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TWO HORNICH-HLAWKA-TYPE AND GRAM MATRIX-BASED INEQUALITIES

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Abstract. We establish two inequalities in real inner product spaces. The first is a multiplicative strengthening of the classical Hornich–Hlawka inequality: for all vectors x, y, z in a real inner product space H

$$\|x\|\,\|y\| + \|z\|\,\|x+y+z\| \;\geq\; \|x+z\|\,\|y+z\|.$$

We provide a complete characterization of the equality cases. The second result is a parametric inequality derived from the positive semidefiniteness of Gram matrices: for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in \mathbb{R}$,

$$\alpha^2\|x\|^2\langle y,z\rangle^2+\beta^2\|y\|^2\langle x,z\rangle^2+\gamma^2\|z\|^2\langle x,y\rangle^2+2(\alpha\beta+\alpha\gamma+\beta\gamma)\langle x,y\rangle\langle x,z\rangle\langle y,z\rangle \;\geq\; 0,$$

which, upon optimization over the parameters yields sharp reverse inequalities to the Gram determinant inequality $\det G \geq 0$. As a special case, this recovers and strengthens the classical Cauchy–Schwarz inequality.

1. Introduction

The Hornich–Hlawka inequality — sometimes called Hlawka's inequality or the quadrilateral inequality — is a classical result in the geometry of normed spaces. In its standard form, it asserts that in any normed vector space E in which the inequality holds,

$$\|x\| + \|y\| + \|z\| + \|x+y+z\| \;\geq\; \|x+y\| + \|x+z\| + \|y+z\| \qquad \forall\, x, y, z \in E.$$

The inequality was originally established by Hornich [8] who referred to Hlawka as the originator of the argument, and has since generated a substantial literature, being studied extensively from the perspectives of functional analysis, convex geometry, and the theory of inequalities. Fechner [4] initiated the systematic study of the associated functional inequality $f(x+y)+f(y+z)+f(x+z) \leq f(x+y+z)+f(x)+f(y)+f(z)$,

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