1}^m$ be a frame for $V_L(\mathbb{H})$ with the frame operator S_L. Let $\{u_n\}_{n=1}^m$ be a dual frame of the frame $\{v_n\}_{n=1}^m$, then*

$$\|\{\langle v|S_L^{-1}v_n\rangle\}_{n=1}^m\|_{\mathbb{H}^m} \leq \|\{\langle v|u_n\rangle\}_{n=1}^m\|_{\mathbb{H}^m}, \quad v \in V_L(\mathbb{H}).$$

Proof. Let T be the analysis operator of $\{v_n\}_{n=1}^m$. Then, for each $v \in V_L(\mathbb{H})$, we have

$$\{\langle v|S_L^{-1}v_n\rangle\}_{n=1}^m = \{\langle S_L^{-1}v|v_n\rangle\}_{n=1}^m \in \operatorname{ran} T.$$

Since, for each $v \in V_L(\mathbb{H})$, $v = \sum_{n=1}^{m} \langle v|u_n\rangle v_n$. So, we have

$$\{\langle v|u_n\rangle - \langle v|S_L^{-1}v_n\rangle\}_{n=1}^m \in \ker T^* = (\operatorname{ran} T)^{\perp}.$$

Therefore, considering the decomposition

$$\{\langle v|u_n\rangle\}_{n=1}^m = \{\langle v|S_L^{-1}v_n\rangle\}_{n=1}^m + \{\langle v|u_n\rangle - \langle v|S_L^{-1}v_n\rangle\}_{n=1}^m$$

we obtain

$$\sum_{n=1}^{m} |\langle v|u_n\rangle|^2 = \sum_{n=1}^{m} |\langle v|S_L^{-1}v_n\rangle|^2 + \sum_{n=1}^{m} |\langle v|u_n\rangle - \langle v|S_L^{-1}v_n\rangle|^2$$

Hence

$$\|\{\langle v|S_L^{-1}v_n\rangle\}_{n=1}^m\|_{\mathbb{H}^m}\leq \|\{\langle v|u_n\rangle\}_{n=1}^m\|_{\mathbb{H}^m}, \quad v\in V_L(\mathbb{H}). \qquad \square$$

Finally, we give an equivalent condition for dual frames for a left quaternionic Hilbert spaces.

Theorem 3.11. *Let $V_L(\mathbb{H})$ be a left quaternionic Hilbert space and $\{v_n\}_{n=1}^m$ be a frame for $V_L(\mathbb{H})$ with analysis operator T and frame operator S_L. Then, the following statements are equivalent:*

(1) $\{u_n\}_{n=1}^m$ is a dual frame for $\{v_n\}_{n=1}^m$.

(2) The analysis operator W of the sequence $\{u_n - S_L^{-1}v_n\}$ satisfy

ran $T\perp$ ran W.

Proof. Let $w_n = u_n - S_L^{-1}v_n, \quad n=1,2,\cdots,m$. Then, for each $v\in V_L(\mathbb{H})$

$$\begin{aligned}\sum_{n=1}^m\langle v|u_n\rangle v_n &= \sum_{n=1}^m\langle v|w_n+S_L^{-1}v_n\rangle v_n\\ &= v+\sum_{n=1}^m\langle v|w_n\rangle v_n\\ &= v+T^*W(v).\end{aligned}$$

Hence, $\{u_n\}_{n=1}^m$ is a dual frame for $\{v_n\}_{n=1}^m$ if and only if the analysis operator W of the sequence $\{u_n - S_L^{-1}v_n\}$ satisfy ran $T\perp$ ran W. $\square$

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